INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

# Modeling of the unsteady flow through a channel with an artificial outflow condition by the Navier-Stokes variational inequality

*Stanislav Kračmar*

*Jiří Neustupa*

Preprint No. 38-2017

PRAHA 2017

# Modeling of the Unsteady Flow through a Channel with an Artificial Outflow Condition by the Navier–Stokes Variational Inequality

Stanislav Kračmar, Jiří Neustupa

**Abstract**

We prove the global in time existence of a weak solution to the variational inequality of the Navier–Stokes type, simulating the unsteady flow of a viscous fluid through the channel, with the so called "do nothing" boundary condition on the outflow. The condition that the solution lies in a certain given, however arbitrarily large, convex set and the use of the variational inequality enables us to derive an energy–type estimate of the solution. We also discuss the use of a series of other possible outflow "do nothing" boundary conditions.

*AMS 2010 Subject Classification:* 35Q30, 65N30, 76D05.

*Key words:* Variational inequality, Navier–Stokes equation, "do nothing" outflow boundary conditions.

## 1 Introduction and notation

**1.1. The Navier–Stokes initial–boundary value problem.** Let $T > 0$ and $\Omega$ be a bounded Lipschitzian domain in $\mathbb{R}^3$. The flow of an incompressible Newtonian fluid in $\Omega$ in the time interval $(0,T)$ is described by the system of equations

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \operatorname{div}\mathbb{S} \;=\; \boldsymbol{f}, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} \;=\; 0, \tag{1.2}$$

where $\boldsymbol{u}$ is the velocity, $\mathbb{S}$ is the stress tensor and $\boldsymbol{f}$ is the acting volume force. The density of the fluid is assumed to be equal to one. Tensor $\mathbb{S}$, in the Newtonian fluid, has the form $\mathbb{S} = -p\mathbb{I} + \nu\,[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]$, where $p$ is the pressure and $\nu$ is the coefficient of viscosity. We assume that domain $\Omega$ represents a channel, where the fluid inflows through the part $\Gamma_1$ of the boundary $\partial\Omega$ and outflows through the part $\Gamma_2$ of $\partial\Omega$. (See Fig. 1.) It is logical to assume that the flow on $\Gamma_1$ is known, which leads to the Dirichlet boundary condition

$$\boldsymbol{u} \;=\; \boldsymbol{u}^* \qquad \text{on } \Gamma_1\times(0,T), \tag{1.3}$$

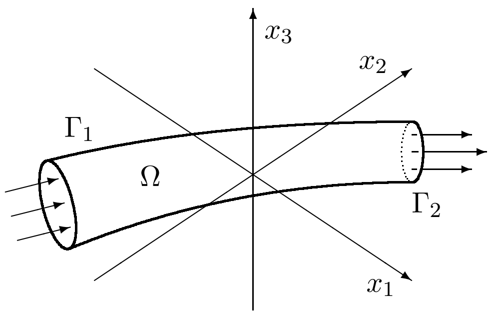


Fig. 1: the channel

where $\boldsymbol{u}^*$ is a given function. (The part of $\Gamma_1$ may coincide with a fixed wall, where $\boldsymbol{u}^*$ equals zero.) On the other hand, since the velocity profile on $\Gamma_2$ is not known in advance, the authors here usually use some "artificial" boundary condition of non–Dirichlet type. One can find artificial boundary conditions of various forms in literature, see e.g. [1], [2], [4], [6], [7], [8], [11], [22]. Boundary conditions that naturally follow from an appropriate weak formulation of the considered boundary value or initial–boundary value problem are often called the "do nothing" boundary conditions. (See e.g. [1], [11], [14] for more details.) In this paper, we use the inhomogeneous "do nothing" boundary condition

$$-p\boldsymbol{n} + \nu\nabla\boldsymbol{u}\cdot\boldsymbol{n} \;=\; \boldsymbol{g} \qquad \text{on } \Gamma_2\times(0,T), \tag{1.4}$$

where $\boldsymbol{n}$ denotes the outer normal vector field and $\boldsymbol{g}$ is a given function. (We explain later, in subsection 1.5, how condition (1.4) follows from the weak formulation. We also present some other "do nothing" boundary conditions and compare them with (1.4) in subsection 1.5.) The problem is completed by the initial condition

$$\boldsymbol{u} = \boldsymbol{u}_0 \qquad \text{in } \Omega \times \{0\}. \tag{1.5}$$

**1.2. The question of solvability of the problem (1.1)–(1.5) and related results.** If one wants to prove the existence of a solution of the problem (1.1)–(1.4) then the first logical step is the derivation of an a priori estimate. However, since the boundary condition (1.4) does not exclude backward flows on $\Gamma_2$ that might possibly bring back to $\Omega$ an uncontrollable amount of kinetic energy, the derivation of the usual energy inequality fails. This is the reason why the known existential results for the problem (1.1)–(1.5) assume that the given data of the problem are in some sense "small", or the time interval $(0, T)$ is "sufficiently short". (See [1], [14], [15].) The global in time existence of a weak solution of the problem (1.1)–(1.5) for "large" data, which is well known for the Navier–Stokes equations with other boundary conditions than (1.4), is an open problem. The situation is similar if one studies a flow through a 2D profile cascade, see [6], [7]. Some authors consider boundary conditions on $\Gamma_2$, modified by artificial terms that enable one to control the kinetic energy of the fluid entering $\Omega$ through $\Gamma_2$. (Such a modification was proposed e.g. in [4]. The same and other modifications have also been used in papers [8], [22] which deal with profile cascades. A modification of condition (1.4) by certain nonlinear terms, elaborated into a numerical algorithm, can also be found in paper [5]. In paper [16], the authors use the modified "do nothing" boundary condition in connection with the flow of a shear–thinning fluid. A nonlinearly modified condition (1.4) also plays an important role in [17], where the authors prove the solvability of the steady Navier–Stokes variational inequality.) Another approach has been used in papers [12], [13], where the authors consider the steady problems and impose an additional condition on $\Gamma_2$, that enables them to derive an a priori energy estimate. However, the additional condition means that the solution is from the beginning sought for in a certain closed convex subset of the Sobolev space $\boldsymbol{W}^{1,2}(\Omega)$, and the momentum equation (1.1) (or more precisely, its weak form) must be replaced by a variational inequality. A modification of the boundary condition (1.4) is also used in paper [2], where the authors study the flow of an incompressible viscous mixture with non–constant density.

**1.3. Aims of this paper.** We present several types of "do nothing" boundary conditions in subsection 1.4, and discuss them from the point of view of energy estimates and comparison with the steady state Poiseuille flow through a pipe. Then we apply a similar approach as in [12] and [13], however to the non–stationary flow. In Section 2, we formally derive the variational inequality and formulate the main result on the global in time existence of its weak solution. (See Theorem 1.) We also show that if the weak solution and an associated pressure are "smooth" then they satisfy the Navier–Stokes system (1.1), (1.2) in $\Omega \times (0, T)$ and the variational inequality is reduced only to set $\Gamma_2 \times (0, T)$. Moreover, if the solution finds itself in the interior of the aforementioned convex set then the boundary condition (1.4) is satisfied point-wise in $\Gamma_2 \times (0, T)$. (See Theorem 2.) The proof of Theorem 1 is given in Section 3.

Everywhere in the paper, we focus especially on points where the use of the variational inequality brings something new or requires a different approach or technique and we do not repeat the parts (estimates, procedures, arguments) that are well known from the theory of weak solutions to the Navier–Stokes equations.

**1.4. A discussion on boundary conditions of the "do nothing" type.** We consider, for simplicity, only the steady–state problem in this subsection. The term $\operatorname{div} \mathbb{S}$ in equation (1.1) can be written in any of these forms:

1a) $\operatorname{div} \mathbb{S} = -\operatorname{div}(p\mathbb{I}) + \nu \operatorname{div}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) = -\nabla p + \nu \operatorname{div}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]$,

1b) $\operatorname{div} \mathbb{S} = -\operatorname{div}(p\mathbb{I}) + \nu \Delta \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}$ (using the identity $\operatorname{div}(\nabla \boldsymbol{u})^T = \mathbf{0}$),

1c) $\operatorname{div} \mathbb{S} = -\operatorname{div}(p\mathbb{I}) - \nu \operatorname{\mathbf{curl}}^2 \boldsymbol{u} = -\nabla p - \nu \operatorname{\mathbf{curl}}^2 \boldsymbol{u}$ (using the formula $\Delta \boldsymbol{u} = -\operatorname{\mathbf{curl}}^2 \boldsymbol{u}$).

If we want to derive formally a weak form of the system (1.1), (1.2) with the boundary condition (1.3), we multiply equation (1.1) by a "smooth" divergence–free test function $\boldsymbol{\phi}$ and integrate in $\Omega$. It is reasonable to assume that $\boldsymbol{\phi} = \mathbf{0}$ on $\Gamma_1$ due to the Dirichlet boundary condition (1.3), but we impose no condition on $\boldsymbol{\phi}$ on $\Gamma_2$. Then the cases 1a) – 1c) successively yield

2a) $\displaystyle\int_\Omega \operatorname{div}\mathbb{S}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} = \int_{\Gamma_2}\big[-p\boldsymbol{n}+\nu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)\cdot\boldsymbol{n}\big]\cdot\boldsymbol{\phi}\,\mathrm{d}S - \nu\int_\Omega\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right):\nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},$

2b) $\displaystyle\int_\Omega \operatorname{div}\mathbb{S}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} = \int_{\Gamma_2}[-p\boldsymbol{n}+\nu\nabla\boldsymbol{u}\cdot\boldsymbol{n}]\cdot\boldsymbol{\phi}\,\mathrm{d}S - \nu\int_\Omega\nabla\boldsymbol{u}:\nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},$

2c) $\displaystyle\int_\Omega \operatorname{div}\mathbb{S}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x} = \int_{\Gamma_2}[-p\boldsymbol{n}-\nu\,\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}]\cdot\boldsymbol{\phi}\,\mathrm{d}S - \nu\int_\Omega\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}.$

The integrals on $\Gamma_2$ cannot be involved into the weak formulation, because the integrands cannot be reasonably interpreted on the level of weak solutions. Thus, they are usually neglected or replaced by $\int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{\phi}\,\mathrm{d}S$, where function $\boldsymbol{g}$ can be appropriately chosen. Then the weak variants of the system (1.1), (1.2) take the forms

3a) $\displaystyle\int_\Omega\big[\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{\phi}-\nu\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right):\nabla\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}+\int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{\phi}\,\mathrm{d}S=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},$

3b) $\displaystyle\int_\Omega\big[\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{\phi}-\nu\,\nabla\boldsymbol{u}:\nabla\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}+\int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{\phi}\,\mathrm{d}S=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},$

3c) $\displaystyle\int_\Omega\big[\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{\phi}-\nu\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}+\int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{\phi}\,\mathrm{d}S=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}.$

(The equations are required to be satisfied for all test functions $\boldsymbol{\phi}$ with the mentioned properties and $\boldsymbol{u}$ is also required to satisfy the condition $\boldsymbol{u}=\boldsymbol{u}^*$ on $\Gamma_1$.) If a weak solution $\boldsymbol{u}$ exists and is sufficiently smooth then one, applying the backward integration by parts, can reconstruct an associated pressure $p$ and successively show that $\boldsymbol{u}$ and $p$ satisfy the boundary conditions

4a) $-p\boldsymbol{n}+\nu\,[\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T]\cdot\boldsymbol{n}=\boldsymbol{g},$

4b) $-p\boldsymbol{n}+\nu\nabla\boldsymbol{u}\cdot\boldsymbol{n}=\boldsymbol{g},$

4c) $-p\boldsymbol{n}-\nu\,\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{g},$

respectively, on $\Gamma_2$. (These conditions retroactively certify that it was correct to neglect the integrals on $\Gamma_2$ in 1a) – 1c).) However, although the pressure involved in tensor $\mathbb{S}$ in equation (1.1) can be modified by an arbitrary additional constant, with no effect on the validity of the equation, the same assertion does not hold for $p$ in the boundary conditions 4a) – 4c). Here, one can deduce from the weak formulation that there exists just one pressure $p$ (in the class of pressures that differ by additive constants) that satisfies the boundary condition. (The reasons are the same as the reasons for the presence of function $\vartheta$ in formula (2.6) in Theorem 2.)

