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The Interior Regularity of Pressure Associated with a Weak Solution to the Navier–Stokes Equations with the Navier–Type Boundary Conditions

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Abstract

We prove that if $\boldsymbol{u}$ is a weak solution to the Navier–Stokes system with the Navier–type boundary conditions in $\Omega \times (0,T)$, satisfying the strong energy inequality in $\Omega \times (0,T)$ and Serrin's integrability conditions in $\Omega' \times (t_1,t_2)$ (where Ω' is a sub-domain of Ω and $0 \leq t_1 < t_2 \leq T$) then p and $\partial_t\boldsymbol{u}$ have spatial derivatives of all orders essentially bounded in $\Omega'' \times (t_1+\epsilon, t_2-\epsilon)$ for any bounded sub-domain $\Omega'' \subset \overline{\Omega''} \subset \Omega'$ and $\epsilon > 0$ so small that $t_1+\epsilon < t_2-\epsilon$. (See Theorem 1.) We show an application of Theorem 1 to the procedure of localization.

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Key words: Navier–Stokes equation, Navier–type boundary conditions, interior regularity.

1 Introduction

1.1. Notation. We assume that Ω is a bounded domain in $\mathbb{R}^3$ with the boundary of the class $C^{2+(h)}$ for some $h>0$. We denote vector functions and spaces of vector functions by boldface letters. Furthermore,

- the scalar product in $\boldsymbol{L}^2(\Omega)$ is denoted by $(\,.\,,\,.\,)_2$ and the induced norm is denoted by $\|\,.\,\|_2$.
- c is a generic constant, i.e. a constant whose value may vary from line to line.
- $\boldsymbol{L}^s_{\tau,\sigma}(\Omega)$ (where $1<s<\infty$) is the closure of $\boldsymbol{C}^\infty_{0,\sigma}(\Omega)$ (the linear space of infinitely differentiable divergence–free vector functions in Ω, with a compact support in Ω) in $\boldsymbol{L}^s(\Omega)$. $\boldsymbol{L}^s_{\tau,\sigma}(\Omega)$ can be characterized as a space of functions from $\boldsymbol{L}^s(\Omega)$ that are divergence–free in the sense of distributions (which is the sense of the subscript σ) and their normal component on $\partial\Omega$ is equal to zero in the sense of traces, see e.g. [8] for the detailed explanation. (I.e. the functions are tangent to $\partial\Omega$, which is the meaning of the subscript τ.)
- $\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega) := \boldsymbol{W}^{1,2}(\Omega) \cap \boldsymbol{L}^2_{\tau,\sigma}(\Omega)$; the dual space to $\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)$ is denoted by $\boldsymbol{W}^{-1,2}_{\tau,\sigma}(\Omega)$ and the duality between elements of $\boldsymbol{W}^{-1,2}_{\tau,\sigma}(\Omega)$ and $\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)$ is denoted by $\langle\,.\,,\,.\,\rangle_\sigma$.
- $\boldsymbol{n}$ denotes the outer normal vector field on $\partial\Omega$.
- $\boldsymbol{W}^{1,2}_{\tau}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}$; the dual space to $\boldsymbol{W}^{1,2}_{\tau}(\Omega)$ is denoted by $\boldsymbol{W}^{-1,2}_{\tau}(\Omega)$ and the duality between elements of $\boldsymbol{W}^{-1,2}_{\tau}(\Omega)$ and $\boldsymbol{W}^{1,2}_{\tau}(\Omega)$ is denoted by $\langle\,.\,,\,.\,\rangle$.

1.2. The initial–boundary value problem, a weak solution. Let $T>0$. We consider the Navier–Stokes initial–boundary value problem

$$\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p \;=\; \nu\Delta\boldsymbol{u} \qquad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} \;=\; 0 \qquad \text{in } \Omega\times(0,T), \tag{1.2}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} \;=\; 0 \qquad \text{on } \partial\Omega\times(0,T), \tag{1.3}$$

$$\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} \ = \ \boldsymbol{0} \qquad \text{on } \partial\Omega\times(0,T), \tag{1.4}$$

$$\boldsymbol{u}(\,.\,,0) \ = \ \boldsymbol{u}_0 \qquad \text{in } \Omega\times 0, \tag{1.5}$$

where $\boldsymbol{u}=(u_1,u_2,u_3)$ is the velocity, p is the pressure and ν is the kinematic coefficient of viscosity. (It is a positive constant.) Boundary conditions (1.3), (1.4) are often called the *Navier–type boundary conditions.* (See e.g. [3] for a more detailed explanation.)

Denote by $\mathscr{A}_\sigma$ the linear operator from $\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)$ to $\boldsymbol{W}^{-1,2}_{\tau,\sigma}(\Omega)$, defined by the equation

$$\langle \mathscr{A}_\sigma \boldsymbol{v},\boldsymbol{\phi}\rangle_\sigma \ := \ (\mathbf{curl}\,\boldsymbol{v},\mathbf{curl}\,\boldsymbol{\phi})_2 \qquad \text{for all } \boldsymbol{v},\ \boldsymbol{\phi}\in\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega).$$

Similarly, denote by $\mathscr{B}_\sigma$ the quadratic operator from $\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)\times\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)$ to $\boldsymbol{W}^{-1,2}_{\tau,\sigma}(\Omega)$, defined by the equation

$$\langle \mathscr{B}_\sigma \boldsymbol{v},\boldsymbol{\phi}\rangle_\sigma \ := \ (\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\phi})_2 \qquad \text{for all } \boldsymbol{v},\ \boldsymbol{\phi}\in\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega).$$

(As the right hand side can be considered to be a duality between $\boldsymbol{v}\cdot\nabla\boldsymbol{v}\in\boldsymbol{W}^{-1,2}_{\tau}(\Omega)$ and $\boldsymbol{\phi}\in\boldsymbol{W}^{1,2}_{\tau}(\Omega)$, we observe that $\mathscr{B}_\sigma\boldsymbol{v}=\mathcal{P}_\sigma(\boldsymbol{v}\cdot\nabla\boldsymbol{v})$.) By definition, function $\boldsymbol{u}\in L^2(0,T;\,\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega))\cap C_w\big([0,T);\,\boldsymbol{L}^2_{\tau,\sigma}(\Omega)\big)$ is said to be a *weak solution* of the problem (1.1)–(1.5) if $\boldsymbol{u}'$ (the distributional derivative with respect to t of $\boldsymbol{u}$ as a function from $(0,T)$ to $\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)$) is in $L^1(0,T;\,\boldsymbol{W}^{-1,2}_{\tau,\sigma}(\Omega))$ and $\boldsymbol{u}$ satisfies the equation

$$\boldsymbol{u}'+\nu\,\mathscr{A}_\sigma\boldsymbol{u}+\mathscr{B}_\sigma\boldsymbol{u} \ = \ \boldsymbol{0} \tag{1.6}$$

a.e. in $(0,T)$ and it also satisfies the initial condition (1.5).

It is not clear at the first sight whether and how this definition involves the boundary condition (1.4). However, assuming that $\boldsymbol{u}$ is a "smooth" weak solution, one can reconstruct (1.4) from equation (1.6), just writing it in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\boldsymbol{u},\boldsymbol{\phi})_2+\nu\,\langle\mathscr{A}_\sigma\boldsymbol{u},\boldsymbol{\phi}\rangle_\sigma+\langle\mathscr{B}_\sigma\boldsymbol{u},\boldsymbol{\phi}\rangle_\sigma \ = \ 0$$

(a.e. in $(0,T)$, for all $\boldsymbol{\phi}\in\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)$), or equivalently in the form

$$\int_0^T\!\!\int_\Omega\big[-\vartheta'(t)\,\boldsymbol{u}\cdot\boldsymbol{\phi}+\nu\vartheta(t)\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\phi}+\theta(t)\,\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \ = \ \int_\Omega\theta(0)\,\boldsymbol{u}_0\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} \tag{1.7}$$

(for all test functions $\vartheta\in C^\infty\big([0,T]\big)$ such that $\vartheta(T)=0$ and $\boldsymbol{\phi}\in\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)$), and applying appropriately the integration by parts.

