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# Regularity up to the Boundary of a Weak Solution of the Navier–Stokes Equations with Generalized Navier's Slip Boundary Conditions

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**Abstract**

The paper shows that regularity up to the boundary of a weak solution $\mathbf{v}$ of the Navier–Stokes equation with generalized Navier's slip boundary conditions follows from certain rate of integrability of at least one of the functions $\zeta_1$, $(\zeta_2)_+$ (the positive part of $\zeta_2$), $\zeta_3$, where $\zeta_1 \leq \zeta_2 \leq \zeta_3$ are the eigenvalues of the rate of deformation tensor $\mathbb{D}(\mathbf{v})$. A regularity criterion in terms of the principal invariants of tensor $\mathbb{D}(\mathbf{v})$ is also formulated.

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## 1 Introduction

**1.1. Navier–Stokes' initial–boundary value problem.** We assume that $\Omega$ is a bounded domain in $\mathbb{R}^3$ with a smooth boundary and $T$ is a given positive number. The motion of a viscous incompressible fluid with constant density (which is for simplicity assumed to be equal to one) in domain $\Omega$ in the time interval $(0,T)$ is described by the Navier–Stokes equations

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} \;=\; -\nabla p + \operatorname{div}\,[2\nu\mathbb{D}(\mathbf{v})] + \mathbf{f}, \tag{1.1}$$

$$\operatorname{div}\mathbf{v} \;=\; 0 \tag{1.2}$$

(in $\Omega\times(0,T)$) for the unknowns $\mathbf{v}\equiv(v_1,v_2,v_3)$ and $p$ (the velocity and the pressure). Symbol $\nu$ denotes the kinematic coefficient of viscosity (it is supposed to be a positive constant) and $\mathbb{D}(\mathbf{v}) := (\nabla\mathbf{v})_{\mathrm{sym}} := \frac{1}{2}\,[\nabla\mathbf{v} + (\nabla\mathbf{v})^T]$ is the so called "rate of deformation tensor". In this paper, we consider equations (1.1) and (1.2) with generalized Navier's slip boundary conditions

$$\mathbf{v}\cdot\mathbf{n} \;=\; 0, \tag{1.3}$$

$$\big[2\nu\mathbb{D}(\mathbf{v})\cdot\mathbf{n}\big]_\tau + \mathbb{K}\cdot\mathbf{v} \;=\; \mathbf{0} \tag{1.4}$$

(on $\partial\Omega\times(0,T)$). Here, $\mathbf{n}$ is the outer normal vector on $\partial\Omega$, subscript $\tau$ denotes the tangential component and $\mathbb{K}$ is a non–negative 2nd–order tensor defined a.e. on $\partial\Omega$ such that $\mathbb{K}(\mathbf{x})\cdot\mathbf{a}$ is tangential to $\partial\Omega$ at point $\mathbf{x}\in\partial\Omega$ if vector $\mathbf{a}$ is tangential to $\partial\Omega$ at point $\mathbf{x}$. Condition (1.4) generalizes the "classical" Navier boundary condition $[2\nu\mathbb{D}(\mathbf{v})\cdot\mathbf{n}]_\tau + \kappa\mathbf{v} = \mathbf{0}$, where $\kappa \geq 0$ is the coefficient of friction between the fluid and the boundary. The replacement of $\kappa\mathbf{v}$ by $\mathbb{K}\cdot\mathbf{v}$ reflects the fact that the microscopic structure of $\partial\Omega$ can vary from point to point, it need not produce the same resistance in all tangential directions, and it may therefore divert the flow to the side. In this paper, we assume that $\mathbb{K}$ in (1.4) is a trace (on $\partial\Omega$) of a tensor–valued function from $W^{1,2}(\Omega)^{3\times 3}$, which is also denoted by $\mathbb{K}$. The problem (1.1)–(1.4) is completed by the initial condition

$$\mathbf{v}\big|_{t=0} = \mathbf{v}_0 \qquad \text{in } \Omega. \tag{1.5}$$

**1, 2. Shortly on regularity criteria for weak solutions to the system (1.1), (1.2).** Existence of a global regular solution and uniqueness of a weak solution are still the fundamental open questions in the theory of the Navier–Stokes equation in 3D. There exist a series of a posteriori assumptions on weak solutions that exclude the development of possible singularities. (They are usually called the "criteria of regularity".) The assumptions concern various quantities, like e.g. the velocity or some of its components (see e.g. [6], [14], [18], [22], etc.), the gradient of velocity or some of its components (see e.g. [18], [17], etc.), the vorticity or only two of its components (see e.g. [1], [6]), the direction of vorticity (see [2], [3]) and the pressure (see e.g. [4], [13], [20], etc.). The absence of a blow up (i.e. the non–existence of singularities) in a weak solution has also been proven under certain assumptions on the integrability of the positive part of the middle eigenvalue of the rate of deformation tensor $\mathbb{D}(\mathbf{v})$ in [15].