None of the conditions 4a) – 4c) prevents the existence of a backward flow on $\Gamma_2$, that could theoretically bring an uncontrollable amount of the kinetic energy from the outside to $\Omega$. The flow of the kinetic energy through $\Gamma_2$ comes from the nonlinear term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ in equation (1.1), if it is formally multiplied by $\boldsymbol{u}$ and integrated over $\Omega$. It yields:

$$\int_\Omega\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \;=\; \int_{\partial\Omega}(\boldsymbol{u}\cdot\boldsymbol{n})\,\tfrac12|\boldsymbol{u}|^2\,\mathrm{d}S \;=\; \int_{\Gamma_1}(\boldsymbol{u}^*\cdot\boldsymbol{n})\,\tfrac12|\boldsymbol{u}^*|^2\,\mathrm{d}S+\int_{\Gamma_2}(\boldsymbol{u}\cdot\boldsymbol{n})\,\tfrac12|\boldsymbol{u}|^2\,\mathrm{d}S. \tag{1.6}$$

The last integral on the right hand side cannot be dominated by other terms in the energy estimates. It may have a "wrong sign" and act against the other terms if $\boldsymbol{u}\cdot\boldsymbol{n}<0$ on the part of $\Gamma_2$, i.e. in the case of a backward flow. (See e.g. [3] for a more detailed explanation.) The situation changes if one writes the nonlinear term in equation (1.1) in the form $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u}+\nabla\frac12|\boldsymbol{u}|^2$ and considers $\nabla\frac12|\boldsymbol{u}|^2$ together with $p$ as the so called Bernoulli pressure $q\equiv p+\frac12|\boldsymbol{u}|^2$. Then the aforementioned boundary conditions 4a) – 4c) take successively the modified forms

4d) $-q\boldsymbol{n}+\nu\,[\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T]\cdot\boldsymbol{n}=\boldsymbol{g},$

4e) $-q\boldsymbol{n}+\nu\nabla\boldsymbol{u}\cdot\boldsymbol{n}=\boldsymbol{g},$

4f) $-q\boldsymbol{n}-\nu\,\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}=\boldsymbol{g}.$

It is, however, important that the remaining nonlinear term in equation (1.1) is $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u}$. (It leads to

$\int_\Omega \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u} \cdot \boldsymbol{\phi}\, \mathrm{d}\boldsymbol{x}$ in the weak formulation.) If one formally multiplies equation (1.1) by $\boldsymbol{u}$ (which means that one uses $\boldsymbol{\phi} = \boldsymbol{u}$ in the weak formulation) then the nonlinear term disappears, because $(\mathbf{curl}\boldsymbol{u} \times \boldsymbol{u}) \cdot \boldsymbol{u} = 0$. The consequences are: 1) the nonlinear term $\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}$ does not generate a flow of kinetic energy through $\Gamma_2$, 2) one can derive an anergy inequality, 3) one can prove the existence of a global in time weak solution similarly, as in the case of the Dirichlet boundary condition (1.3) on the whole boundary of $\Omega$ (see e.g. [9]).

Thus, there is a natural question which of the boundary conditions 4a) – 4f) on $\Gamma_2$ is the most appropriate one, and why the conditions 4a) – 4c) are considered at all, when, in contrast to 4d) – 4f), they do not enable one to prove the existence of a weak solution. The answer is not quite clear. Many authors prefer condition 4b) because it is satisfied (with $\boldsymbol{g} = \boldsymbol{0}$) by the Poiseuille flow in a circular pipe. For this reason, condition 4b) is being considered to be the most physical one of all the conditions 4a) – 4f). On the other hand, to any of the conditions 4a) – 4f), one can always calculate a non–zero function $\boldsymbol{g}$ so that the Poiseuille flow (or, more generally, any other flow expected on $\Gamma_2$) satisfies the considered condition with this concrete $\boldsymbol{g}$ on the right hand side. So, in our opinion, the choice of the boundary condition depends on a concrete situation. It would be highly interesting to compare numerical results, obtained with various boundary conditions, among themselves and also with results of experiments.

**1.5. Assumptions and notation.** Vector-functions and spaces of vector-functions are denoted by boldface letters.

(i) Let $1 < r < \infty$ and $k \in \{0\} \cup \mathbb{N}$. The norm of a scalar– or vector– or tensor–valued function, with components in $L^r(\Omega)$ (respectively $W^{k,r}(\Omega)$) is denoted by $\|\,.\,\|_r$ (respectively $\|\,.\,\|_{k,r}$). The norm in $L^r(\Gamma_2)$ is denoted by $\|\,.\,\|_{r;\,\Gamma_2}$.

(ii) We assume that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, where $\Gamma_1$ and $\Gamma_2$ are disjoint nonempty subsets of $\partial\Omega$, open in the 2D topology of $\partial\Omega$. Since the 2D measure of $\Gamma_1$ is positive, there exists $c_1 > 0$ such that the Friedrichs inequality

$$\|\,.\,\|_2 \;\leq\; c_1\, \|\nabla.\,\|_2 \tag{1.7}$$

holds for all functions from $W^{1,2}(\Omega)$, whose trace on $\Gamma_1$ is equal to zero. (See [21, Theorem 1.1.9].)

(iii) We assume that $\boldsymbol{u}^*$ is a given function on $\Gamma_1 \times (0,T)$ that can be extended to $\Omega \times (0,T)$ so that the extended function, which is for simplicity also denoted by $\boldsymbol{u}^*$, has these properties: a) $\boldsymbol{u}^* \in L^\infty\big(0,T;\, \boldsymbol{W}^{1,2}(\Omega)\big)$ and $\partial_t \boldsymbol{u}^* \in L^2\big(0,T;\, \boldsymbol{W}^{-1,2}(\Omega)\big)$, b) $\operatorname{div} \boldsymbol{u}^*(t) = 0$ in $\Omega$ for a.a. $t \in (0,T)$. (We denote by $\boldsymbol{W}^{-1,2}(\Omega)$ is the dual to $\boldsymbol{W}^{1,2}(\Omega)$.) It follows from [19, Theorem I.3.1] that $\boldsymbol{u}^* \in C^0\big([0,T];\, \boldsymbol{L}^2(\Omega)\big)$ and $\operatorname{div} \boldsymbol{u}^*(t) = 0$ in $\Omega$ for all $t \in [0,T]$.

(iv) We denote by $\boldsymbol{V}^1$ the linear space of all divergence–free functions from $\boldsymbol{W}^{1,2}(\Omega)$, such that $\boldsymbol{\phi} = \boldsymbol{0}$ on $\Gamma_1$. Then $\boldsymbol{u}^*(t) + \boldsymbol{V}^1$ (for a.a. $t \in (0,T)$) is a linear set of all divergence–free functions from $\boldsymbol{W}^{1,2}(\Omega)$, such that $\boldsymbol{\phi} = \boldsymbol{u}^*(t)$ on $\Gamma_1$.

(v) Let $\epsilon_1 > 0$, $a \in (2,4)$, respectively $\gamma \in L^{l(a)}(0,T)$ (where $l(a) := \max\{4a/(a-2), 4a/(4-a)\}$), be such numbers, respectively function, that

$$\big\|\big(\boldsymbol{u}^*(t)\cdot\boldsymbol{n}\big)_-\big\|_{a;\,\Gamma_2} + \epsilon_1 \;<\; \gamma(t) \tag{1.8}$$

for a.a. $t \in (0,T)$. (The form of $l(a)$ is used in subsection 3.3. The subscript "$-$" denotes the negative part. The negative part is taken "positively", i.e. if $c < 0$ then $c_- = -c$.) We define $\boldsymbol{K}_t^1$ to be the set of all functions $\boldsymbol{\phi} \in \boldsymbol{u}^*(t) + \boldsymbol{V}^1$ such that

$$\big\|(\boldsymbol{\phi}\cdot\boldsymbol{n})_-\|_{a;\,\Gamma_2} \;\leq\; \gamma(t) \qquad \text{for a.a. } t \in (0,T). \tag{1.9}$$

The numbers $\epsilon_1$, $a$ and the functions $\gamma$ and $\boldsymbol{u}^*$ are fixed throughout the paper. Since the inequality in (1.9) is not strong, $\boldsymbol{K}_t^1$ is a closed subset of $\boldsymbol{u}^*(t) + \boldsymbol{V}^1$. Applying Minkowski's inequality, it can be verified that set $\boldsymbol{K}_t^1$ is convex. Due to the presence of positive $\epsilon_1$ in inequality (1.8), one can also show that there exists $\epsilon_2 > 0$ (independent of $t$) such that $\boldsymbol{K}_t^1$ contains the $\epsilon_2$–neighborhood of $\boldsymbol{u}^*(t)$.

(vi) We denote by $\mathscr{W}(0,T)$ the Banach space of functions $\boldsymbol{w} \in L^2(0,T;\, \boldsymbol{W}^{1,2}(\Omega))$ such that $\partial_t \boldsymbol{w} \in L^2(0,T;\, \boldsymbol{W}^{-1,2}(\Omega))$, equipped by the norm

$$|\!|\!| \boldsymbol{w} |\!|\!| \;:=\; \left( \int_0^T \|\boldsymbol{w}\|_{1,2}^2 \, \mathrm{d}t + \int_0^T \|\partial_t \boldsymbol{w}\|_{-1,2}^2 \, \mathrm{d}t \right)^{1/2}.$$

Applying [19, Theorem I.3.1], one can deduce that each function $\boldsymbol{w}$ from $\mathscr{W}(0,T)$ is in $C^0\big([0,T];\, \boldsymbol{L}^2(\Omega)\big)$, too.

(vii) We denote by $\mathscr{K}(0,T)$ the set of functions $\boldsymbol{w} \in \mathscr{W}(0,T)$ such that $\boldsymbol{w}(t) \in \boldsymbol{K}_t^1$ for a.a. $t \in (0,T)$.