If $\boldsymbol{u}_0\in\boldsymbol{L}^2_{\tau,\sigma}(\Omega)$ then, due to [23, Theorem 6.3], the problem (1.1)–(1.5) has at least one weak solution, that satisfies the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{2}\,\|\boldsymbol{u}(\,.\,,t)\|_2^2+\nu\,\|\mathbf{curl}\,\boldsymbol{u}\|_2^2 \ \leq \ 0 \tag{1.8}$$

in the sense of distributions in $(0,T)$.

1.3. Previous results on the interior regularity of velocity and pressure and aims of this paper. The next lemma recalls the well known Serrin's result on the interior regularity of weak solutions to the system (1.1), (1.2). (See e.g. [16] or [9].) It concerns weak solutions in $\Omega'\times(t_1,t_2)$, where Ω' is a sub-domain of Ω, independently of boundary conditions satisfied on $\partial\Omega\times(0,T)$.

Lemma 1. *Let Ω' be a sub-domain of Ω, $0\leq t_1<t_2\leq T$ and let $\boldsymbol{u}$ be a weak solution to the system (1.1), (1.2) in $\Omega'\times(t_1,t_2)$. (It means that $\boldsymbol{u}$ satisfies (1.7) for all test functions $\vartheta\in C^\infty(t_1,t_2)$, such that $\operatorname{supp}\vartheta\subset(t_1,t_2)$, and $\boldsymbol{\phi}\in\boldsymbol{C}^\infty_{0,\sigma}(\Omega)$, with the support in Ω'.) Let $\boldsymbol{u}\in L^r(t_1,t_2;\,\boldsymbol{L}^s(\Omega'))$, where $r\in[2,\infty)$, $s\in(3,\infty]$ and $2/r+3/s=1$. Then, if Ω'' is a sub-domain of Ω' such that $\Omega''\subset\overline{\Omega''}\subset\Omega'$ and $0<2\epsilon<t_2-t_1$, solution $\boldsymbol{u}$ has all spatial derivatives (of all orders) bounded in $\Omega''\times(t_1+\epsilon,t_2-\epsilon)$.*

Note that it is not important whether Ω and Ω' are bounded or unbounded in the lemma, but it is important that Ω'' is bounded. Also note that the same statement on the associated pressure p (see subsection 1.6) or the time derivative $\partial_t \boldsymbol{u}$ is not known to hold. If $\boldsymbol{u}$ satisfies the no–slip boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega \times (0,T)$ then p is only known to have any spatial derivative in $L^q(t_1+\epsilon, t_2-\epsilon;\, L^\infty(\Omega''))$ for any $q \in (1,2)$, see [13], [15] or [18]. The analogous statement also holds on $\partial_t \boldsymbol{u}$, because ∇p and $\partial_t \boldsymbol{u}$ are interconnected through the Navier–Stokes equation (1.1). If $\Omega = \mathbb{R}^3$ then the statement on the regularity of p can be improved so that p has all spatial derivatives in $L^\infty(t_1+\epsilon, t_2-\epsilon;\, L^\infty(\Omega''))$, see [18]. These results confirm the well known fact that p is a global quantity, and its behavior in a sub-domain Ω'' of Ω is influenced by the boundary conditions, satisfied by $\boldsymbol{u}$ on $\partial\Omega$, independently of the distance between Ω'' and $\partial\Omega$. The aim of this paper is to show that if $\boldsymbol{u}$ satisfies the Navier–type boundary conditions (1.3), (1.4) then we can derive the same estimates of spatial derivatives of p and $\partial_t \boldsymbol{u}$ in $\Omega'' \times (t_1+\epsilon, t_2-\epsilon)$ as in the case $\Omega = \mathbb{R}^3$. Our main result is formulated in Theorem 1 in Section 2. We need a series of auxiliary results (on the uniqueness of weak solutions, strong energy inequality, existence of an associated pressure, etc.), which are well known for weak solutions with the no–slip boundary condition. We recall or reprove these results for weak solutions with the boundary conditions (1.3), (1.4) in the next subsections. To illustrate an application of Theorem 1, we explain the procedure of localization in Section 3 and show how Theorem 1 improves the regularity of the right hand side of the localized Navier–Stokes equation.

1.4. Uniqueness of weak solutions. By analogy with [9, Theorem 4.2] or [20, Theorem V.1.5.1] (on the uniqueness of weak solutions), which concern the Navier–Stokes problem with the no–slip boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega \times (0,T)$, one can also prove the same result for weak solutions to the problem with the Navier–type boundary conditions (1.3), (1.4). We state this result without proof because the proof would be more or less a straightforward copy of the procedures from [9] or [20]:

Lemma 2. *Let $\boldsymbol{u}_0 \in \boldsymbol{L}^2_{\tau,\sigma}(\Omega)$. Let $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ be two weak solutions to the problem (1.1)–(1.5). Suppose that $\boldsymbol{u}^1$ satisfies the energy inequality*

$$\frac{1}{2}\,\|\boldsymbol{u}^1(t)\|_2^2 + \nu \int_0^t \|\mathbf{curl}\,\boldsymbol{u}^1(s)\|_2^2\,\mathrm{d}s \;\le\; \frac{1}{2}\|\boldsymbol{u}_0\|_2^2 \tag{1.9}$$

(for all $t \in (0,T)$) and $\boldsymbol{u}^2 \in L^r(0,T;\, \boldsymbol{L}^s(\Omega))$ for some r, $s \in \mathbb{R}$ such that $s > 3$ and $2/r + 3/s = 1$. Then $\boldsymbol{u}^1 = \boldsymbol{u}^2$.

1.5. The strong energy inequality. Let χ be an infinitely differentiable function on $(-\infty,\infty)$ such that $\chi = 0$ on $(-\infty,0]$, χ is nondecreasing on $(0,1)$ and $\chi = 1$ on $[1,\infty)$. Let $0 \le t_1 < t_2 < T$ and $m, n \in \mathbb{N}$ be so large that $t_1 + 1/n < t_2$, $t_2 + 1/m < T$. Then

$$\chi_{1,m}(t) \;:=\; \chi\big(m(t-t_1)\big) \qquad \begin{cases} =\; 0 & \text{for } t \le t_1, \\ \in (0,1) & \text{for } t_1 < t < t_1 + 1/m, \\ =\; 1 & \text{for } t_1 + 1/m \le t. \end{cases},$$

$$\chi_{2,n}(t) \;:=\; \chi\big(n(t_2-t)+1\big) \begin{cases} =\; 1 & \text{for } t \le t_2, \\ \in (0,1) & \text{for } t_2 < t < t_2 + 1/n, \\ = 0 & \text{for } t_2 + 1/n \le t. \end{cases}$$

Observe that $\chi'_{1,m}(t) = m\,\chi'\big(m(t-t_1)\big) \ge 0$ and $\chi'_{2,n}(t) = -n\,\chi'\big(n(t_2-t)+1\big) \le 0$. Testing inequality (1.8) by the product $\chi_{1,m}(t)\,\chi_{2,n}(t)$, we obtain

$$\begin{aligned} -\frac{1}{2}\int_{t_2}^{t_2+1/n} \chi'_{2,n}(t)\,\|\boldsymbol{u}(\,.\,,t)\|_2^2\,\mathrm{d}t + \nu \int_{t_1}^{t_2+1/n} \chi_{1,m}(t)\,\chi_{2,n}(t)\,\|\mathbf{curl}\,\boldsymbol{u}(\,.\,,t)\|_2^2\,\mathrm{d}t \\ \le\; \frac{1}{2}\int_{t_1}^{t_1+1/m} \chi'_{1,m}(t)\,\|\boldsymbol{u}(\,.\,,t)\|_2^2\,\mathrm{d}t. \end{aligned}$$

Considering the limit inferior for $n \to \infty$ and using the weak continuity of $\boldsymbol{u}$ from $(0,T)$ to $\boldsymbol{L}^2(\Omega)$ and the classical property of weak limits, we obtain

$$\frac{1}{2}\,\|\boldsymbol{u}(t_2)\|_2^2 + \nu \int_{t_1}^{t_2} \chi_{1,m}(t)\,\|\mathbf{curl}\,\boldsymbol{u}(t)\|_2^2\;\mathrm{d}t \;\le\; \frac{1}{2}\int_{t_1}^{t_1+1/m} \chi'_{1,m}(t)\,\|\boldsymbol{u}(t)\|_2^2\;\mathrm{d}t. \tag{1.10}$$

