Flux reconstructions in Lehmann–Goerisch method for lower bounds on eigenvalues

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Lower bounds on eigenvalues

Laplace eigenvalue problem

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\begin{aligned}
-\Delta u_i &= \lambda_i u_i && \text{in } \Omega \\
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Weak formulation

$\lambda_i > 0, u_i \in V: \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V$

Notation:

$V = H_0^1(\Omega)$

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Finite element method

$\Lambda_{h,i} > 0, u_{h,i} \in V_h : \quad (\nabla u_{h,i}, \nabla v_h) = \Lambda_{h,i}(u_{h,i}, v_h) \quad \forall v_h \in V_h$

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Upper bound:

$$\lambda_i \le \Lambda_{h,i}$$

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Can we do lower bound?

$$\ell_i \le \lambda_i \le \Lambda_{h,i} \quad \Rightarrow \quad |\Lambda_{h,i} - \lambda_i| \le \Lambda_{h,i} - \ell_i$$

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Lower bounds on eigenvalues of differential operators

Standard (conforming) approach:
Temple (1928), Weinstein (1937), Kato (1949), Lehmann (1949), Goerisch (1985), ...

Nonconforming FEM:
Carstensen (2013), Gedicke (2013), Gallistl (2013), Xuefeng LIU (2015), ...

Lehmann–Goerisch method

Input: $\gamma > 0$ and $\ell_{m+1} \leq \lambda_{m+1}$

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Algorithm:

- FEM eigenpairs: $\Lambda_{h,i} \in \mathbb{R}$, $u_{h,i} \in V_h$, $i = 1, 2, \ldots, m$

[Behnke, Mertins, Plum, Wieners 2000]

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- Flux reconstructions: $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h$, $i = 1, 2, \ldots, m$
- For $n = m, m-1, \ldots, 2, 1$ do

$$\rho = \ell_{n+1} + \gamma$$
$$\mathbf{M}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - \rho)(u_{h,i}, u_{h,j})$$
$$\mathbf{N}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - 2\rho)(u_{h,i}, u_{h,j}) + \rho^2(\boldsymbol{\sigma}_{h,i}, \boldsymbol{\sigma}_{h,j}) + (\rho^2/\gamma)(u_{h,i} + \operatorname{div}\boldsymbol{\sigma}_{h,i}, u_{h,j} + \operatorname{div}\boldsymbol{\sigma}_{h,j})$$
$$\mu_1 \leq \cdots \leq \mu_n : \quad \mathbf{M}\mathbf{y}_i = \mu_i \mathbf{N}\mathbf{y}_i, \quad i = 1, 2, \ldots, n$$

If $\mathbf{N}$ is s.p.d. and if $\mu_{n+1-j} < 0$ then
$$\ell^*_{j,n} = \rho - \gamma - \rho/\left(1 - \mu_{n+1-j}\right) \leq \lambda_j, \quad j = 1, 2, \ldots, n.$$
$$\ell_n = \max\{\ell^*_{n,i},\ i = n, n+1, \ldots, m\} \leq \lambda_n.$$

end for

[Behnke, Mertins, Plum, Wieners 2000]

Global problem for $\boldsymbol{\sigma}_{h,i}$

(a) Mixed FEM:

Find $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h,\ q_{h,i} \in Q_h,\ i = 1, 2, \ldots, m$

$$(\boldsymbol{\sigma}_{h,i}, \mathbf{w}_h) + (q_{h,i}, \operatorname{div} \mathbf{w}_h) = \left(\frac{\nabla u_{h,i}}{\Lambda_{h,i} + \gamma}, \mathbf{w}_h \right) \qquad \forall \mathbf{w}_h \in \mathbf{W}_h$$

$$(\operatorname{div} \boldsymbol{\sigma}_{h,i}, \varphi_h) = \left(-\frac{\Lambda_{h,i} u_{h,i}}{\Lambda_{h,i} + \gamma}, \varphi_h \right) \quad \forall \varphi_h \in Q_h$$

[Behnke, Mertins, Plum, Wieners 2000]

Spaces:

$\mathbf{W}_h = \{\boldsymbol{\sigma}_h \in \mathbf{H}(\operatorname{div}, \Omega) : \boldsymbol{\sigma}_h|_K \in \mathbf{RT}_1(K) \quad \forall K \in \mathcal{T}_h\}$

$Q_h = \{q_h \in L^2(\Omega) : q_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\}$

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(b) Positive definite problem:

Find $\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h,\ i = 1, 2, \dots, m$

$$(\boldsymbol{\sigma}_{h,i}, \mathbf{w}_h) + \frac{1}{\gamma} (\operatorname{div} \boldsymbol{\sigma}_{h,i}, \operatorname{div} \mathbf{w}_h) = \left(\frac{\nabla u_{h,i}}{\Lambda_{h,i} + \gamma}, \mathbf{w}_h \right) - \frac{1}{\gamma} \left(\frac{\Lambda_{h,i} u_{h,i}}{\Lambda_{h,i} + \gamma}, \operatorname{div} \mathbf{w}_h \right)$$

$$\forall \mathbf{w}_h \in \mathbf{W}_h$$

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Local problems on patches

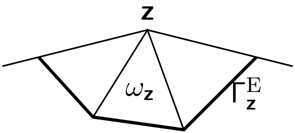
Partition of unity: $\sum_{\mathbf{z}\in\mathcal{N}_h} \psi_{\mathbf{z}} \equiv 1$ in Ω

[Braess, Schöberl 2000], [Ern, Vohralík 2013]

Construct:

$$\boldsymbol{\sigma}_{h,i} = \sum_{\mathbf{z}\in\mathcal{N}_h} \boldsymbol{\sigma}_{\mathbf{z},i},$$

where $\boldsymbol{\sigma}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$ solve local problems on patches of elements.