## 2 The Navier–Stokes variational inequality and its global in time solution

**2.1. A formal derivation of the variational inequality.** Suppose that $\boldsymbol{u}$, $p$ is a "smooth" solution of the problem (1.1)–(1.5). Let $\boldsymbol{w}$ be a "smooth" function from $[0,T]$ such that $\boldsymbol{w}(t) \in \boldsymbol{K}_t^1$ for a.a. $t \in [0,T]$. Using the formula $\operatorname{div} \mathbb{S} = -\nabla p + \nu \Delta \boldsymbol{u}$, multiplying equation (1.1) by $\boldsymbol{w} - \boldsymbol{u}$, integrating over $\Omega \times (0,T)$, using the identity $\boldsymbol{w} - \boldsymbol{u} = \boldsymbol{0}$ on $\Gamma_1 \times (0,T)$ and applying the boundary condition (1.4), we obtain

$$\int_0^T \int_\Omega [\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}] \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^T \int_\Omega \nu \nabla \boldsymbol{u} \cdot \nabla (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$
$$= \int_0^T \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^T \int_{\Gamma_2} \boldsymbol{g} \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}S \, \mathrm{d}t. \tag{2.1}$$

The term with the time derivative satisfies

$$\int_0^T \int_\Omega \partial_t \boldsymbol{u} \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \;=\; \int_0^T \int_\Omega \partial_t (\boldsymbol{u} - \boldsymbol{w}) \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^T \int_\Omega \partial_t \boldsymbol{w} \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$
$$= \frac{1}{2} \, \|\boldsymbol{w}(0) - \boldsymbol{u}(0)\|_2^2 - \frac{1}{2} \, \|\boldsymbol{w}(T) - \boldsymbol{u}(T)\|_2^2 + \int_0^T \int_\Omega \partial_t \boldsymbol{w} \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$
$$\leq \frac{1}{2} \, \|\boldsymbol{w}(0) - \boldsymbol{u}_0\|_2^2 + \int_0^T \int_\Omega \partial_t \boldsymbol{w} \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \tag{2.2}$$

From now on, we consider $\boldsymbol{w} \in \mathscr{K}(0,T)$. Thus, the integral of $\partial_t \boldsymbol{w} \cdot (\boldsymbol{w} - \boldsymbol{u})$ in $\Omega$ can be written as the duality $\langle \partial_t \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{u} \rangle$. The integral of $\boldsymbol{f} \cdot (\boldsymbol{w} - \boldsymbol{u})$ in $\Omega$ can be written as the duality $\langle \boldsymbol{f}, \boldsymbol{w} - \boldsymbol{u} \rangle$, too. Substituting from (2.2) to (2.1), we obtain

$$\int_0^T \big\langle \partial_t \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{u} \big\rangle \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^T \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^T \int_\Omega \nu \nabla \boldsymbol{u} \cdot \nabla (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$
$$\geq \int_0^T \int_\Omega \langle \boldsymbol{f}, \boldsymbol{w} - \boldsymbol{u} \rangle \, \mathrm{d}t + \int_0^T \int_{\Gamma_2} \boldsymbol{g} \cdot (\boldsymbol{w} - \boldsymbol{u}) \, \mathrm{d}S \, \mathrm{d}t - \frac{1}{2} \, \|\boldsymbol{w}(0) - \boldsymbol{u}_0\|_2^2. \tag{2.3}$$

The fact that (2.3) is an inequality, and not an equation, gives us the freedom to impose an additional condition on the solution $\boldsymbol{u}$: we require $\boldsymbol{u}(t) \in \boldsymbol{K}_t^1$ for a.a. $t \in (0,T)$.

**2.2. The initial–boundary value problem $(\mathcal{P})$.** *Given $\boldsymbol{u}^*$ as in subsection 1.5, $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$ satisfying $\operatorname{div} \boldsymbol{u}_0 = 0$ in $\Omega$ (in the sense of distributions) and $\boldsymbol{u}_0 \cdot \boldsymbol{n} = \boldsymbol{u}^*(0) \cdot \boldsymbol{n}$ on $\Gamma_1$ (in the sense of traces), $\boldsymbol{f} \in L^2(0,T;\, \boldsymbol{W}^{-1,2}(\Omega))$ and $\boldsymbol{g} \in L^2(0,T;\, \boldsymbol{L}^{4/3}(\Gamma_2))$. We look for $\boldsymbol{u} \in L^\infty(0,T;\, \boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\, \boldsymbol{W}^{1,2}(\Omega))$ such that $\boldsymbol{u}(t) \in \boldsymbol{K}_t^1$ for a.a. $t \in (0,T)$ and $\boldsymbol{u}$ satisfies the inequality (2.3) for all test functions $\boldsymbol{w} \in \mathscr{K}(0,T)$.*

Recall that normal components of divergence–free (in the sense of distributions) functions from $\boldsymbol{L}^2(\Omega)$ belong to $W^{-1/2,2}(\partial\Omega)$, see [10, Theorem III.2.2]. This gives a sense to the condition of compatibility $\boldsymbol{u}_0 \cdot \boldsymbol{n} = \boldsymbol{u}^*(0) \cdot \boldsymbol{n}$ on $\Gamma_1$. Moreover, since the traces of $\boldsymbol{w}(t)$ and $\boldsymbol{u}(t)$ are in $\boldsymbol{L}^4(\partial\Omega)$ at a.a. $t \in (0,T)$, the assumption on function $\boldsymbol{g}$ guarantees the convergence of the second integral on the right hand side of (2.3).

The main result of this paper is formulated in the next theorem:

**Theorem 1.** *A solution of problem* ($\mathcal{P}$) *exists. Moreover, the solution can be constructed in the form* $\boldsymbol{u} = \boldsymbol{u}^* + \boldsymbol{v}$, *where* $\boldsymbol{v} \in L^\infty(0,T;\, \boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\, \boldsymbol{V}^1)$ *and* $\boldsymbol{v}$ *satisfies the energy–type inequality*

$$\frac{1}{2}\,\|\boldsymbol{v}(t)\|_2^2 + \nu \int_0^t \|\nabla \boldsymbol{v}(s)\|_2^2 \,\mathrm{d}s \;\le\; \frac{1}{2}\,\|\boldsymbol{v}(0)\|_2^2 + c_2 \int_0^t \bigl[\gamma^{\frac{1}{1-\kappa}}(s) + 1\bigr]\, \|\boldsymbol{v}(s)\|_2^2 \,\mathrm{d}s$$
$$+ \int_0^t \bigl[c_3\, \|\boldsymbol{f}(s)\|_{-1,2}^2 + c_4\, \|\boldsymbol{u}^*(s)\|_{1,2}^2 + c_5\, \|\partial_t \boldsymbol{u}^*(s)\|_{-1,2}^2 + c_6\, \|\boldsymbol{g}(s)\|_{4/3;\,\Gamma_2}^2\bigr] \,\mathrm{d}s \tag{2.4}$$

*for all* $t \in (0,T)$. *(The constants* $c_2$*–*$c_6$ *depend only on* $\Omega$, $\Gamma_2$ *and number* $a$.*)*

The proof is briefly described (with stress on parts where the variational inequality requires a new technique) in Section 3.

**2.3. Some a posteriori properties of a solution to problem ($\mathcal{P}$).** If $\boldsymbol{u}$ is a solution of problem ($\mathcal{P}$) then, considering the test functions $\boldsymbol{w}$ of the form $\boldsymbol{w} = \boldsymbol{u}^* \pm \boldsymbol{v}$, where $\boldsymbol{v}$ is infinitely differentiable, divergence–free and with a compact support in $\Omega \times (0,T)$, one can show (by analogy with the Navier–Stokes equations) that there exists a distribution $p$ in $\Omega \times (0,T)$ (the so called *associated pressure*) such that $\boldsymbol{u}$ and $p$ satisfy equation (1.1) in the sense of distributions in $\Omega \times (0,T)$. It follows from the definition on problem ($\mathcal{P}$) that $\Delta\boldsymbol{u} \in L^2(0,T;\, \boldsymbol{W}^{-1,2}(\Omega))$ and $\boldsymbol{u}\cdot\nabla\boldsymbol{u} \in L^{4/3}(0,T;\, \boldsymbol{W}^{-1,2}(\Omega))$. If, moreover, $\partial_t\boldsymbol{u} \in L^1(0,T;\, \boldsymbol{W}^{-1,2}(\Omega))$ then $\nabla p$ (the distributional gradient of $p$) belongs to $L^1(0,T;\, \boldsymbol{W}^{-1,2}(\Omega))$, too. Consequently, $p$ can be chosen so that it belongs to $L^1(0,T;\, \boldsymbol{L}^2(\Omega))$. Since $p$ is unique up to an additive function of $t$, this function can be chosen so that $p$ satisfies the condition $\int_\Omega p(t)\, \mathrm{d}\boldsymbol{x} = \overline{p}(t)$ a.e. in $(0,T)$, where $\overline{p}$ is any given function in $L^1(0,T)$.

In the theory of partial differential equations, one can usually show that if a weak solution is "sufficiently smooth" then it coincides with a strong (or classical) solution. It is, however, not clear at the first sight whether some analogue of this also holds for the variational inequality (2.3), and in which sense problem ($\mathcal{P}$) involves the boundary condition (1.4). Thus, assume that $\boldsymbol{f} \in L^2(0,T\; \boldsymbol{L}^2(\Omega))$ and $\boldsymbol{u}$ is such a solution of problem ($\mathcal{P}$) that $\boldsymbol{u} \in L^2(0,T;\, \boldsymbol{W}^{2,2}(\Omega))$ and the terms $\partial_t\boldsymbol{u}$, $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ also belong to $L^2(0,T;\, \boldsymbol{L}^2(\Omega))$. Then the associated pressure $p$ exists as a function from $L^2(0,T;\, W^{1,2}(\Omega))$. Applying the integration by parts with respect to $t$ to the integral of $\langle \partial\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{u}\rangle$ (i.e. applying the procedure, reverse to the first two lines of (2.2)), we obtain

$$\int_0^T \langle \partial_t\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{u}\rangle \,\mathrm{d}t \;=\; \int_0^T \langle \partial_t\boldsymbol{u}, \boldsymbol{w}-\boldsymbol{u}\rangle \,\mathrm{d}t + \frac{1}{2}\,\|\boldsymbol{w}(T)-\boldsymbol{u}(T)\|_2^2 - \frac{1}{2}\,\|\boldsymbol{w}(0)-\boldsymbol{u}_0\|_2^2.$$

Substituting this to (2.3), using the inclusions $\partial_t\boldsymbol{u}$, $\Delta\boldsymbol{u} \in L^2(0,T;\, \boldsymbol{L}^2(\Omega))$ and applying the integration by parts to the third integral on the left hand side of (2.3), we get

$$\int_0^T \int_\Omega \bigl[\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} - \boldsymbol{f}\bigr]\cdot(\boldsymbol{w}-\boldsymbol{u}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{t}$$
$$\ge\; \int_0^T \int_{\Gamma_2} (\boldsymbol{g} - \nu\nabla\boldsymbol{u}\cdot\boldsymbol{n})\cdot(\boldsymbol{w}-\boldsymbol{u}) \,\mathrm{d}S\,\mathrm{d}t - \frac{1}{2}\,\|\boldsymbol{w}(T)-\boldsymbol{u}(T)\|_2^2.$$