One can deduce from inequality (1.8) that the norm $\|\boldsymbol{u}(\,.\,,t)\|_2$ is a.e. in $(0,T)$ a non-increasing function of t. Hence $\frac{1}{2}\,\|\boldsymbol{u}(t_1)\|_2^2$ is greater than or equal to the limit superior of the right hand side of (1.10) for $m \to \infty$ at a.a. points t_1 in $[0,t_2)$. Consequently, $\boldsymbol{u}$ satisfies

$$\frac{1}{2}\,\|\boldsymbol{u}(t_2)\|_2^2 + \nu \int_{t_1}^{t_2} \|\mathbf{curl}\,\boldsymbol{u}(t)\|_2^2\;\mathrm{d}t \;\le\; \frac{1}{2}\|\boldsymbol{u}(t_1)\|_2^2\;\mathrm{d}t \tag{1.11}$$

for a.a. $t_1 \in [0,T)$ and all $t_2 \in (t_1,T)$. Inequality (1.11) is called the *strong energy inequality,* in contrast to the energy inequality (1.9). Note that (1.11) is a direct consequence of (1.8). It is generally not known if every weak solution of (1.1)–(1.5) satisfies (1.9) or (1.11).

1.6. An associated pressure. Let $\boldsymbol{u}$ be a weak solution to the problem (1.1)–(1.5). We say that p is an *associated pressure* if $\boldsymbol{u}$ and p satisfy equation (1.1) in the sense of distributions in Q_T. The purpose of this subsection is to show that an associated pressure exists.

Let $\mathcal{P}_\sigma$ be a linear mapping of $\boldsymbol{W}_\tau^{-1,2}(\Omega)$ to $\boldsymbol{W}_{\tau,\sigma}^{-1,2}(\Omega)$, defined by the equation

$$\langle \mathcal{P}_\sigma \boldsymbol{f}, \boldsymbol{\phi}\rangle_\sigma \;:=\; \langle \boldsymbol{f}, \boldsymbol{\phi}\rangle \qquad \text{for all } \boldsymbol{\phi} \in \boldsymbol{W}_{\tau,\sigma}^{1,2}(\Omega).$$

By analogy with [17, Lemma 4], one can prove that mapping $\mathcal{P}_\sigma$ is continuous, its range is the whole space $\boldsymbol{W}_{\tau,\sigma}^{1,2}(\Omega)$ and it is not one–to–one. Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ then $\mathcal{P}_\sigma$ coincides with the Helmholtz projection in $\boldsymbol{L}^2(\Omega)$. Consequently, $\mathcal{P}_\sigma \boldsymbol{f} = \boldsymbol{f}$ for $\boldsymbol{f} \in \boldsymbol{L}_{\tau,\sigma}^2(\Omega)$.

Let the operator $\mathscr{A}$ from $\boldsymbol{W}_\tau^{1,2}(\Omega)$ to $\boldsymbol{W}_\tau^{-1,2}(\Omega)$ be defined by the equation

$$\langle \mathscr{A}\boldsymbol{v}, \boldsymbol{\phi}\rangle \;:=\; (\mathbf{curl}\,\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{\phi})_2 + (\operatorname{div}\boldsymbol{v}, \operatorname{div}\boldsymbol{\phi})_2$$

for all $\boldsymbol{v},\ \boldsymbol{\phi} \in \boldsymbol{W}_\tau^{1,2}(\Omega)$. Then, obviously, $\mathscr{A}_\sigma \boldsymbol{v} = \mathcal{P}_\sigma \mathscr{A}\boldsymbol{v}$. Thus, equation (1.6) can also be written in the form

$$\boldsymbol{u}' + \nu\,\mathcal{P}_\sigma \mathscr{A}\boldsymbol{u} + \mathcal{P}_\sigma(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) \;=\; \mathbf{0}.$$

Integrating this equation with respect to time from 0 to t and using the identities $\boldsymbol{u}(\,.\,,t) = \mathcal{P}_\sigma \boldsymbol{u}(\,.\,,t)$ and $\boldsymbol{u}_0 = \mathcal{P}_\sigma \boldsymbol{u}_0$, we obtain

$$\mathcal{P}_\sigma \boldsymbol{u}(\,.\,,t) - \mathcal{P}_\sigma \boldsymbol{u}_0 + \int_0^t \mathcal{P}_\sigma\big[\nu\,\mathscr{A}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\big](\,.\,,s)\;\mathrm{d}s \;=\; \mathbf{0}.$$

Hence the linear functional $\mathcal{F}(\boldsymbol{u})$ on $\boldsymbol{W}_\tau^{1,2}(\Omega)$, defined by the equation

$$\big\langle \mathcal{F}(\boldsymbol{u}), \boldsymbol{\phi}\big\rangle \;:=\; \int_\Omega \big[\boldsymbol{u}(\,.\,,t) - \boldsymbol{u}_0\big]\cdot\boldsymbol{\phi}\;\mathrm{d}\boldsymbol{x} + \int_\Omega \int_0^t \big[\nu\,\mathscr{A}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\big](\,.\,,s)\;\mathrm{d}s\;\cdot\boldsymbol{\phi}\;\mathrm{d}\boldsymbol{x}$$

vanishes on the subspace $\boldsymbol{W}_{\tau,\sigma}^{1,2}(\Omega)$ of $\boldsymbol{W}_\tau^{1,2}(\Omega)$. Functional $\mathcal{F}(\boldsymbol{u})$ can also be considered to be a distribution in Ω, that vanishes on $\boldsymbol{C}_{0,\sigma}^\infty(\Omega)$. By [22, Proposition I.1.1] (which originally comes from G. De Rham), there exists a distribution $P(t)$ in Ω such that

$$\boldsymbol{u}(\,.\,,t) - \boldsymbol{u}_0 + \int_0^t \big[\nu\,\mathscr{A}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\big](\,.\,,s)\;\mathrm{d}s \;=\; -\nabla P(t), \tag{1.12}$$

which is now the equation in the space of distributions in Ω. Observe that if $\mathscr{A}$ is considered to be a distribution in Ω then it coincides with $-\Delta$, where Δ is the distributional Laplace operator. Each term in equation (1.12) can also be considered to be a distribution in $Q_T := \Omega \times (0,T)$, i.e. a distribution acting on functions from $\boldsymbol{C}_0^\infty(Q_T)$. Differentiating equation (1.12) in the sense of distributions with respect to t, denoting $p := \partial_t P$, and using the identity $A = -\Delta$, we obtain the Navier–Stokes equation (1.1), which is now satisfied by $\boldsymbol{u}$ and p in the sense of distributions in Q_T. Thus, p is a pressure associated with the weak solution $\boldsymbol{v}$.

Note that we have not used any assumptions on domain Ω in this subsection. It means that the associated pressure exists (as a distribution) to a weak solution to (1.1)–(1.5) in any domain Ω. On the other hand, if Ω is a smooth bounded domain in $\mathbb{R}^3$, $\boldsymbol{u}_0$ is in the domain of a certain fractional power of the Stokes operator and $\boldsymbol{u}$ is a weak solution to the Navier–Stokes system (1.1), (1.2) with the initial condition (1.5) and with the boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega \times (0,T)$ then the associated pressure is known to be a function e.g. from $L^{5/3}(Q_T)$, see [19]. (Similar conclusions can also be found in paper [21].) To our knowledge, the same results are not known for weak solutions with the boundary conditions (1.3), (1.4), and it is not our aim to prove them in this paper. Here, we shall use another kind of "local" regularity of p, obtained in subsections 1.7 and 1.8.