Spaces:

$\mathbf{W}_{\mathbf{z}} = \{\boldsymbol{\sigma}_{\mathbf{z}} \in \mathbf{H}(\mathrm{div}, \omega_{\mathbf{z}}) : \boldsymbol{\sigma}_{\mathbf{z}}|_K \in \mathbf{RT}_1(K)\ \forall K \in \mathcal{T}_{\mathbf{z}}$ and $\boldsymbol{\sigma}_{\mathbf{z}} \cdot \mathbf{n}_{\mathbf{z}} = 0$ on $\Gamma_{\mathbf{z}}^{\mathrm{E}}\}$

$Q_{\mathbf{z}} = \{q_{\mathbf{z}} \in L^2(\omega_{\mathbf{z}}) : q_{\mathbf{z}}|_K \in P_1(K) \quad \forall K \in \mathcal{T}_{\mathbf{z}}\}$

Local problems on patches

(c) Local mixed FEM:

Find $\boldsymbol{\sigma}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}},\ q_{\mathbf{z},i} \in Q_{\mathbf{z}},\ i = 1, 2, \dots, m$

$$(\boldsymbol{\sigma}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} + (q_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} = \left(\psi_{\mathbf{z}} \frac{\nabla u_{h,i}}{\Lambda_{h,i} + \gamma}, \mathbf{w}_h \right)_{\omega_{\mathbf{z}}} \qquad \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}}$$

$$(\operatorname{div} \boldsymbol{\sigma}_{\mathbf{z},i}, \varphi_h)_{\omega_{\mathbf{z}}} = \left(-\frac{\Lambda_{h,i} \psi_{\mathbf{z}} u_{h,i}}{\Lambda_{h,i} + \gamma}, \varphi_h \right)_{\omega_{\mathbf{z}}} + \left(\frac{\nabla \psi_{\mathbf{z}} \cdot \nabla u_{h,i}}{\Lambda_{h,i} + \gamma}, \varphi_h \right)_{\omega_{\mathbf{z}}} \qquad \forall \varphi_h \in Q_{\mathbf{z}}$$

(d) Local positive definite problem:

Find $\boldsymbol{\sigma}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}},\ i = 1, 2, \dots, m$

$$(\boldsymbol{\sigma}_{\mathbf{z},i}, \mathbf{w}_h)_{\omega_{\mathbf{z}}} + \frac{1}{\gamma} (\operatorname{div} \boldsymbol{\sigma}_{\mathbf{z},i}, \operatorname{div} \mathbf{w}_h)_{\omega_{\mathbf{z}}} = \left(\psi_{\mathbf{z}} \frac{\nabla u_{h,i}}{\Lambda_{h,i} + \gamma}, \mathbf{w}_h \right)_{\omega_{\mathbf{z}}} - \frac{1}{\gamma} \left(\frac{\Lambda_{h,i} \psi_{\mathbf{z}} u_{h,i}}{\Lambda_{h,i} + \gamma}, \operatorname{div} \mathbf{w}_h \right)_{\omega_{\mathbf{z}}} \qquad \forall \mathbf{w}_h \in \mathbf{W}_{\mathbf{z}}$$

Example: Dumbbell shaped domain

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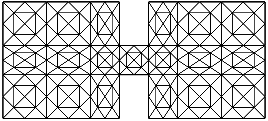
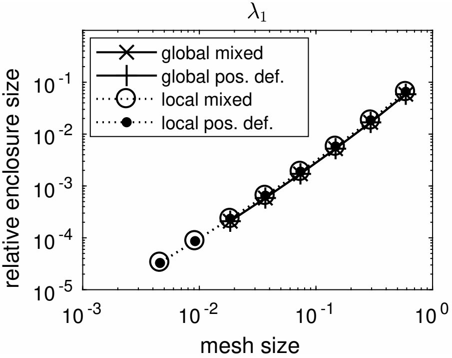
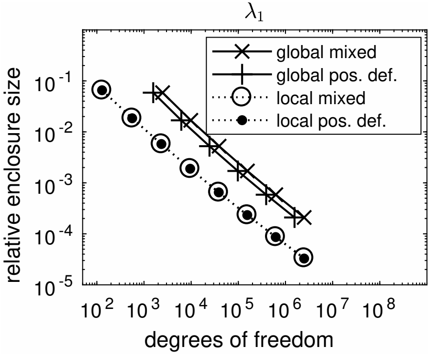


Uniformly refined meshes:





- $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}, \quad \ell_{11} = 8.9383 \leq \lambda_{11} \approx 10.0017$

Conclusions

There are known flux reconstructions for source problems.