Let $\boldsymbol{q}$ be a function from the same class as $\boldsymbol{w}$, i.e. $\boldsymbol{q} \in \mathscr{K}(0,T)$. Let $\xi \in (0,1)$. We use $\boldsymbol{w}$ in in the form $\boldsymbol{w} = \xi\boldsymbol{q} + (1-\xi)\boldsymbol{u}$, divide the inequality by $\xi$ and consider $\xi \to 0+$. In this way, we get rid of the term $\frac{1}{2}\,\|\boldsymbol{w}(T)-\boldsymbol{u}(T)\|_2^2$ on the right hand side. Furthermore, we use the equation $\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\nu\Delta\boldsymbol{u}-\boldsymbol{f} = -\nabla p$ and apply the integration by parts to the integral with $\nabla p$. Thus, we obtain

$$\int_0^T \int_{\Gamma_2} (\nu\nabla\boldsymbol{u}\cdot\boldsymbol{n} - p\boldsymbol{n} - \boldsymbol{g})\cdot(\boldsymbol{q}-\boldsymbol{u}) \,\mathrm{d}S\,\mathrm{d}t \;\ge\; 0. \tag{2.5}$$

The next theorem summarizes these findings (in items a) and b)) and adds a new item c):

**Theorem 2.** *Let* $\boldsymbol{u}$ *be a solution of problem* ($\mathcal{P}$). *Then* *a) there exists a distribution $p$ in $\Omega\times(0,T)$ (the so called associated pressure) such that $\boldsymbol{u}$, $p$ satisfy the Navier–Stokes system (1.1), (1.2 in the sense of distributions in $\Omega\times(0,T)$. If $\partial_t\boldsymbol{u}\in L^1(0,T;\,\boldsymbol{W}^{-1,2}(\Omega))$ and $\overline{p}\in L^1(0,T)$ is a given function then $p$ can be chosen so that $\int_\Omega p(t)\,\mathrm{d}\boldsymbol{x}=\overline{p}(t)$ for a.a. $t\in(0,T)$.*

*b) If $\boldsymbol{u}\in L^2(0,T;\,\boldsymbol{W}^{2,2}(\Omega))$ and $\partial_t\boldsymbol{u},\ \boldsymbol{u}\cdot\nabla\boldsymbol{u},\ \boldsymbol{f}\in L^2(0,T;\,\boldsymbol{L}^2(\Omega))$ then the associated pressure $p$ exists as a function from $L^2(0,T;\,W^{1,2}(\Omega))$ and satisfies the inequality (2.5) for all $\boldsymbol{q}\in\mathscr{K}(0,T)$.*

*c) If, in addition to the assumptions of item b), $\boldsymbol{u}(t)$ lies uniformly in the interior of $\boldsymbol{K}_t^1$ in the sense that there exists $\epsilon_3>0$ such that all $\boldsymbol{\phi}\in\boldsymbol{u}^*(t)+\boldsymbol{V}^1$ such that $\|\boldsymbol{\phi}-\boldsymbol{u}^*(t)\|_{1,2}<\epsilon_3$ belong to $\boldsymbol{K}_t^1$ (for a.a. $t\in(0,T)$) then there exists a function $\vartheta\in L^2(0,T)$ such that*

$$\nu\nabla\boldsymbol{u}\cdot\boldsymbol{n}-(p+\vartheta)\boldsymbol{n}\;=\;\boldsymbol{g} \tag{2.6}$$

*holds true point-wise a.e. in $\Gamma_2\times(0,T)$.*

**Proof.** We only need to prove the statement in item c). Let $\boldsymbol{h}\in\mathscr{W}(0,T)$ such that $\operatorname{div}\boldsymbol{h}=0$ a.e. in $\Omega\times(0,T)$, $\boldsymbol{h}=\boldsymbol{0}$ on $\Gamma_1\times(0,T)$ and $\|\boldsymbol{h}(t)\|_{1,2}<\epsilon_3$ for a.a. $t\in(0,T)$. Then $\boldsymbol{q}=\boldsymbol{u}\pm\boldsymbol{h}$ are admissible test functions in (2.5). Using these $\boldsymbol{q}$ in (2.5), we obtain

$$\int_0^T\int_{\Gamma_2}\bigl(\nu\nabla\boldsymbol{u}\cdot\boldsymbol{n}-p\boldsymbol{n}-\boldsymbol{g}\bigr)\cdot\boldsymbol{h}\,\mathrm{d}S\,\mathrm{d}t\;=\;0. \tag{2.7}$$

Since the left hand side depends linearly on $\boldsymbol{h}$, (2.7) holds for all $\boldsymbol{h}$ with the aforementioned properties, and not only for those $\boldsymbol{h}$ that differ from $\boldsymbol{0}$ by less that $\epsilon_3$. As the flux of $\boldsymbol{h}$ through $\Gamma_2$ equals zero at a.a. time instants $t\in(0,T)$, the space of traces of $\boldsymbol{h}$ on $\Gamma_2\times(0,T)$ annihilates the space of functions of the type $\vartheta\boldsymbol{n}$, where $\vartheta\in L^2(0,T)$. Hence $\nu\nabla\boldsymbol{u}\cdot\boldsymbol{n}-p\boldsymbol{n}-\boldsymbol{g}=\vartheta\boldsymbol{n}$ for some $\vartheta\in L^2(0,T)$. The proof is completed. $\square$

The statement in item c) is in a coincidence with what has been said about the pressure in subsection 1.4: the concrete pressure that can be reconstructed from the variational formulation (of an equation or an inequality) and that satisfies the outflow boundary condition on $\Gamma_2\times(0,T)$, cannot be arbitrarily modified by an additive constant (or more generally, by an additive function of $t$). On the other hand, just one representant from the class of all pressures, associated with a concrete weak solution $\boldsymbol{u}$, satisfies the outflow boundary condition.

# 3 Proof of Theorem 1

**3.1. The operator $\boldsymbol{\Psi}_t$.** The approximations of a solution $\boldsymbol{u}$ of problem $(\mathcal{P})$ are constructed in the next subsections by means of a penalization, where the main role plays an operator $\boldsymbol{\Psi}_t$. This operator is defined by the equation $\boldsymbol{\Psi}_t(\boldsymbol{\phi}):=\boldsymbol{\phi}-P_t^1(\boldsymbol{\phi})$ (for $\boldsymbol{\phi}\in\boldsymbol{u}^*(t)+\boldsymbol{V}^1$), where $P_t^1$ is the projector in $\boldsymbol{u}^*(t)+\boldsymbol{V}^1$, which assigns to each element of $\boldsymbol{u}^*(t)+\boldsymbol{V}^1$ the nearest element in $\boldsymbol{K}_t^1$. Due to the convexity of $\boldsymbol{K}_t^1$, $P_t^1$ is a bounded and continuous mapping of $\boldsymbol{u}^*(t)+\boldsymbol{V}^1$ into itself.

**Lemma 1.** *Operator $\boldsymbol{\Psi}_t$ is monotone and satisfies the inequalities*

$$\bigl(\boldsymbol{\Psi}_t(\boldsymbol{\phi}),\boldsymbol{\phi}-\boldsymbol{u}^*(t)\bigr)_{1,2}\geq\|\boldsymbol{\Psi}_t(\boldsymbol{\phi})\|_{1,2}^2\,,\qquad\bigl(\boldsymbol{\Psi}_t(\boldsymbol{\phi}),\boldsymbol{\phi}-\boldsymbol{u}^*(t)\bigr)_{1,2}\geq\epsilon_2\,\|\boldsymbol{\Psi}_t(\boldsymbol{\phi})\|_{1,2} \tag{3.1}$$

*for a.a. $t\in(0,T)$ and for all $\boldsymbol{\phi}\in\boldsymbol{u}^*(t)+\boldsymbol{V}^1$, where $(\,.\,,\,.\,)_{1,2}$ is the scalar product in $\boldsymbol{W}^{1,2}(\Omega)$ and $\epsilon_2$ is the number from paragraph 1.5 (v).*

**Proof.** If $\boldsymbol{\phi}_1,\ \boldsymbol{\phi}_2\in\boldsymbol{u}^*(t)+\boldsymbol{V}^1$ then, due to the convexity of $\boldsymbol{K}_t^1$, $\|P_t^1(\boldsymbol{\phi}_1)-P_t^1(\boldsymbol{\phi}_2)\|_{1,2}\leq\|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{1,2}$. Hence

$$\begin{aligned}\bigl(\boldsymbol{\Psi}_t(\boldsymbol{\phi}_1)-\boldsymbol{\Psi}_t(\boldsymbol{\phi}_2),\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\bigr)_{1,2}\;&=\;\|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{1,2}^2-\bigl(P_t^1(\boldsymbol{\phi}_1)-P_t^1(\boldsymbol{\phi}_2),\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\bigr)_{1,2}\\&\geq\;\|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{1,2}^2-\|P_t^1(\boldsymbol{\phi}_1)-P_t^1(\boldsymbol{\phi}_2)\|_{1,2}\,\|\boldsymbol{\phi}_1-\boldsymbol{\phi}_2\|_{1,2}\;\geq\;0.\end{aligned}$$

This proves the monotonicity of $\boldsymbol{\Psi}_t$. Furthermore, using the inequality $\big(\boldsymbol{\phi} - P_t^1(\boldsymbol{\phi}), P_t^1(\boldsymbol{\phi}) - \boldsymbol{u}^*(t)\big)_{1,2} \geq 0$, we get

$$\begin{aligned}\big(\boldsymbol{\Psi}_t(\boldsymbol{\phi}), \boldsymbol{\phi} - \boldsymbol{u}^*(t)\big)_{1,2} \;&=\; \big(\boldsymbol{\phi} - P_t^1(\boldsymbol{\phi}), \boldsymbol{\phi} - P_t^1(\boldsymbol{\phi})\big)_{1,2} + \big(\boldsymbol{\phi} - P_t^1(\boldsymbol{\phi}), P_t^1(\boldsymbol{\phi}) - \boldsymbol{u}^*(t)\big)_{1,2} \\ &\geq\; \big(\boldsymbol{\phi} - P_t^1(\boldsymbol{\phi}), \boldsymbol{\phi} - P_t^1(\boldsymbol{\phi})\big)_{1,2} \;=\; \|\boldsymbol{\Psi}_t(\boldsymbol{\phi})\|_{1,2}^2.\end{aligned}$$

This proves the first inequality in (3.1). The second inequality obviously holds if $\boldsymbol{\Psi}_t(\boldsymbol{\phi}) = \mathbf{0}$. Thus, assume that $\boldsymbol{\Psi}_t(\boldsymbol{\phi}) \neq \mathbf{0}$ and put $\boldsymbol{h} := \boldsymbol{u}^*(t) + \epsilon_2\, \boldsymbol{\Psi}_t(\boldsymbol{\phi})/\|\boldsymbol{\Psi}_t(\boldsymbol{\phi})\|_{1,2}$. Then

$$\begin{aligned}&\big(\boldsymbol{\Psi}_t(\boldsymbol{\phi}), \boldsymbol{\phi} - \boldsymbol{u}^*(t)\big)_{1,2} \\ &\quad= \big(\boldsymbol{\phi} - P_t^1(\boldsymbol{\phi}), \boldsymbol{\phi} - P_t^1(\boldsymbol{\phi})\big)_{1,2} + \big(\boldsymbol{\phi} - P_t^1(\boldsymbol{\phi}), P_t^1(\boldsymbol{\phi}) - \boldsymbol{h}\big)_{1,2} + \big(\boldsymbol{\phi} - P_t^1(\boldsymbol{\phi}), \boldsymbol{h} - \boldsymbol{u}^*(t)\big)_{1,2}.\end{aligned}$$