1.7. The local in time existence of a strong solution. By analogy with the case of the no–slip boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$, the local in time existence of a strong solution to the Navier–Stokes system (1.1), (1.2) has also been proven in the case of the Navier–type boundary conditions (1.3), (1.4). We can cite e.g. the works [1], [23], [6] and [3]. The least assumptions on the initial velocity $\boldsymbol{u}_0$ are imposed in paper [1], where $\boldsymbol{u}_0$ is supposed to be in $\boldsymbol{L}^q_{\tau,\sigma}(\Omega)$ for some $q \geq 3$ and the solution is obtained in $C^0\big([0,T_*];\, \boldsymbol{L}^q_{\tau,\sigma}(\Omega)\big) \cap L^r(0,T_0;\, \boldsymbol{L}^s_{\tau,\sigma}(\Omega))$ for $r > q$, $s > q$, $2/r + 3/s = 3/q$ and some $T_* > 0$. In [23], the initial velocity $\boldsymbol{u}_0$ is assumed to be in $\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)$ and the solution is in $L^2(0,T_*;\boldsymbol{W}^{2,2}(\Omega)) \cap C^0\big([0,T_*];\, \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)\big)$, with the time derivative in $L^2(0,T_*;\, \boldsymbol{L}^2_{\tau,\sigma}(\Omega))$. The authors of [6] assume that $\boldsymbol{u}_0 \in \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega))$ and the solution is shown to be in $C^1\big([0,T_*);\, \boldsymbol{L}^2_{\tau,\sigma}(\Omega)\big) \cap C^0\big([0,T_*);\, \boldsymbol{W}^{2,2}(\Omega)\big))$. The pressure satisfies the Neumann problem

$$\Delta p = -(\nabla \boldsymbol{u}) : (\nabla \boldsymbol{u})^T \quad \text{in } \Omega, \qquad \frac{\partial p}{\partial \boldsymbol{n}} = \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{n} \quad \text{on } \partial\Omega. \tag{1.13}$$

In [3], the authors construct a ν–continuous family of strong solutions of the Euler or Navier–Stokes equations on the time interval $(0,T_*)$, provided that the initial velocity is in $\boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega) \cap \boldsymbol{W}^{4,2}(\Omega)$.

The boundary $\partial\Omega$ is assumed to be of the class $C^{3,1}$ in [3]. Although the authors of [6] and [23] only assume that the boundary of Ω is "smooth", a closer study of the proofs shows that the aforementioned results from [6] and [23] are applicable to our case (i.e. a bounded domain Ω with the boundary of class $C^{2+(h)}$ for some $h > 0$). Paper [23] also brings theorems that provide (locally in time) strong solutions with higher regularity. Concretely, if $\boldsymbol{u}_0 \in \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega)$ then $\boldsymbol{u} \in L^2(0,T_*;\, \boldsymbol{W}^{3,2}(\Omega)) \cap C^0\big([0,T_*];\, \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega)\big)$ and the time derivative is in $L^2(0,T_*;\, \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega))$. If, moreover, $\boldsymbol{u}_0 \in \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega) \cap \boldsymbol{W}^{2,3}(\Omega)$ then $\boldsymbol{u} \in L^2(0,T_*;\, \boldsymbol{W}^{4,2}(\Omega)) \cap C^0\big([0,T_*];\, \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega) \cap \boldsymbol{W}^{3,2}(\Omega)\big)$ and the time derivative is in $L^2(0,T_*;\, \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega) \cap \boldsymbol{W}^{2,2}(\Omega))$. However, these results require a higher smoothness of $\partial\Omega$ than we assume in this paper.

1.8. The structure of the weak solution $\boldsymbol{u}$. Let us denote by $\mathcal{T}_0$ the set of time instants $t_1 \in (0,T)$ such that $\boldsymbol{u}(\,.\,,t_1) \in \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega))$ and the strong energy inequality (1.11) holds for all $t_2 \in (t_1,T)$. Since the weak solution $\boldsymbol{u}$ is in $L^2(0,T;\, \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega))$ for a.a. $t_1 \in (0,T)$ and inequality (1.11) holds for a.a. $t_1 \in (0,T)$, the Lebesgue measure of $(0,T) \smallsetminus \mathcal{T}_0 = 0$. Considering $\boldsymbol{u}(\,.\,,t_1)$ (for $t_1 \in \mathcal{T}_0$) to be an initial value for a new solution, we obtain from [23] that there exists $\delta(t_1) > 0$ such that $t_1 + \delta(t_1) \leq T$ and the Navier–Stokes system (1.1), (1.2) with the boundary conditions (1.3), (1.4) has a strong solution $\widetilde{\boldsymbol{u}} \in L^2(t_1,t_1+\delta(t_1);\boldsymbol{W}^{2,2}(\Omega)) \cap C\big([t_1,t_1+\delta(t_1)];\, \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)\big)$. Due to the regularity of $\widetilde{\boldsymbol{u}}$ and the energy inequality (1.11) satisfied by $\boldsymbol{u}$, the solution $\widetilde{\boldsymbol{u}}$ can be identified with $\boldsymbol{u}$ on the time interval $(t_1,t_1+\delta(t_1))$. (See Lemma 2.) Put $\mathcal{T}_1 := \cup_{t_1 \in \mathcal{T}_0}(t_1,t_1+\delta(t_1))$. Then $\mathcal{T}_1$ has the form $\mathcal{T}_1 = \cup_{\gamma\in\Gamma}(a_\gamma,b_\gamma)$, where set Γ is at most countable. It follows from [23] that solution $\boldsymbol{u}$ is smooth on each of the intervals (a_γ,b_γ) in the sense that $\boldsymbol{u} \in L^2(t_1,t_2;\, \boldsymbol{W}^{2,2}(\Omega)) \cap C^0\big([t_1,t_2];\, \boldsymbol{W}^{1,2}_{\tau,\sigma}(\Omega)\big)$ and the time derivative is in $L^2(t_1,t_2;\, \boldsymbol{L}^2_{\tau,\sigma}(\Omega))$ for all $a_\gamma < t_1 < t_2 < b_\gamma$. Applying [6], one can even state that $\boldsymbol{u} \in C^1\big([t_1,t_2];\, \boldsymbol{L}^2_{\tau,\sigma}(\Omega)\big) \cap C^0\big([t_1,t_2];\, \boldsymbol{W}^{2,2}(\Omega)\big)$. Consequently, $\nabla p \in C^0(t_1,t_2;\boldsymbol{L}^2(\Omega))$ and p satisfies (1.13) for each $t \in [t_1,t_2]$.

2 Estimates of the pressure

2.1. Assumptions on solution $\boldsymbol{u}$, our aims and technical preliminaries. We suppose that $\boldsymbol{u}$ is a weak solution to the problem (1.1)–(1.5), that satisfies Serrin's condition $\boldsymbol{u} \in L^r(t_1, t_2;\, \boldsymbol{L}^s(\Omega'))$, where Ω' is a sub-domain if Ω, $0 \leq t_1 < t_2 \leq T$, $r \in [2, \infty)$, $s \in (3, \infty]$ and $2/r + 3/s = 1$.

Let $\Omega'' \subset \overline{\Omega''} \subset \Omega'$. Let $\epsilon > 0$ be so small that $t_1 + \epsilon < t_2 - \epsilon$. Our first aim is to show that all spatial derivatives of ∇p are in $L^\infty(\Omega'' \times (t_1 + \epsilon, t_2 - \epsilon))$.

Let Ω'_* be a sub-domain of Ω' such that $\Omega'' \subset \overline{\Omega''} \subset \Omega'_* \subset \overline{\Omega'_*} \subset \Omega'$. Let η be an infinitely–differentiable cut–off function in $\mathbb{R}^3$ such that $0 \leq \eta \leq 1$ in $\mathbb{R}^3$, $\eta = 0$ in $\mathbb{R}^3 \smallsetminus \Omega'_*$, $\eta = 1$ in Ω'' and $\mathrm{dist}(\mathrm{supp} \nabla\eta;\, \partial\Omega'') > 0$.