- They can be used for eigenvalue problems
- Savings in memory requirements
- Parallelization

Generalizations:

- General symmetric elliptic operators
- Higher-order approximations
- Adaptivity

Thank you for your attention

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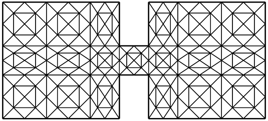
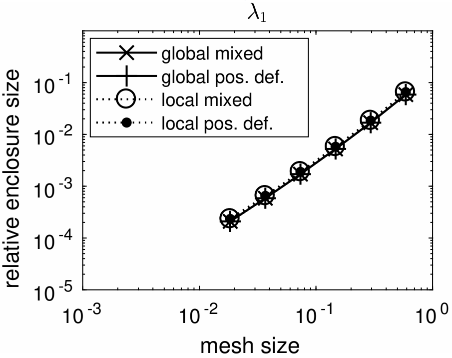
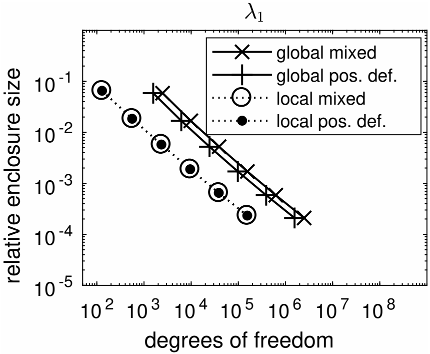


Dependence on γ:





- $\gamma = 10^{-3}$, $\quad \ell_{11} = 8.9383 \leq \lambda_{11} \approx 10.0017$

Example: Dumbbell shaped domain

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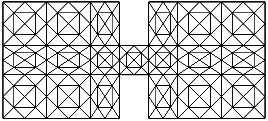

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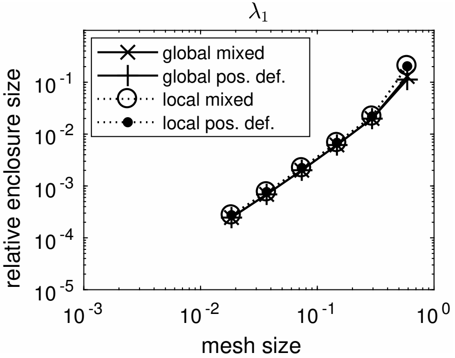

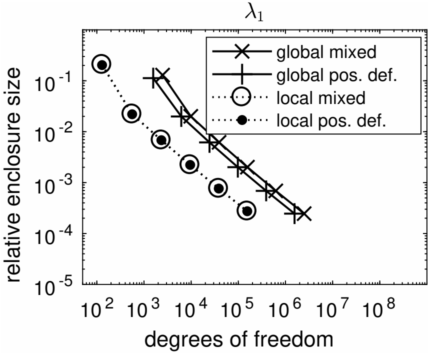

- $\gamma = 1, \quad \ell_{11} = 8.9383 \leq \lambda_{11} \approx 10.0017$

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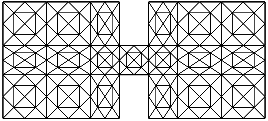

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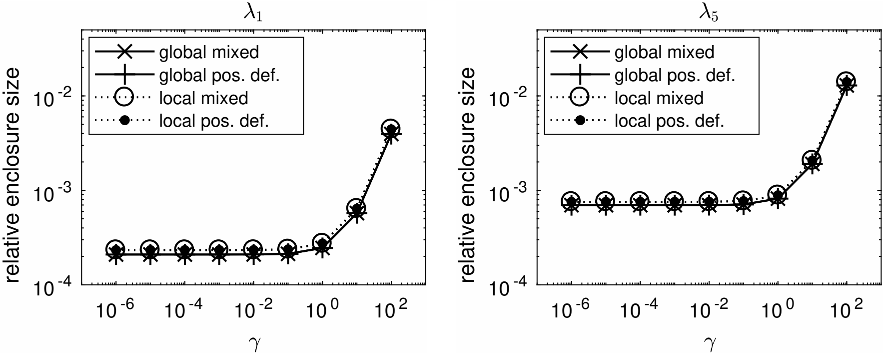

- 6th mesh, $\ell_{11} = 8.9383 \leq \lambda_{11} \approx 10.0017$

Minimization

It is natural to minimize:

$$\left\| \frac{\nabla u_{h,i}}{\Lambda_{h,i} + \gamma} - \boldsymbol{\sigma}_i \right\|_0^2 + \frac{1}{\gamma} \left\| \frac{\Lambda_{h,i} u_{h,i}}{\Lambda_{h,i} + \gamma} + \operatorname{div} \boldsymbol{\sigma}_i \right\|_0^2$$

over a suitable subspace $\mathbf{W}_h \subset \mathbf{H}(\operatorname{div}, \Omega)$.