The first term on the right hand side is nonnegative. The second term is also nonnegative, because $\boldsymbol{h} \in \boldsymbol{K}_t^1$ and $\boldsymbol{K}_t^1$ is convex. Thus, substituting for $\boldsymbol{h}$, we obtain

$$\big(\boldsymbol{\Psi}_t(\boldsymbol{\phi}), \boldsymbol{\phi} - \boldsymbol{u}^*(t)\big)_{1,2} \;\geq\; \epsilon_2 \Big(\boldsymbol{\Psi}_t(\boldsymbol{\phi}),\, \frac{\boldsymbol{\Psi}_t(\boldsymbol{\phi})}{\|\boldsymbol{\Psi}_t(\boldsymbol{\phi})\|_{1,2}}\Big)_{1,2} \;=\; \epsilon_2\, \|\boldsymbol{\Psi}_t(\boldsymbol{\phi})\|_{1,2}. \qquad \square$$

**3.2. Construction of approximations.** Put $\boldsymbol{V}^2 := \boldsymbol{V}^1 \cap \boldsymbol{W}^{2,2}(\Omega)$. $\boldsymbol{V}^2$ is a Hilbert space with the scalar product $(\,.\,,\,.\,)_{2,2}$, identical with the scalar product in $\boldsymbol{W}^{2,2}(\Omega)$. Let $\boldsymbol{e}_1, \boldsymbol{e}_2 \dots$ be a basis in $\boldsymbol{V}^2$, orthonormal in $\boldsymbol{L}^2(\Omega)$.

Put

$$\kappa \;:=\; \begin{cases} \frac{3}{4} + \frac{1}{2a} & \text{if } 2 < a \leq 3, \\ \frac{5}{4} - \frac{1}{a} & \text{if } 3 \leq a < 4. \end{cases} \tag{3.2}$$

Then $\max\{\frac{1}{2} + \frac{1}{a}, \frac{3}{2} - \frac{2}{a}\} \leq \kappa < 1$ and there exists a continuous operator of traces from the Sobolev–Slobodeckij space $\boldsymbol{W}^{\kappa,2}(\Omega)$ to the Besov space $\boldsymbol{B}_{2,2}^{\kappa-1/2}(\partial\Omega)$, see [20]. Applying the partition of unity on $\partial\Omega$, a local representation of $\partial\Omega$ by graphs of Lipschitz functions and the continuous imbedding $\boldsymbol{B}_{2,2}^{\kappa-1/2}(\mathbb{R}^2) \hookrightarrow \boldsymbol{L}^{4/(3-2\kappa)}(\mathbb{R}^2)$ (see [23, p. 36]), we deduce that $\boldsymbol{B}_{2,2}^{\kappa-1/2}(\partial\Omega) \hookrightarrow \boldsymbol{L}^{4/(3-2\kappa)}(\partial\Omega)$. Hence there exists a continuous operator of traces from $\boldsymbol{W}^{\kappa,2}(\Omega)$ to $\boldsymbol{L}^{4/(3-2\kappa)}(\partial\Omega)$. By analogy with $\boldsymbol{V}^1$, we denote by $\boldsymbol{V}^\kappa$ be the space of all divergence–free (in the sense of distributions) functions $\boldsymbol{\phi}$ from $\boldsymbol{W}^{\kappa,2}(\Omega)$, such that $\boldsymbol{\phi} = \mathbf{0}$ on $\Gamma_1$. Furthermore, we define $\boldsymbol{K}_t^\kappa$ to be the set of all functions $\boldsymbol{\phi} \in \boldsymbol{u}^*(t) + \boldsymbol{V}^\kappa$ that satisfy inequality (1.9). (The norm in (1.9) has a sense because $a \leq 4/(3-2\kappa)$.) By analogy with $\boldsymbol{K}_t^1$, set $\boldsymbol{K}_t^\kappa$ is convex and closed in $\boldsymbol{V}^\kappa$. Denote by $P_t^\kappa$ the projector in $\boldsymbol{u}^*(t) + \boldsymbol{V}^\kappa$, which assigns to each element of $\boldsymbol{u}^*(t) + \boldsymbol{V}^\kappa$ the nearest element in $\boldsymbol{K}_t^\kappa$. Projector $P_t^\kappa$ is a continuous mapping of $\boldsymbol{u}^*(t) + \boldsymbol{V}^\kappa$ into itself. Moreover, since $\boldsymbol{K}_t^\kappa$ is convex, $P_t^\kappa$ satisfies $\|P_t^\kappa(\boldsymbol{\phi}) - \boldsymbol{u}^*(t)\|_{\kappa,2} \leq \|\boldsymbol{\phi} - \boldsymbol{u}^*(t)\|_{\kappa,2}$ for each $\boldsymbol{\phi} \in \boldsymbol{V}^\kappa$ and a.a $t \in (0,T)$. Consequently, $\|P_t^\kappa(\boldsymbol{\phi})\|_{\kappa,2} \leq \|\boldsymbol{\phi} - \boldsymbol{u}^*(t)\|_{\kappa,2} + \|\boldsymbol{u}^*(t)\|_{\kappa,2}$.

Let $n \in \mathbb{N}$. We look for the coefficients $a_k^{(n)} \in C^1([0,T])$, ($k = 1, 2, \dots, n$) such that the functions $\boldsymbol{u}^{(n)} := \boldsymbol{u}^* + \boldsymbol{v}^{(n)}$, where

$$\boldsymbol{v}^{(n)} \;:=\; \sum_{k=1}^{n} a_k^{(n)}\, \boldsymbol{e}_k, \tag{3.3}$$

satisfy the initial conditions $\boldsymbol{u}^{(n)}(0) \equiv \boldsymbol{u}^*(0) + \boldsymbol{v}^{(n)}(0) = \sum_{k=1}^{n} \big(\boldsymbol{u}_0 - \boldsymbol{u}^*(0), \boldsymbol{e}_k\big)_2\, \boldsymbol{e}_k$, (where $(\,.\,,\,.\,)_2$ denotes the scalar product in $\boldsymbol{L}^2(\Omega)$), and the integral equations

$$\begin{aligned}\langle \partial_t \boldsymbol{u}^{(n)}, \boldsymbol{e}_k\rangle + \int_\Omega \big[P_t^\kappa(\boldsymbol{u}^{(n)}) \cdot \nabla \boldsymbol{u}^{(n)} \cdot \boldsymbol{e}_k + \nu \nabla \boldsymbol{u}^{(n)} : \nabla \boldsymbol{e}_k\big]\, \mathrm{d}\boldsymbol{x} + n\, \big(\boldsymbol{\Psi}_t(\boldsymbol{u}^{(n)}), \boldsymbol{e}_k\big)_{1,2} \\ = \langle \boldsymbol{f}, \boldsymbol{e}_k\rangle + \int_{\Gamma_2} \boldsymbol{g} \cdot \boldsymbol{e}_k\, \mathrm{d}S,\end{aligned} \tag{3.4}$$

hold for all $k=1,\dots,n$. The last term in the integral on the left hand side plays the role of a penalization. Substituting here from (3.3), we obtain a system of $n$ ordinary differential equations for the unknown coefficients $a_k^{(n)}$ ($k=1,\dots,n$). The system is completed by the initial condition

$$a_k^{(n)}(0) \;=\; \big(\boldsymbol{u}_0-\boldsymbol{u}^*(0),\boldsymbol{e}_k\big)_2. \tag{3.5}$$

The local solvability of the system follows from Caratheodory's theorem. In order to prove the global solvability on the time interval $(0,T)$, one needs estimates of $a_k^{(n)}$ ($k=1,\dots,n$), valid on the whole interval $(0,T)$.

**3.3. A priori estimates and existence of the approximations.** Multiplying the $k$–th equation in (2.5) by $a_k^{(n)}$, writing $\boldsymbol{u}^{(n)}$ in the form $\boldsymbol{u}^*+\boldsymbol{v}^{(n)}$ and summing over $k$ from 1 to $n$, we obtain

$$\begin{aligned}\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\boldsymbol{v}^{(n)}\|_2^2+\nu\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+n\,\big(\boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{v}^{(n)}),\boldsymbol{v}^{(n)}\big)_{1,2}\;&=\;-\big\langle\partial_t\boldsymbol{u}^*,\boldsymbol{v}^{(n)}\big\rangle-\nu\int_\Omega\nabla\boldsymbol{u}^*:\nabla\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\\ -\int_\Omega P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\nabla(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}&+\big\langle\boldsymbol{f},\boldsymbol{v}^{(n)}\big\rangle+\int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}S.\end{aligned} \tag{3.6}$$

The third term on the right hand side equals

$$\begin{aligned}&-\int_\Omega P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\nabla\boldsymbol{v}^{(n)}\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}-\int_\Omega P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\nabla\boldsymbol{u}^*\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\\ &\quad=\;-\frac{1}{2}\int_{\Gamma_2}P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\boldsymbol{n}\,|\boldsymbol{v}^{(n)}|^2\,\mathrm{d}S-\int_\Omega P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\nabla\boldsymbol{u}^*\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\\ &\quad\leq\;\frac{1}{2}\bigg(\int_{\Gamma_2}\big[P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\boldsymbol{n}\big]_-^a\,\mathrm{d}S\bigg)^{\frac{1}{a}}\bigg(\int_{\Gamma_2}|\boldsymbol{v}^{(n)}|^{\frac{2a}{a-1}}\,\mathrm{d}S\bigg)^{\frac{a-1}{a}}-\int_\Omega P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\cdot\nabla\boldsymbol{u}^*\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\\ &\quad\leq\;C\,\gamma(t)\,\|\boldsymbol{v}^{(n)}\|_{4/(3-2\kappa);\,\Gamma_2}^2+\big\|P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\big\|_{r_2}\,\|\nabla\boldsymbol{u}^*\|_2\,\|\boldsymbol{v}^{(n)}\|_{s_2},\\ &\quad\leq\;C\,\gamma(t)\,\|\boldsymbol{v}^{(n)}\|_{\kappa,2}^2+\big\|P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\big\|_{r_2}\,\|\nabla\boldsymbol{u}^*\|_2\,\|\boldsymbol{v}^{(n)}\|_{s_2},\end{aligned}$$