Let $\boldsymbol{x} \in \Omega''$ and $t \in (t_1 + \epsilon, t_2 - \epsilon) \cap (a_\gamma, b_\gamma)$ for some $\gamma \in \Gamma$. Since t is fixed in the next considerations, we write only $p(\boldsymbol{x})$ instead of $p(\boldsymbol{x}, t)$ and $\boldsymbol{u}(\boldsymbol{x})$ instead of $\boldsymbol{u}(\boldsymbol{x}, t)$ in the rest of this section. Although function p is defined only in Ω, we may naturally consider the product ηp to be a function in $\mathbb{R}^3$, equal to zero in $\mathbb{R}^3 \smallsetminus \Omega'_*$. Let $\boldsymbol{e}$ be a unit vector in $\mathbb{R}^3$. Then $\boldsymbol{e} \cdot \nabla_{\boldsymbol{x}} p(\boldsymbol{x})$ satisfies the automatic formula

$$
\begin{aligned}
\boldsymbol{e} \cdot \nabla_{\boldsymbol{x}} p(\boldsymbol{x}) \ &= \ \eta(\boldsymbol{x}) \big(\boldsymbol{e} \cdot \nabla_{\boldsymbol{x}} p(\boldsymbol{x}) \big) \ = \ -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, \Delta_{\boldsymbol{y}} \big[\eta(\boldsymbol{y}) \big(\boldsymbol{e} \cdot \nabla_{\boldsymbol{y}} p(\boldsymbol{y}) \big) \big]\, \mathrm{d}\boldsymbol{y} \\
&= \ -\frac{1}{4\pi} \big[P^{(1)}(\boldsymbol{x}) + 2P^{(2)}(\boldsymbol{x}) + P^{(3)}(\boldsymbol{x}) \big],
\end{aligned}
$$

where

$$
\begin{aligned}
P^{(1)}(\boldsymbol{x}) \ &= \ \int_\Omega \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, \Delta_{\boldsymbol{y}} \eta(\boldsymbol{y}) \big(\boldsymbol{e} \cdot \nabla_{\boldsymbol{y}} p(\boldsymbol{y}) \big)\, \mathrm{d}\boldsymbol{y}, \\
P^{(2)}(\boldsymbol{x}) \ &= \ \int_\Omega \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} \big[\nabla_{\boldsymbol{y}} \eta(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \big(\boldsymbol{e} \cdot \nabla_{\boldsymbol{y}} p(\boldsymbol{y}) \big) \big]\, \mathrm{d}\boldsymbol{y}, \\
P^{(3)}(\boldsymbol{x}) \ &= \ \int_\Omega \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, \eta(\boldsymbol{y})\, \Delta_{\boldsymbol{y}} \big(\boldsymbol{e} \cdot \nabla_{\boldsymbol{y}} p(\boldsymbol{y}) \big)\, \mathrm{d}\boldsymbol{y}.
\end{aligned}
$$

Note that as p satisfies (1.13) and the right hand side of the first equation in (1.13) is of the class C^∞ in Ω' (due to Lemma 1), it follows from results on the interior regularity of solutions of elliptic equations (see e.g. [7, paragraph II.3]) that p is of the class C^∞ in $\overline{\Omega'_*}$, where the integrands in $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ are supported.

We denote by D^α the spatial derivative of order $|\alpha|$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index and $|\alpha| \equiv \alpha_1 + \alpha_2 + \alpha_3 = 0, 1, 2, 3, \dots$. If we want to specify whether the differentiation is considered with respect to $\boldsymbol{x}$ or $\boldsymbol{y}$, we use the notation $D^\alpha_{\boldsymbol{x}}$ or $D^\alpha_{\boldsymbol{y}}$.

2.2. Estimates of $D^\alpha P^{(3)}$. The term $D^\alpha P^{(3)}$ satisfies

$$
\begin{aligned}
|D^\alpha_{\boldsymbol{x}} P^{(3)}(\boldsymbol{x})| \ &= \ \bigg| D^\alpha_{\boldsymbol{x}} \int_\Omega \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, \eta(\boldsymbol{y}) \big(\boldsymbol{e} \cdot \nabla_{\boldsymbol{y}} \Delta_{\boldsymbol{y}} p(\boldsymbol{y}) \big)\, \mathrm{d}\boldsymbol{y} \bigg| \\
&= \ \bigg| \int_{\Omega'_*} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, D^\alpha_{\boldsymbol{y}} \big\{ \eta(\boldsymbol{y})\, \boldsymbol{e} \cdot \nabla_{\boldsymbol{y}} \big[\nabla_{\boldsymbol{y}} \boldsymbol{u}(\boldsymbol{y}) : \big(\nabla_{\boldsymbol{y}} \boldsymbol{u}(\boldsymbol{y}) \big)^T \big] \big\}\, \mathrm{d}\boldsymbol{y} \bigg| \\
&\leq \ c \int_{\Omega'_*} \frac{\mathrm{d}\boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|} \ \leq \ c
\end{aligned}
\tag{2.1}
$$

due to Lemma 1. Constant c on the right hand side of (2.1) is independent of $\boldsymbol{e}$, $\boldsymbol{x}$ and t.

2.3. Estimates of $D^\alpha P^{(2)}$. In order to estimate $D^\alpha P^{(2)}$, we at first apply the integration by parts

$$
\begin{aligned}
D^\alpha_{\boldsymbol{x}} P^{(2)}(\boldsymbol{x}) \ &= \ -D^\alpha_{\boldsymbol{x}} \int_\Omega \mathrm{div}_{\boldsymbol{y}} \Big(\frac{\nabla_{\boldsymbol{y}} \eta(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \Big) \big(\boldsymbol{e} \cdot \nabla_{\boldsymbol{y}} p(\boldsymbol{y}) \big)\, \mathrm{d}\boldsymbol{y} \\
&= \ -\int_\Omega D^\alpha_{\boldsymbol{x}}\, \mathrm{div}_{\boldsymbol{y}} \Big(\frac{\nabla_{\boldsymbol{y}} \eta(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \Big)\, \boldsymbol{e} \cdot \nabla_{\boldsymbol{y}} p(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}
\end{aligned}
$$

and then use the Helmholtz decomposition

$$D_{\boldsymbol{x}}^{\alpha}\operatorname{div}_{\boldsymbol{y}}\Big(\frac{\nabla_{\boldsymbol{y}}\eta(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\Big)\,\boldsymbol{e} \;=\; \nabla_{\boldsymbol{y}}\varphi_1^{\boldsymbol{x}}(\boldsymbol{y})+\boldsymbol{w}_1^{\boldsymbol{x}}(\boldsymbol{y}), \tag{2.2}$$

where

$$\Delta_{\boldsymbol{y}}\varphi_1^{\boldsymbol{x}}=\operatorname{div}_{\boldsymbol{y}}\Big[D_{\boldsymbol{x}}^{\alpha}\operatorname{div}_{\boldsymbol{y}}\Big(\frac{\nabla_{\boldsymbol{y}}\eta(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\Big)\,\boldsymbol{e}\Big]\quad\text{in }\Omega,\qquad \frac{\partial\varphi_1^{\boldsymbol{x}}}{\partial\boldsymbol{n}}=D_{\boldsymbol{x}}^{\alpha}\operatorname{div}_{\boldsymbol{y}}\Big(\frac{\nabla_{\boldsymbol{y}}\eta(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\Big)\,\boldsymbol{e}\cdot\boldsymbol{n}=0\quad\text{on }\partial\Omega.$$

Note that $D_{\boldsymbol{x}}^{\alpha}\operatorname{div}_{\boldsymbol{y}}\big(\nabla_{\boldsymbol{y}}\eta(\boldsymbol{y})/|\boldsymbol{x}-\boldsymbol{y}|\big)\,\boldsymbol{e}$ is, in dependence on $\boldsymbol{y}$, an infinitely differentiable function with a compact support in Ω, whose all derivatives are bounded in Ω independently of $\boldsymbol{e}$ and $\boldsymbol{x}\in\Omega''$. Since domain Ω is bounded and the boundary $\partial\Omega$ is of the class $C^{2+(h)}$, function $\varphi_1^{\boldsymbol{x}}$ satisfies the estimate

$$|\nabla_{\boldsymbol{y}}^2\varphi_1^{\boldsymbol{x}}(\boldsymbol{y})|_{0+(h)}\;\le\; c\,\Big|\operatorname{div}_{\boldsymbol{y}}\Big[D_{\boldsymbol{x}}^{\alpha}\operatorname{div}_{\boldsymbol{y}}\Big(\frac{\nabla_{\boldsymbol{y}}\eta(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\Big)\,\boldsymbol{e}\Big]\Big|_{0+(h)}\;\le\; c, \tag{2.3}$$