where $r_2^{-1}+s_2^{-1}=\frac{1}{2}$. If $r_2$ is chosen so that $3<r_2<6/(3-2\kappa)$ (which is $<6$) then $\boldsymbol{V}^\kappa\hookrightarrow\hookrightarrow\boldsymbol{L}^{r_2}(\Omega)$ and $\boldsymbol{V}^1\hookrightarrow\hookrightarrow\boldsymbol{L}^{s_2}(\Omega)$ (because $s_2<6$). (Here and further on, $C$ denotes a generic constant.) Moreover, interpolating the norm $\|\boldsymbol{v}^{(n)}\|_{\kappa,2}$ between $\|\boldsymbol{v}^{(n)}\|_2$ and $\|\boldsymbol{v}^{(n)}\|_{1,2}$ and using Friedrichs' inequality (1.7), the continuous imbedding $\boldsymbol{W}^{1,2}(\Omega)\hookrightarrow\boldsymbol{L}^6(\Omega)$, Young's inequality and the inequalities $\|P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\|_{r_2}\leq C\,\|P_t^\kappa(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\|_{\kappa,2}\leq C\,\big(\|\boldsymbol{u}^*\|_{\kappa,2}+\|\boldsymbol{v}^{(n)}\|_{\kappa,2}\big)\leq C\,\big(\|\boldsymbol{u}^*\|_{1,2}+\|\nabla\boldsymbol{v}^{(n)}\|_2\big)$, we observe that the right hand side of the last inequality is

$$\begin{aligned}&\leq\;C\,\gamma(t)\,\|\boldsymbol{v}^{(n)}\|_2^{2(1-\kappa)}\,\|\boldsymbol{v}^{(n)}\|_{1,2}^{2\kappa}+C\,\big(\|\boldsymbol{u}^*\|_{1,2}+\|\nabla\boldsymbol{v}^{(n)}\|_2\big)\,\|\boldsymbol{v}^{(n)}\|_{s_2}\\ &\leq\;C\,\gamma(t)\,\|\boldsymbol{v}^{(n)}\|_2^{2(1-\kappa)}\,\|\nabla\boldsymbol{v}^{(n)}\|_2^{2\kappa}+C\,\big(\|\boldsymbol{u}^*\|_{1,2}+\|\nabla\boldsymbol{v}^{(n)}\|_2\big)\,\|\boldsymbol{v}^{(n)}\|_2^{\frac{6-s_2}{2s_2}}\,\|\boldsymbol{v}^{(n)}\|_6^{\frac{3(s_2-2)}{2s_2}}\\ &\leq\;\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+C\,\delta^{-\frac{\kappa}{1-\kappa}}\,\gamma^{\frac{1}{1-\kappa}}(t)\,\|\boldsymbol{v}^{(n)}\|_2^2+C\,\|\boldsymbol{u}^*\|_{1,2}\,\|\boldsymbol{v}^{(n)}\|_2^{\frac{6-s_2}{2s_2}}\,\|\nabla\boldsymbol{v}^{(n)}\|_2^{\frac{3(s_2-2)}{2s_2}}\\ &\quad+\;C\,\|\boldsymbol{v}\|_2^{\frac{6-s_2}{2s_2}}\,\|\nabla\boldsymbol{v}^{(n)}\|_2^{\frac{5s_2-6}{2s_2}}\\ &\leq\;3\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\gamma^{\frac{1}{1-\kappa}}(t)\,\|\boldsymbol{v}^{(n)}\|_2^2+C(\delta)\,\|\boldsymbol{u}^*\|_{1,2}^2+C(\delta)\,\|\boldsymbol{v}^{(n)}\|_2^2,\end{aligned} \tag{3.7}$$

where $\delta>0$ can be chosen arbitrarily small. The exponent $1/(1-\kappa)$ equals $4a/(a-2)$ for $2<a\leq 3$ and $4a/(4-a)$ for $3\leq a<4$. Hence it is less than or equal to $l(a)$ (the number defined in subsection 1.5). Consequently, $\gamma^{1/(1-\kappa)}\in L^1(0,T)$. The estimates of the other terms on the right hand side of (3.6) are standard:

$$\bigg|\nu\int_\Omega\nabla\boldsymbol{u}^*:\nabla\boldsymbol{v}^{(n)}\,\mathrm{d}\boldsymbol{x}\bigg|\;\leq\;\delta\,\|\nabla\boldsymbol{v}^{(n)}\|_2^2+\frac{\nu^2}{4\delta}\,\|\nabla\boldsymbol{u}^*\|_2^2, \tag{3.8}$$

$$\big|\big\langle\partial_t\boldsymbol{u}^*,\boldsymbol{v}^{(n)}\big\rangle\big| \ \le\ \|\partial_t\boldsymbol{u}^*\|_{-1,2}\, c_1\, \|\nabla\boldsymbol{v}^{(n)}\|_2 \ \le\ \delta\, \|\nabla\boldsymbol{v}^{(n)}\|_2^2+\frac{c_1^2}{4\delta}\, \|\partial_t\boldsymbol{u}^*\|_{-1,2}^2, \tag{3.9}$$

$$\big|\big\langle\boldsymbol{f},\boldsymbol{v}^{(n)}\big\rangle\big| \ \le\ \|\boldsymbol{f}\|_{-1,2}\, c_1\, \|\nabla\boldsymbol{v}^{(n)}\|_2 \ \le\ \delta\, \|\nabla\boldsymbol{v}^{(n)}\|_2^2+\frac{c_1^2}{4\delta}\, \|\boldsymbol{f}\|_{-1,2}^2, \tag{3.10}$$

$$\begin{aligned}\left|\int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{v}^{(n)}\,\mathrm{d}S\right| \ &\le\ \|\boldsymbol{g}\|_{4/3;\,\Gamma_2}\, \|\boldsymbol{v}^{(n)}\|_{4;\,\Gamma_2} \ \le\ \|\boldsymbol{g}\|_{4/3;\,\Gamma_2}\, c_7\, c_1\, \|\nabla\boldsymbol{v}^{(n)}\|_2 \\ &\le\ \delta\, \|\nabla\boldsymbol{v}^{(n)}\|_2^2+\frac{c_7^2c_1^2}{4\delta}\, \|\boldsymbol{g}\|_{4/3;\,\Gamma_2}^2.\end{aligned} \tag{3.11}$$

(We have again used inequality (1.7), Young's inequality and the continuity of the operator of traces from $\boldsymbol{W}^{1,2}(\Omega)$ to $\boldsymbol{L}^4(\partial\Omega)$. The norm of this operator is denoted by $c_7$.) Substituting now from (3.7)–(3.11) to (3.6) and choosing $\delta$ sufficiently small, we calculate that there exist positive constants $c_2,\ c_3,\ c_4,\ c_5,\ c_6$, independent of $n$, such that

$$\begin{aligned}\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\, \|\boldsymbol{v}^{(n)}\|_2^2+\nu\, \|\nabla\boldsymbol{v}^{(n)}\|_2^2+n\, \big(\boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{v}^{(n)}),\boldsymbol{v}^{(n)}\big)_{1,2} \ &\le\ c_2\, \big[\gamma^{\frac{1}{1-\kappa}}(t)+1\big]\, \|\boldsymbol{v}^{(n)}\|_2^2+c_3\, \|\boldsymbol{f}\|_{-1,2}^2 \\ &\quad +c_4\, \|\boldsymbol{u}^*\|_{1,2}^2+c_5\, \|\partial_t\boldsymbol{u}^*\|_{-1,2}^2+c_6\, \|\boldsymbol{g}\|_{4/3;\,\Gamma_2}^2.\end{aligned} \tag{3.12}$$

Integrating this inequality on the time interval $(0,T)$, we derive the estimates

$$\|\boldsymbol{v}^{(n)}(t)\|_2 \ \le\ c_8 \qquad \text{for all } t\in(0,T) \text{ and } n\in\mathbb{N}, \tag{3.13}$$

$$\int_0^T \|\nabla\boldsymbol{v}^{(n)}\|_2^2\, \mathrm{d}t \ \le\ c_9 \qquad \text{for all } n\in\mathbb{N}, \tag{3.14}$$

$$\int_0^T n\, \big(\boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{v}^{(n)}),\boldsymbol{v}^{(n)}\big)_{1,2}\, \mathrm{d}t \ \le\ c_{10} \qquad \text{for all } n\in\mathbb{N}. \tag{3.15}$$

The upper bounds $c_8,\ c_9$ and $c_{10}$ are independent of $n$. Note that the inequalities (3.1) and (3.15) yield

$$\int_0^T n\, \|\boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\|_{1,2}^2\, \mathrm{d}t\le c_{10}, \qquad \int_0^T n\, \|\boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{v}^{(n)})\|_{1,2}\, \mathrm{d}t\le\frac{c_{10}}{\epsilon_1}. \tag{3.16}$$

Estimates (3.13) imply that $\sum_{k=1}^n a_k^{(n)}(t)^2\le c_8^2$ for all $t\in(0,T)$ and $n\in\mathbb{N}$. From this, one can deduce that the system of ordinary differential equations for the unknowns $a_k^{(n)}$ ($k=1,\dots,n$), which we obtain if we use (3.3) and substitute $\boldsymbol{u}^{(n)}=\boldsymbol{u}^*+\boldsymbol{v}^{(n)}$ to (3.4), is uniquely solvable on the whole time interval $(0,T)$. Consequently, the approximations $\boldsymbol{u}^{(n)}\equiv\boldsymbol{u}^*+\boldsymbol{v}^{(n)}$ also exist on the whole interval $(0,T)$ and satisfy estimates (3.13)–(3.16).

**3.4. An estimate of a fractional derivative of $\boldsymbol{v}^{(n)}$.** In order to obtain later a strong convergence of a subsequence of $\{\boldsymbol{v}^{(n)}\}$, we also need an estimate of at least a fractional derivative of $\boldsymbol{v}^{(n)}$ with respect to $t$. Choose $r\in(0,\frac{1}{2})$ and put

$$\begin{aligned}\boldsymbol{\mathcal{H}}^r \ &:=\ \big\{\boldsymbol{v}\in L^2(0,T;\,\boldsymbol{V}^1);\ |\tau|^r\,\widehat{\boldsymbol{v}}(\tau)\in L^2\big(-\infty,\infty;\,\boldsymbol{V}^{-2}(\Omega)\big)\big\}, \\ \|\boldsymbol{v}\|_{\boldsymbol{\mathcal{H}}^r}^2 \ &:=\ \int_0^T \|\boldsymbol{v}(t)\|_{1,2}^2\, \mathrm{d}t+\int_{-\infty}^{\infty}|\tau|^{2r}\, \|\widehat{\boldsymbol{v}}(\tau)\|_{-2,2}^2\, \mathrm{d}\tau,\end{aligned}$$

where $\boldsymbol{V}^{-2}$ denotes the dual space to $\boldsymbol{V}^2$, $\|\,.\,\|_{-2,2}$ is the norm in $\boldsymbol{V}^{-2}$ and $\widehat{\boldsymbol{v}}$ is the Fourier transform of $\boldsymbol{v}$ in variable $t$. (In order to calculate the Fourier transform, we extend $\boldsymbol{v}(t)$ by zero for $t\in(-\infty,0)\cup(T,\infty)$.) Recall that $\boldsymbol{V}^2\ :=\ \boldsymbol{V}^1\cap\boldsymbol{W}^{2,2}(\Omega)$. By analogy with the proof of the existence of a weak solution to the Navier–Stokes equations, one can derive the estimate

$$\|\boldsymbol{v}^{(n)}\|_{\boldsymbol{\mathcal{H}}^r}^2 \ \le\ C+\int_{-\infty}^{\infty}\|\widehat{\boldsymbol{v}}^{(n)}(\tau)\|_2^2\, \mathrm{d}\tau \ \le\ C+\int_0^T \|\boldsymbol{v}^{(n)}(t)\|_2^2\, \mathrm{d}t \ \le\ c_{11}. \tag{3.17}$$

(See e.g. [18, Sec. 1.6.5] for more details.) It should be only mentioned that one needs to apply (3.15) in order to control the terms that contain the penalization $\boldsymbol{\Psi}_t$ in equation (3.4).