where $|\,.\,|_{0+(h)}$ is the norm in Hölder's space $C^{0+(h)}(\overline{\Omega})$, see [12]. The last constant c on the right hand side is independent of $\boldsymbol{e}$ and of $\boldsymbol{x}$ for $\boldsymbol{x}\in\Omega''$. Function $\boldsymbol{w}_1^{\boldsymbol{x}}$ is divergence–free and $\boldsymbol{w}_1^{\boldsymbol{x}}\cdot\boldsymbol{n}=0$ on $\partial\Omega$. The term $P^{(2)}(\boldsymbol{x})$ now satisfies

$$\begin{aligned} D_{\boldsymbol{x}}^{\alpha}P^{(2)}(\boldsymbol{x}) \;&=\; \int_{\Omega}\big[\nabla_{\boldsymbol{y}}\varphi_1^{\boldsymbol{x}}+\boldsymbol{w}_1^{\boldsymbol{x}}\big]\cdot\nabla_{\boldsymbol{y}}p\;\mathrm{d}\boldsymbol{y}\;=\;\int_{\Omega}\nabla_{\boldsymbol{y}}\varphi_1^{\boldsymbol{x}}\cdot\nabla_{\boldsymbol{y}}p\;\mathrm{d}\boldsymbol{y}\\ &=\;\int_{\Omega}\nabla_{\boldsymbol{y}}\varphi_1^{\boldsymbol{x}}\cdot\big[-\partial_t\boldsymbol{u}-\boldsymbol{u}\cdot\nabla_{\boldsymbol{y}}\boldsymbol{u}+\Delta_{\boldsymbol{y}}\boldsymbol{u}\big]\;\mathrm{d}\boldsymbol{y}\;=\;-\int_{\Omega}\nabla_{\boldsymbol{y}}\varphi_1^{\boldsymbol{x}}\cdot[\boldsymbol{u}\cdot\nabla_{\boldsymbol{y}}\boldsymbol{u}]\;\mathrm{d}\boldsymbol{y}\\ &=\;-\int_{\partial\Omega}\big(\nabla_{\boldsymbol{y}}\varphi_1^{\boldsymbol{x}}\cdot\boldsymbol{u}\big)\,(\boldsymbol{u}\cdot\boldsymbol{n})\;\mathrm{d}\boldsymbol{y}+\int_{\Omega}\nabla_{\boldsymbol{y}}^2\varphi_1^{\boldsymbol{x}}(\boldsymbol{y}):[\boldsymbol{u}\otimes\boldsymbol{u}]\;\mathrm{d}\boldsymbol{y}. \end{aligned} \tag{2.4}$$

We have used the boundary conditions (1.3), (1.4), which guarantee that $\Delta\boldsymbol{u}\cdot\boldsymbol{n}=0$ on $\partial\Omega$. (This property of the Laplace operator, applied to functions that satisfy (1.2)–(1.4), is well known, see e.g. [2] or [6]. The reason is simple: the boundary condition (1.4) implies that $\mathbf{curl}\,\boldsymbol{u}$ is normal to $\partial\Omega$. Hence $\mathbf{curl}^2\boldsymbol{u}\equiv-\Delta\boldsymbol{u}$ is tangent to $\partial\Omega$, which means that $\Delta\boldsymbol{u}\cdot\boldsymbol{n}=0$ on $\partial\Omega$.) The first integral on the right hand side of (2.4) equals zero because $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on $\partial\Omega$. The second integral satisfies

$$\Big|\int_{\Omega}\nabla_{\boldsymbol{y}}^2\varphi_1^{\boldsymbol{x}}(\boldsymbol{y}):[\boldsymbol{u}\otimes\boldsymbol{u}](\boldsymbol{y})\;\mathrm{d}\boldsymbol{y}\Big|\;\le\; c\int_{\Omega}|\boldsymbol{u}(\boldsymbol{y})|^2\;\mathrm{d}\boldsymbol{y}\;\le\; c.$$

Hence we have

$$|D_{\boldsymbol{x}}^{\alpha}P^{(2)}(\boldsymbol{x})|\;\le\; c, \tag{2.5}$$

where c is independent of $\boldsymbol{e}$, t and $\boldsymbol{x}$.

2.4. Estimates of $D^{\alpha}P^{(1)}$. It remains to estimate $D_{\boldsymbol{x}}^{\alpha}P^{(1)}$. We use the Helmholtz decomposition

$$D_{\boldsymbol{x}}^{\alpha}\Big(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\Big)\,\Delta_{\boldsymbol{y}}\eta(\boldsymbol{y})\,\boldsymbol{e}\;=\;\nabla_{\boldsymbol{y}}\varphi_2^{\boldsymbol{x}}(\boldsymbol{y})+\boldsymbol{w}_2^{\boldsymbol{x}}(\boldsymbol{y}),$$

where

$$\Delta_{\boldsymbol{y}}\varphi_2^{\boldsymbol{x}}=\operatorname{div}_{\boldsymbol{y}}D_{\boldsymbol{x}}^{\alpha}\Big(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\Big)\,\Delta_{\boldsymbol{y}}\eta(\boldsymbol{y})\,\boldsymbol{e}\quad\text{in }\Omega,\qquad\frac{\partial\varphi_2^{\boldsymbol{x}}}{\partial\boldsymbol{n}}=D_{\boldsymbol{x}}^{\alpha}\Big(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\Big)\,\Delta_{\boldsymbol{y}}\eta(\boldsymbol{y})\,\boldsymbol{e}\cdot\boldsymbol{n}=0\quad\text{on }\partial\Omega.$$

By analogy with (2.3), function $\varphi_2^{\boldsymbol{x}}$ satisfies the inequalities

$$|\nabla_{\boldsymbol{y}}^2\varphi_2^{\boldsymbol{x}}(\boldsymbol{y})|_{0+(h)}\;\le\; c\,\Big|D_{\boldsymbol{x}}^{\alpha}\Big(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\Big)\,\Delta_{\boldsymbol{y}}\eta(\boldsymbol{y})\,\boldsymbol{e}\Big|_{0+(h)}\;\le\; c, \tag{2.6}$$

where the last constant c is independent of $\boldsymbol{e}$ and $\boldsymbol{x}$ for $\boldsymbol{x} \in \Omega''$. Then

$$D_{\boldsymbol{x}}^{\alpha} P^{(1)}(\boldsymbol{x}) \;=\; \int_{\Omega} \big[\nabla_{\boldsymbol{y}} \varphi_2^{\boldsymbol{x}} + \boldsymbol{w}_2^{\boldsymbol{x}}\big] \cdot \nabla_{\boldsymbol{y}} p \, \mathrm{d}\boldsymbol{y} \;=\; \int_{\Omega} \nabla_{\boldsymbol{y}} \varphi_2^{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{y}} p \, \mathrm{d}\boldsymbol{y}.$$

The last integral can be estimated in the same way as (2.4). Hence we obtain

$$|D_{\boldsymbol{x}}^{\alpha} P^{(1)}(\boldsymbol{x})| \;\leq\; c, \tag{2.7}$$

where c is independent of $\boldsymbol{x}$ and t.

2.5. A consequence of (2.1), (2.5), (2.7). The inequalities (2.1), (2.5) and (2.7) yield the estimate

$$\big| D_{\boldsymbol{x}}^{\alpha} \left(\boldsymbol{e} \cdot \nabla_{\boldsymbol{x}} p(\boldsymbol{x}) \right) \big| \;\leq\; c, \tag{2.8}$$

where c is independent of $\boldsymbol{e}$, $\boldsymbol{x}$ and t for $\boldsymbol{x} \in \Omega''$ and $t \in (t_1+\epsilon, t_2-\epsilon) \cap \big[\bigcup_{\gamma\in\Gamma}(a_\gamma, b_\gamma)\big]$. We have proven the theorem:

Theorem 1. *Let $\boldsymbol{u}$ be a weak solution to the problem (1.1)–(1.5) in $\Omega \times (0,T)$ that satisfies the strong energy inequality (1.11), and let p be an associated pressure. Let $0 \leq t_1 < t_2 \leq T$, Ω' be a sub-domain of Ω and let $\boldsymbol{u}$ satisfy Serrin's integrability condition in $\Omega' \times (t_1, t_2)$, which means that $\boldsymbol{u} \in L^r(t_1, t_2;\, \boldsymbol{L}^s(\Omega'))$ for some $r \in [2, \infty)$ and $s \in (3, \infty]$, such that $2/r + 3/s = 1$. Let $0 < 2\epsilon < t_2 - t_1$ and Ω'' be a sub-domain of Ω' such that $\Omega'' \subset \overline{\Omega''} \subset \Omega'$. Then ∇p has all spatial derivatives (of all orders) essentially bounded in $\Omega'' \times (t_2 - \epsilon, t_1 + \epsilon)$. Consequently, $\partial_t \boldsymbol{u}$ has all spatial derivatives essentially bounded in $\Omega'' \times (t_2 - \epsilon, t_1 + \epsilon)$, too.*