**3.5. Convergence of the approximations.** It follows from (3.13), (3.14), (3.17), the compact imbedding $\boldsymbol{\mathcal{H}}^r \hookrightarrow\hookrightarrow L^2(0,T;\, \boldsymbol{V}^\kappa)$ (see e.g. [18, Chap. I.5.2]) and the continuity of the operator of traces from $\boldsymbol{V}^\kappa$ to $\boldsymbol{L}^{4/(3-2\kappa)}(\partial\Omega)$ that there exist $\boldsymbol{v} \in L^\infty(0,T;\, \boldsymbol{L}^2(\Omega)) \cap \boldsymbol{\mathcal{H}}^r$ and a subsequence of $\{\boldsymbol{v}^{(n)}\}$ (which we again denote by $\{\boldsymbol{v}^{(n)}\}$) such that

$$\boldsymbol{v}^{(n)} \longrightarrow \boldsymbol{v} \qquad \text{weakly in } \boldsymbol{\mathcal{H}}^r \text{ and weakly--}* \text{ in } L^\infty(0,T;\, \boldsymbol{L}^2(\Omega)), \tag{3.18}$$

$$\boldsymbol{v}^{(n)} \longrightarrow \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;\, \boldsymbol{V}^\kappa), \tag{3.19}$$

$$\boldsymbol{v}^{(n)} \longrightarrow \boldsymbol{v} \qquad \text{strongly in } L^2(0,T;\, \boldsymbol{L}^{4/(3-2\kappa)}(\Gamma_2)). \tag{3.20}$$

**3.6. The inclusion $\boldsymbol{u}^*(t)+\boldsymbol{v}(t) \in \boldsymbol{K}^1_t$.** Due to the monotonicity of operator $\boldsymbol{\Psi}$ in $\boldsymbol{W}^{1,2}$, we have

$$\int_0^T \big(\boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{v}^{(n)}) - \boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{z}), \boldsymbol{v}^{(n)}-\boldsymbol{z}\big)_{1,2}\, \mathrm{d}t \;\geq\; 0 \tag{3.21}$$

for all $n \in \mathbb{N}$ and $\boldsymbol{z} \in L^2(0,T;\, \boldsymbol{V}^1)$. Using (3.1), (3.16) and (3.18), we obtain

$$\lim_{n\to\infty} \int_0^T \big(\boldsymbol{\Psi}_t(\boldsymbol{u}^*(t)+\boldsymbol{v}^{(n)}), \boldsymbol{v}^{(n)}-\boldsymbol{z}\big)_{1,2}\, \mathrm{d}t \;=\; 0,$$
$$\lim_{n\to\infty} \int_0^T \big(\boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{z}), \boldsymbol{v}^{(n)}\big)_{1,2}\, \mathrm{d}t \;=\; \int_0^T \big(\boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{z}), \boldsymbol{v}\big)_{1,2}\, \mathrm{d}t.$$

Thus, passing to the limit for $n \to \infty$ in (3.21), we obtain $\int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{u}^*+\boldsymbol{z}), \boldsymbol{v}-\boldsymbol{z}\big)_{1,2}\, \mathrm{d}t \leq 0$. Consider $\boldsymbol{z}$ in the form $\boldsymbol{z} := \boldsymbol{v} - \xi\, \boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{v})$ where $\xi > 0$. Dividing the inequality by $\xi$ and passing to the limit for $\xi \to 0+$, we get $\int_0^T \big(\boldsymbol{\Psi}(\boldsymbol{u}^*+\boldsymbol{v}), \boldsymbol{\Psi}_t(\boldsymbol{u}^*+\boldsymbol{v})\big)_{1,2}\, \mathrm{d}t \leq 0$, which means that $\boldsymbol{\Psi}_t(\boldsymbol{u}^*(t)+\boldsymbol{v}(t)) = \boldsymbol{0}$ for a.a. $t \in (0,T)$. This implies that $\boldsymbol{u}^*(t)+\boldsymbol{v}(t) \in \boldsymbol{K}^1_t$ for a.a. $t \in (0,T)$.

**3.7. The limit transition in equation (2.5) for $\boldsymbol{w} \in \mathscr{K}_m(0,T)$.** For $m \in \mathbb{N}$, we denote by $\mathscr{W}_m(0,T)$ the set of functions $\boldsymbol{w} \in \mathscr{W}(0,T)$ that have a finite expansion $\boldsymbol{w}(t) = \boldsymbol{u}^*(t) + \sum_{k=1}^m \mu_k(t)\boldsymbol{e}_k$, and by $\mathscr{K}_m(0,T)$ the set of functions $\boldsymbol{w} \in \mathscr{W}_m(0,T)$ such that $\boldsymbol{w}(t) \in \boldsymbol{K}^1_t$ for a.a. $t \in (0,T)$.

We claim that the function $\boldsymbol{u} \equiv \boldsymbol{u}^*+\boldsymbol{v}$ satisfies the inequality (2.3). Assume at first that the test function $\boldsymbol{w}$ in (2.3) is chosen from set $\mathscr{K}_m(0,T)$ and $n > m$. Recall that $\boldsymbol{v}^{(n)}$ has the expansion (3.3) and $\boldsymbol{u}^{(n)} = \boldsymbol{u}^*+\boldsymbol{v}^{(n)}$. Let us multiply equation (3.4) by $\mu_k - a_k^{(n)}$ if $k \leq m$ and by $-a_k^{(n)}$ if $m < k \leq n$ and sum the equations for $k = 1,\dots,n$. Then we integrate the resulting equation with respect to time on $(0,T)$. We obtain

$$\begin{aligned}\int_0^T \big\langle \partial_t\boldsymbol{u}^{(n)}, \boldsymbol{w}-\boldsymbol{u}^{(n)}\big\rangle\, \mathrm{d}t + \int_0^T\!\!\int_\Omega \big[P^\kappa_t(\boldsymbol{u}^{(n)})\cdot\nabla\boldsymbol{u}^{(n)}\cdot(\boldsymbol{w}-\boldsymbol{u}^{(n)}) + \nu\nabla\boldsymbol{u}^{(n)} : \nabla(\boldsymbol{w}-\boldsymbol{u}^{(n)})\big]\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \\ + \int_0^T n\,\big(\boldsymbol{\Psi}_t(\boldsymbol{u}^{(n)}), \boldsymbol{w}-\boldsymbol{u}^{(n)}\big)_{1,2}\, \mathrm{d}t \;=\; \int_0^T \langle \boldsymbol{f}, \boldsymbol{w}-\boldsymbol{u}^{(n)}\rangle\, \mathrm{d}t + \int_0^T\!\!\int_{\Gamma_2} \boldsymbol{g}\cdot(\boldsymbol{w}-\boldsymbol{u}^{(n)})\, \mathrm{d}S\, \mathrm{d}t.\end{aligned} \tag{3.22}$$

The first integral on the left hand side satisfies:

$$\begin{aligned}\int_0^T \big\langle \partial_t\boldsymbol{u}^{(n)}, \boldsymbol{w}-\boldsymbol{u}^{(n)}\big\rangle\, \mathrm{d}t \;&=\; \int_0^T \big\langle \partial_t\boldsymbol{u}^{(n)}-\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{u}^{(n)}\big\rangle\, \mathrm{d}t + \int_0^T \big\langle \partial_t\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{u}^{(n)}\big\rangle\, \mathrm{d}t \\ &=\; -\frac{1}{2}\,\|\boldsymbol{u}^n(T)-\boldsymbol{w}(T)\|_2^2 + \frac{1}{2}\,\|\boldsymbol{u}^n(0)-\boldsymbol{w}(0)\|_2^2 + \int_0^T \big\langle \partial_t\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{u}^{(n)}\big\rangle\, \mathrm{d}t \\ &\leq\; \frac{1}{2}\,\|\boldsymbol{u}^n(0)-\boldsymbol{w}(0)\|_2^2 + \int_0^T \big\langle \partial_t\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{u}^{(n)}\big\rangle\, \mathrm{d}t.\end{aligned} \tag{3.23}$$

The integral of $n\big(\boldsymbol{\Psi}_t(\boldsymbol{u}^{(n)}), \boldsymbol{w}-\boldsymbol{u}^{(n)}\big)_{1,2}$ can be estimated by means of the monotonicity of operator $\boldsymbol{\Psi}_t$ and the identity $\boldsymbol{\Psi}_t(\boldsymbol{w})(t)) = \boldsymbol{0}$ (which holds because $\boldsymbol{w}(t) \in \boldsymbol{K}_t^1$) as follows:

$$\int_0^T n\left(\boldsymbol{\Psi}(\boldsymbol{u}^{(n)}), \boldsymbol{w}-\boldsymbol{u}^{(n)}\right)_{1,2} \mathrm{d}t \;=\; -n\int_0^T \left(\boldsymbol{\Psi}(\boldsymbol{w})-\boldsymbol{\Psi}(\boldsymbol{u}^{(n)}), \boldsymbol{w}-\boldsymbol{u}^{(n)}\right)_{1,2} \mathrm{d}t \;\leq\; 0. \tag{3.24}$$

Thus, (3.22)–(3.24) yield

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{w}, \boldsymbol{w}-\boldsymbol{u}^{(n)}\rangle \,\mathrm{d}t + \int_0^T\int_\Omega \big[P_t^\kappa(\boldsymbol{u}^{(n)})\cdot\nabla\boldsymbol{u}^{(n)}\cdot(\boldsymbol{w}-\boldsymbol{u}^{(n)}) + \nu\nabla\boldsymbol{u}^{(n)} : \nabla(\boldsymbol{w}-\boldsymbol{u}^{(n)})\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ \geq\; \int_0^T \langle \boldsymbol{f}, \boldsymbol{w}-\boldsymbol{u}^{(n)}\rangle\,\mathrm{d}t + \int_0^T\int_{\Gamma_2} \boldsymbol{g}\cdot(\boldsymbol{w}-\boldsymbol{u}^{(n)})\,\mathrm{d}S\,\mathrm{d}t - \frac{1}{2}\,\big\|\boldsymbol{w}(0)-\boldsymbol{u}^{(n)}(0)\big\|_2^2.\end{aligned} \tag{3.25}$$

The next step is the passage to the limit for $n\to\infty$ in (3.25). Here, we apply all types of convergence (3.18)–(3.20). We explain the limit transition only in the two terms in (3.25), because the transition in the other terms is analogous or the same as in the proof of the existence of weak solutions of the Navier–Stokes equations, see e.g. [9]) or [18]. a) The inequality

$$\liminf_{n\to\infty}\;\left(-\nu\int_0^T\int_\Omega \nabla\boldsymbol{u}^{(n)} : \nabla\boldsymbol{u}^{(n)}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right) \;\leq\; -\nu\int_0^T\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{3.26}$$

follows from (3.18) and the identity $\boldsymbol{u}^{(n)} = \boldsymbol{u}^* + \boldsymbol{v}^{(n)}$. b) The integral of $P_t^\kappa(\boldsymbol{u}^{(n)})\cdot\nabla\boldsymbol{u}^{(n)}\cdot(\boldsymbol{w}-\boldsymbol{u}^{(n)})$ equals

$$\int_0^T\int_\Omega P_t^\kappa(\boldsymbol{u}^{(n)})\cdot\nabla\boldsymbol{u}^{(n)}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \frac{1}{2}\int_0^T\left(\int_{\Gamma_1}(\boldsymbol{u}^*\cdot\boldsymbol{n})\,|\boldsymbol{u}^*|^2\,\mathrm{d}S + \int_{\Gamma_2}\big(P_t^\kappa(\boldsymbol{u}^{(n)})\cdot\boldsymbol{n}\big)\,|\boldsymbol{u}^{(n)}|^2\,\mathrm{d}S\right)\mathrm{d}t$$

The integral $\int_{\Gamma_2}\big(P_t^\kappa(\boldsymbol{u}^{(n)})\cdot\boldsymbol{n}\big)\,|\boldsymbol{u}^{(n)}|^2\,\mathrm{d}S$ converges to $\int_{\Gamma_2}\big(P_t^\kappa(\boldsymbol{u})\cdot\boldsymbol{n}\big)\,|\boldsymbol{u}|^2\,\mathrm{d}S$ point-wise for a.a. $t\in(0,T)$ due to (3.20). Thus, applying Fatou's lemma on the interval $(0,T)$, we get

$$\liminf_{n\to\infty}\;\left(-\frac{1}{2}\int_0^T\int_{\Gamma_2}\big(P_t^\kappa(\boldsymbol{u}^{(n)})\cdot\boldsymbol{n}\big)\,|\boldsymbol{u}^{(n)}|^2\,\mathrm{d}S\,\mathrm{d}t\right) \;\leq\; -\frac{1}{2}\int_0^T\int_{\Gamma_2}(\boldsymbol{u}\cdot\boldsymbol{n})\,|\boldsymbol{u}|^2\,\mathrm{d}S\,\mathrm{d}t.$$

Since $\boldsymbol{K}_t^\kappa$ is convex, $\boldsymbol{u}(t)\in\boldsymbol{K}_t^\kappa$ and $\|P_t^\kappa\boldsymbol{u}^{(n)}(t)-\boldsymbol{u}(t)\|_{\kappa,2}\leq\|\boldsymbol{u}^{(n)}(t)-\boldsymbol{u}(t)\|_{\kappa,2}$ for a.a. $t\in(0,T)$, (3.19) implies that $P_t^\kappa\boldsymbol{u}^{(n)}\to\boldsymbol{u}$ strongly in $L^2(0,T;\,\boldsymbol{V}^\kappa)$. Hence

$$\int_0^T\int_\Omega P_t^\kappa(\boldsymbol{u}^{(n)})\cdot\nabla\boldsymbol{u}^{(n)}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \;\longrightarrow\; \int_0^T\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \qquad (\text{for } n\to\infty).$$

Consequently,

$$\begin{aligned}&\liminf_{n\to\infty}\;\int_0^T\int_\Omega P_t^\kappa(\boldsymbol{u}^{(n)})\cdot\nabla\boldsymbol{u}^{(n)}\cdot(\boldsymbol{w}-\boldsymbol{u}^{(n)})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &\leq\; -\frac{1}{2}\int_0^T\int_{\Gamma_1}(\boldsymbol{u}^*\cdot\boldsymbol{n})\,|\boldsymbol{u}^*|^2\,\mathrm{d}S\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_{\Gamma_2}(\boldsymbol{u}\cdot\boldsymbol{n})\,|\boldsymbol{u}|^2\,\mathrm{d}S\,\mathrm{d}t + \int_0^T\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &=\; \int_0^T\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot(\boldsymbol{w}-\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned} \tag{3.27}$$

Thus, we observe that $\boldsymbol{u}$ satisfies inequality (2.3) for all test functions $\boldsymbol{w}\in\mathscr{K}_m(0,T)$. Since $m$ was an arbitrary number from $\mathbb{N}$, (2.3) holds for all $\boldsymbol{w}\in\bigcup_{m=1}^\infty\mathscr{K}_m(0,T)$.

**3.8. Completion of the proof.** We still need to show that (2.3) is satisfied for all $\boldsymbol{w}\in\mathscr{K}(0,T)$. For this purpose, it is sufficient to show that $\bigcup_{m=1}^\infty\mathscr{K}_m(0,T)$ is dense in $\mathscr{K}(0,T)$ in the norm $|||\,.\,|||$. Obviously, $\bigcup_{m=1}^\infty\mathscr{W}_m(0,T)$ is dense in $\mathscr{W}(0,T)$. Set $\mathscr{K}(0,T)$ is closed in $\mathscr{W}(0,T)$, with the property that

$\mathscr{K}(0,T)$ is equal to the closure of its interior. Hence $\left(\bigcup_{m=1}^{\infty}\mathscr{W}_m(0,T)\right)\cap\mathscr{K}(0,T)$ (which coincides with $\bigcup_{m=1}^{\infty}\mathscr{K}_m(0,T)$) is dense in $\mathscr{W}(0,T)\cap\mathscr{K}(0,T)$ (which coincides with $\mathscr{K}(0,T)$).

The validity of the energy inequality (2.4) can be deduced from inequality (3.12) by means of the same arguments as in the case of the Leray–Hopf weak solution to the Navier–Stokes equations, see e.g. [9].

The proof of Theorem 1 is completed.

**Acknowledgement.** The research has been supported by the Grant Agency of the Czech Republic (grant No. GA16-03230S) and by the Academy of Sciences of the Czech Republic (RVO 67985840).

## References

[1] M. Beneš, P. Kučera: Solutions of the Navier-Stokes equations with various types of boundary conditions. *Arch. Math.* **98** (2012), 487-497.

[2] F. Boyer, P. Fabrie: Outflow boundary conditions for the incompressible non–homogeneous Navier–Stokes equations. *DCDS – Series B* **7** (2007), No. 2, 219–250.

[3] M. Braack, P. P. Mucha: Directional do–nothing condition for the Navier–Stokes equations. *J. Comp. Math.* **32** (2014), No. 5, 507–521.

[4] C. H. Bruneau and P. Fabrie, New efficient boundary conditions for incompressible Navier–Stokes equations: A well–posedness result, *Mathematical Modelling and Numerical Analysis* **30** (1996), No. 7, 815–840.

[5] S. Dong, G. E. Karniadakis, C. Chryssostomidis: A robust and accurate outflow boundary condition for incompressible flow simulations on severely-truncated unbounded domains. *J. of Comp. Physics* **261** (2014), 83-105.

[6] M. Feistauer, T. Neustupa: On some aspects of analysis of incompressible flow through cascades of profiles. *Operator Theory, Advances and Applications*, Vol. 147, Birkhauser, Basel, 2004, 257–276.

[7] M. Feistauer, T. Neustupa: On non-stationary viscous incompressible flow through a cascade of profiles. *Math. Meth. Appl. Sci.* **29** (2006), No. 16, 1907-1941.

[8] M. Feistauer, T. Neustupa: On the existence of a weak Solution of viscous incompressible flow past a cascade of profiles with an arbitrarily large inflow. *J. Math. Fluid Mech.* **15** (2013), No. 4, 701–715.

[9] G. P. Galdi: An Introduction to the Navier–Stokes initial–boundary value problem. In *Fundamental Directions in Mathematical Fluid Mechanics*, editors G. P. Galdi, J. Heywood, R. Rannacher, series "Advances in Mathematical Fluid Mechanics", Birkhauser–Verlag, Basel 2000, 1–98.

[10] G. P. Galdi: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Steady–State Problems.* Springer–Verlag, New York–Berlin–Heidelberg, 2nd edition 2011.

[11] J. G. Heywood, R. Rannacher, S. Turek: Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations. *Int. J. for Num. Meth. in Fluids* **22** (1996), 325–352.

[12] S. Kračmar, J. Neustupa: Modelling of flows of a viscous incompressible fluid through a channel by means of variational inequalities. *ZAMM* **74** (1994), No. 6, 637-639.

[13] S. Kračmar, J. Neustupa: A weak solvability of a steady variational inequality of the NavierStokes type with mixed boundary conditions. *Nonlinear Analysis* **47** (2001), No. 6, 4169-4180.

[14] P. Kučera, Z. Skalák: Solutions to the NavierStokes equations with mixed boundary conditions. *Acta Appl. Math.* **54** (1998), No. 3, 275-288.

[15] P. Kučera: Basic properties of the non-steady Navier–Stokes equations with mixed boundary conditions ina bounded domain. *Ann. Univ. Ferrara* **55** (2009), 289–308.

[16] M. Lanzendörfer, J. Stebel: On pressure boundary conditions for steady flows of incompressible fluids with pressure and shear rate dependent viscosities. *Nečas Center for Mathematical Modeling,* Preprint No. 2008–015, 2015.

[17] Y. Li, K. Li: Existence of the solution to stationary Navier-Stokes equations with nonlinear slip boundary conditions. *J. Math. Anal. Appl.* **381** (2011), 1-9.

[18] J. L. Lions: *Quelques méthodes de résolution des problèmes âux limites non linéaire.* Dunod, Gauthier–Villars, Paris 1969.

[19] J. L. Lions, E. Magenes: *Nonhomogeneous Boundary Value Problems and Applications I.* Springer–Verlag, New York, 1972.

[20] J. Marschall: The trace of Sobolev–Slobodeckij spaces on Lipschitz domains. *Manuscipta Mathematica* **58** (1987), 47–65.

[21] J. Nečas: *Direct Methods in the Theory of Elliptic Equations.* Springer–Verlag, Berlin, Heidelberg 2012.

[22] T. Neustupa: A steady flow through a plane cascade of profiles with an arbitrarily large inflow – the mathematical model, existence of a weak solution. *Applied Mathematics and Computation* **272** (2016), 687–691.

[23] T. Runst, W. Sickel: *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations.* Walter de Gruyter, Berlin–New York 1996.

*Authors' addresses:*

Stanislav Kračmar
Czech Academy of Sciences
Institute of Mathematics
Žitná 25, 115 67 Praha 1
Czech Republic
e–mail: stanislav.kracmar@fs.cvut.cz

and

Czech Technical University
Faculty of Mechanical Engineering
Department of Technical Mathematics
Karlovo nám. 13, 121 35 Praha 2
Czech Republic

Jiří Neustupa
Czech Academy of Sciences
Institute of Mathematics
Žitná 25, 115 67 Praha 1
Czech Republic
e–mail: neustupa@math.cas.cz