Remark 2.1. According to Theorem 3.1 in [12], one can replace $|\nabla_{\boldsymbol{y}}^2 \varphi_1^{\boldsymbol{x}}|_{0+(h)}$ by the Hölder norm $|\varphi_1^{\boldsymbol{x}}|_{2+(h)}$ on the left hand side of (2.3), provided that $\varphi_1^{\boldsymbol{x}}$ is chosen so that $\int_\Omega \varphi_1^{\boldsymbol{x}}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = 0$. The same can also be said on function $\varphi_2^{\boldsymbol{x}}$ in inequality (2.6). Theorem 3.1 in [12] requires Ω to be a bounded domain with the boundary of the class $C^{2+(h)}$. This is the main reason why we impose the same condition on domain Ω in this paper. However, due to the personal communication of our colleague Dagmar Medková, the same inequalities as (2.3) and (2.6) also hold if Ω is an exterior domain with the boundary of the class $C^{2+(h)}$. (The proof follows from the contents of a book, which Dagmar Medková is currently completing.) Consequently, Theorem 1 also holds if Ω is a "smooth" exterior domain in $\mathbb{R}^3$. In this case, however, one must assume that Ω'' is bounded.

3 Application of Theorem 1 to the procedure of localization

In this section, we describe a situation, in which the application of Theorem 1 plays an important role.

3.1. A suitable weak solution and its partial regularity. The so called suitable weak solutions are usually considered in literature in connection with the boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega \times (0,T)$, see e.g. [5], [11] or [10]. A weak solution $\boldsymbol{u}$ is said to be a *suitable weak solution*, if there exists an associated pressure $p \in L^{5/3}(\Omega \times (0,T))$ such that the pair $(\boldsymbol{u}, p)$ satisfies the localized energy inequality

$$2\nu \int_0^T \int_\Omega |\nabla \boldsymbol{u}|^2 \, \phi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \;\leq\; \int_0^T \int_{\mathbb{R}^3} \big[\, |\boldsymbol{u}|^2 \left(\partial_t \phi + \nu \Delta \phi \right) + \left(|\boldsymbol{u}|^2 + 2p \right) \boldsymbol{u} \cdot \nabla \phi \big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \tag{3.1}$$

for every non–negative function ϕ from $C_0^\infty\big(\Omega \times (0,T)\big)$. This definition can also be extended to weak solutions of the problem (1.1)–(1.5), i.e. the problem with the Navier–type boundary conditions. The localized energy inequality enables one to prove the "local regularity criterion", which says that if the identity

$$\limsup_{\rho \to 0+} \frac{1}{\rho} \int_{t_0 - \frac{7}{8}\rho^2}^{t_0 + \frac{1}{8}\rho^2} \int_{|\boldsymbol{x} - \boldsymbol{x}_0| < \rho} |\nabla \boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \;=\; 0 \tag{3.2}$$

holds for some $\boldsymbol{x}_0 \in \Omega$ and $t_0 \in (0,T)$ then the space–time point $(\boldsymbol{x}_0, t_0)$ is a regular point of solution $\boldsymbol{u}$. (See [5].) The point $(\boldsymbol{x}_0, t_0)$ is, by definition, said to be a *regular point* of weak solution $\boldsymbol{u}$, if there exists a neighborhood $U(\boldsymbol{x}_0, t_0)$ in $\Omega \times (0,T)$, where $\boldsymbol{u}$ is essentially bounded. Applying (3.2), one can deduce that the set of hypothetical *singular points* of solution $\boldsymbol{u}$ (i.e. points from $\Omega \times (0,T)$ that are not regular) – let us denote this set by $\mathcal{S}(\boldsymbol{u})$ – has the 1–dimensional Hausdorff measure equal to zero. (See [5] for the details.) This conclusion also holds for a suitable weak solution of the problem with the Navier–type boundary conditions, because the boundary conditions play no role in the arguments used in [5].

3.2. The procedure of localization. A suitable weak solution $\boldsymbol{u}$ is often assumed, in addition to the properties that directly follow from its definition, to have some additional properties (like e.g. a "better" integrability) in a sub-domain Ω' of Ω in a time interval $(t_1, t_2) \subset (0,T)$. In order to exploit these properties, one usually needs to localize all the equations to $\Omega' \times (t_1, t_2)$ and to formulate a "new" problem, whose solution is in the spatial variable supported only in Ω'. The usual way one can do it is to assume that $\Omega'' \subset \overline{\Omega''} \subset \Omega'$ and to multiply solution $\boldsymbol{u}$ by a cut–off function $\zeta \in C^\infty(\mathbb{R}^3)$ such that $0 \leq \zeta \leq 1$ in $\mathbb{R}^3$, $\zeta = 0$ in $\mathbb{R}^3 \smallsetminus \Omega'$, $\zeta = 1$ in Ω'' and $\operatorname{supp}(\nabla\zeta)$ is contained in a "smooth" domain $\mathcal{D} \subset \Omega' \smallsetminus \overline{\Omega''}$.The boundary of $\mathcal{D}$ can be split to two closed disjoint surfaces: $\partial\mathcal{D} = (\partial\mathcal{D})_1 \cup (\partial\mathcal{D})_2$, where $\zeta = 0$ on $(\partial\mathcal{D})_1$, $\zeta = 1$ on $(\partial\mathcal{D})_2$ and $\Omega'' \subset \operatorname{Int}(\partial\mathcal{D})_2 \subset \operatorname{Int}(\partial\mathcal{D})_1 \subset \Omega'$.

It is shown in [14] that since the 1–dimensional Hausdorff measure of the singular set $\mathcal{S}(\boldsymbol{u})$ is zero, and set $\mathcal{S}(\boldsymbol{u})$ is closed in $\Omega \times (0,T)$, domain $\mathcal{D}$ which contains $\operatorname{supp}\zeta$ can be chosen so that $\overline{\mathcal{D}} \cap \mathcal{S}(\boldsymbol{u}) = \emptyset$ and solution $\boldsymbol{u}$ is essentially bounded in some neighborhood of $\overline{\mathcal{D}}$ times $(0,T)$. Consequently, applying Lemma 1, we deduce that $\boldsymbol{u}$ has all spatial derivatives (of all orders) essentially bounded in $\mathcal{D} \times (\delta, T-\delta)$ for any $\delta > 0$.

Since the product $\zeta\boldsymbol{u}$ is not divergence–free, one puts $\boldsymbol{v} := \zeta\boldsymbol{u} - \boldsymbol{U}$, where the "corrector" $\boldsymbol{U}$ satisfies $\operatorname{div}\boldsymbol{U} = \nabla\zeta \cdot \boldsymbol{u}$. This equation implies that $\operatorname{div}\boldsymbol{v} = 0$. The existence of an appropriate function $\boldsymbol{U}$ follows e.g. from [8, Theorem III.3.2] or [4, Theorem 2.4]). Due to these theorems, there exists a linear mapping $\mathfrak{B}$ from $W_0^{m,2}(\mathcal{D})$ (for each $m \in \{0\}\cup\mathbb{N}$) to $\boldsymbol{W}_0^{m+1,2}(\mathcal{D})$ such that for all $f \in W_0^{m,2}(\mathcal{D})$, satisfying $\int_{\mathcal{D}} f \, \mathrm{d}\boldsymbol{x} = 0$,

1. $\operatorname{div}\mathfrak{B}f = f$ a.e. in $\mathcal{D}$,
2. $\|\nabla^{m+1}\mathfrak{B}f\|_{2;\mathcal{D}} \leq c\,\|\nabla^m f\|_{2;\mathcal{D}}$.

Mapping $\mathfrak{B}$ is often called the Bogovskij operator. Now, since

$$\begin{aligned} \int_{\mathcal{D}} \nabla\zeta \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \;&=\; \int_{(\partial\mathcal{D})_1} \zeta\,\boldsymbol{u}\cdot\boldsymbol{n} \, \mathrm{d}S + \int_{(\partial\mathcal{D})_2} \zeta\,\boldsymbol{u}\cdot\boldsymbol{n} \, \mathrm{d}S - \int_{\mathcal{D}} \zeta \operatorname{div}\boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \\ &=\; \int_{(\partial\mathcal{D})_2} \boldsymbol{u}\cdot\boldsymbol{n} \, \mathrm{d}S \;=\; -\int_{\operatorname{Int}(\partial\mathcal{D})_2} \operatorname{div}\boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \;=\; 0, \end{aligned}$$

we may put $\boldsymbol{U}(\,.\,,t) := \mathfrak{B}\big[\nabla\zeta\cdot\boldsymbol{u}(\,.\,,t)\big]$ for $t \in (\delta, T-\delta)$. Since $\nabla\zeta\cdot\boldsymbol{u}(\,.\,,t) \in W_0^{m,2}(\mathcal{D})$ for any $m \in \{0\}\cup\mathbb{N}$, we obtain $\boldsymbol{U}(\,.\,,t) \in \boldsymbol{W}_0^{m,2}(\mathcal{D})$ for any $m \in \{0\}\cup\mathbb{N}$. Since all spatial derivatives of $\nabla\zeta\cdot\boldsymbol{u}$ are essentially bounded in $\mathcal{D} \times (\delta, T-\delta)$, we deduce that $\boldsymbol{U} \in L^\infty(\delta, T-\delta;\, \boldsymbol{W}_0^{m,2}(\mathcal{D}))$ for any $m \in \{0\}\cup\mathbb{N}$.

Extending $\boldsymbol{U}$ by zero outside $\mathcal{D}$, and extending also $\zeta\boldsymbol{u}$ by zero outside Ω, we observe that the function $\boldsymbol{v} \equiv \zeta\boldsymbol{u} - \boldsymbol{U}$ is divergence–free in $\mathbb{R}^3 \times (\delta, T-\delta)$, it coincides with $\boldsymbol{u}$ in $\Omega'' \times (\delta, T-\delta)$, it equals zero in $(\mathbb{R}^3 \smallsetminus \Omega') \times (\delta, T-\delta)$ and all spatial derivatives of $\boldsymbol{v}$ are essentially bounded in $(\Omega' \smallsetminus \overline{\Omega''}) \times (\delta, T-\delta)$. One can deduce that if $\vartheta \in L^{5/3}(0,T)$ then the pair $\boldsymbol{v}$, $\zeta(p+\vartheta)$ is a suitable weak solution to the Navier–Stokes system

$$\partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla[\zeta(p+\vartheta)] \;=\; \nu\Delta\boldsymbol{v} + \boldsymbol{g}, \tag{3.3}$$

$$\operatorname{div}\boldsymbol{v} \;=\; 0 \tag{3.4}$$

in $\mathbb{R}^3 \times (\delta, T-\delta)$, where

$$\boldsymbol{g} \;=\; -\partial_t\boldsymbol{U} - \boldsymbol{U}\cdot\nabla(\zeta\boldsymbol{u}) - (\zeta\boldsymbol{u})\cdot\nabla\boldsymbol{U} + \boldsymbol{U}\cdot\nabla\boldsymbol{U} + (\zeta\boldsymbol{u}\cdot\nabla\zeta)\,\boldsymbol{u}$$

$$- \zeta (1-\zeta)\, \boldsymbol{u}\cdot\nabla\boldsymbol{u} - 2\nu\, \nabla\zeta\cdot\nabla\boldsymbol{u} - \nu\,\boldsymbol{u}\,\Delta\zeta + \nu\Delta\boldsymbol{U} + (p+\vartheta)\,\nabla\zeta. \tag{3.5}$$

Of many articles, where the procedure of localization has also been described and applied, we mention e.g. [13].

3.3. Application of Theorem 1. When studying the properties of solution $\boldsymbol{v}$ to the system (3.3), (3.4), it is important to know as much as possible about the right hand side $\boldsymbol{g}$ in equation (3.3). It follows from the definition of the cut–off function ζ and the properties of function $\boldsymbol{U}$ that $\operatorname{supp}\boldsymbol{g} \subset \overline{\mathcal{D}} \times (\delta, \mathrm{T}-\delta)$.

Since $\boldsymbol{U} = \mathfrak{B}[\nabla\eta\cdot\boldsymbol{u}]$, the time derivative $\partial_t\boldsymbol{U}$ equals $\mathfrak{B}[\nabla\zeta\cdot\partial_t\boldsymbol{u}]$ in $\mathcal{D}\times(\delta, T-\delta)$. Moreover, as all spatial derivatives of $\boldsymbol{u}$ are in $\boldsymbol{L}^\infty(\mathcal{D}\times(\delta,T-\delta))$, and due to Theorem 1 all spatial derivatives of $\partial_t\boldsymbol{u}$ are in $\boldsymbol{L}^\infty(\mathcal{D}\times(\delta,T-\delta))$ as well, we observe that all terms on the right hand side of (3.5), except for the last term $(p+\vartheta)\,\nabla\zeta$, have all spatial derivatives in $\boldsymbol{L}^\infty(\mathbb{R}^3\times(\delta,T-\delta))$ (supported in $\overline{\mathcal{D}}\times[0,T]$).

Since $p\in L^{5/3}(\Omega\times(0,T))$ and ∇p has all spatial derivatives in $\boldsymbol{L}^\infty(\mathcal{D}\times(\delta,T-\delta))$ due to Theorem 1, there exists $\vartheta\in L^{5/3}(0,T)$ such that $p+\vartheta\in L^\infty(\mathcal{D}\times(\delta,T-\delta))$, too. Then all spatial derivatives of $(p+\vartheta)\,\nabla\zeta$ are in $\boldsymbol{L}^\infty(\mathbb{R}^3\times(\delta,T-\delta))$ (supported in $\overline{\mathcal{D}}\times[0,T]$) and the same statement now holds on the whole function $\boldsymbol{g}$.

Theorem 2. *Let $\boldsymbol{u}$, p be a suitable weak solution to the problem (1.1)–(1.5) in $\Omega\times(0,T)$. Let Ω' be a sub-domain of Ω and Ω'' be a sub-domain of Ω' such that $\Omega''\subset\overline{\Omega''}\subset\Omega'$. Let $\mathcal{D}$, ζ and $\boldsymbol{U}$ be the set, respectively functions, described in subsection 3.2, and $\boldsymbol{v}:=\zeta\boldsymbol{u}-\boldsymbol{U}$. Then function $\boldsymbol{v}$ satisfies $\boldsymbol{v}=\boldsymbol{u}$ in $\Omega''\times(0,T)$ and $\boldsymbol{v}=\boldsymbol{0}$ in $\mathbb{R}^3\smallsetminus\Omega'$. If $\vartheta\in L^{5/3}(0,T)$ then the pair $\boldsymbol{v}$, $\zeta(p+\vartheta)$ is a suitable weak solution to the system (3.3), (3.4) in $\mathbb{R}^3\times(0,T)$. Function $\boldsymbol{g}$ on the right hand side of (3.3) is supported in $\overline{\mathcal{D}}\times[0,T]$ and function $\vartheta\in L^{5/3}(0,T)$ can be chosen so that $\boldsymbol{g}$ has all spatial derivatives in $\boldsymbol{L}^\infty(\mathbb{R}^3\times(\delta,T-\delta))$ for any $\delta>0$.*

Remark 3.1. If the considered suitable weak solution $\boldsymbol{u}$ is supposed to satisfy Dirichlet's boundary condition $\boldsymbol{u}=\boldsymbol{0}$ on $\partial\Omega\times(0,T)$ instead of the Navier–type boundary conditions (1.3), (1.4), then one can only state that function $\boldsymbol{g}$ has all spatial derivatives in $L^q(\delta,T-\delta;\,\boldsymbol{L}^\infty(\mathbb{R}^3))$ (supported in $\overline{\mathcal{D}}\times[0,T]$), for any $q\in(1,2)$. The reason is that one cannot apply Theorem 1, and instead of it, one has to rely on the results of [13], [15] or [18], mentioned in subsection 1.3.

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