Most of the known regularity criteria can be applied either in the case when $\Omega = \mathbb{R}^3$ (like those from [6], [18], [17]) or they exclude singularities in the interior of $\Omega$, but not the singularities on the boundary. (This concerns e.g. the criteria from [14] and [15]). As to criteria, valid up to the boundary, we can cite e.g. the papers [8] (where the so called suitable weak solution is shown to be bounded locally near the boundary if it satisfies Serrin's conditions near the boundary and the trace of the pressure is bounded on the boundary), [19] (where an analogy of the well known Caffarelli–Kohn–Nirenberg criterion for the regularity of a suitable weak solution at the point $(\mathbf{x}_0, t_0) \in \Omega \times (0,T)$, see [5], is also proven for points on a flat part of the boundary) and [11], [24] (for some generalizations of the criterion from [19], however also valid only on a flat part of the boundary). A generalization of the criterion from [19] for points $(\mathbf{x}_0, t_0)$ on a "smooth" curved part of the boundary can be found in paper [21]. In paper [23], the author shows that if a weak solution satisfies Serrin's integrability conditions in a neighbourhood of a "smooth" part of the boundary then the solution is regular up to this part of the boundary. In all these papers, the authors used the no–slip boundary condition $\mathbf{v} = \mathbf{0}$ on $\partial\Omega \times (0,T)$ (or on the relevant part of this set).

**1.3. On the results of this paper.** In Section 2 of this paper, we consider equations (1.1), (1.2) with generalized Navier's boundary conditions (1.3), (1.4) and we prove results analogous to those from [15], however extended so that they hold up to the boundary of $\Omega$. (See Theorem 1.)

Note that while the regularity criteria, that consider some components of the velocity or the velocity gradient, depend on the observer's frame, the criterion that uses the eigenvalues of tensor $\mathbb{D}(\mathbf{v})$ is frame indifferent. Also note that the study of regularity of a weak solution in the neighborhood of the boundary requires a special technique, which is subtler than the one applied in the interior and closely connected with the used boundary conditions. This can be e.g. documented by the fact that the same result as the one obtained in Section 2 and stated in Theorem 1, for the system (1.1), (1.2) with the no–slip boundary condition, is not known.

**1.4. Notation.** Vector functions and spaces of vector functions are denoted by boldface letters.

- The norms of scalar– or vector– or tensor–valued functions with components in $L^q(\Omega)$ (respectively $W^{k,l}(\Omega)$) are denoted by $\| \, . \, \|_q$ (respectively $\| \, . \, \|_{k,l}$). The norm in $\mathbf{L}^2(\partial\Omega)$ is denoted by $\| \, . \, \|_{2;\, \partial\Omega}$. Norms in other spaces on $\partial\Omega$ are denoted by analogy.
- $\mathbf{L}^2_\sigma(\Omega)$ is the closure in $\mathbf{L}^2(\Omega)$ of the linear space of all infinitely differentiable divergence–free vector functions with a compact support in $\Omega$. The orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{L}^2_\sigma(\Omega)$ is denoted by $P_\sigma$.
- $\mathbf{W}^{1,2}_\sigma(\Omega) := \mathbf{W}^{1,2}(\Omega) \cap \mathbf{L}^2_\sigma(\Omega)$. We denote by $\mathbf{W}^{-1,2}_\sigma(\Omega)$ the dual space to $\mathbf{W}^{1,2}_\sigma(\Omega)$ and by $\langle \, . \, , \, . \, \rangle_\Omega$ the duality between elements of $\mathbf{W}^{-1,2}_\sigma(\Omega)$ and $\mathbf{W}^{1,2}_\sigma(\Omega)$.
- $\| \! | \, . \, | \! \|_{r,s;\, (t',t'')}$ denotes the norm of a vector–valued or a tensor–valued function with the components in in $L^r(t',t'';\, L^s(\Omega))$.

**1.5. A weak solution of the problem (1.1)–(1.5), Theorem on structure.** For $\mathbf{v}_0 \in \mathbf{L}^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2(0,T;\, \mathbf{W}^{-1,2}_\sigma(\Omega))$, a function $\mathbf{v} \in L^2(0,T;\, \mathbf{W}^{1,2}_\sigma(\Omega)) \cap L^\infty(0,T;\, \mathbf{L}^2_\sigma(\Omega))$ is called a *weak solution* of the problem (1.1)–(1.5) if it satisfies

$$\int_0^T \int_\Omega \big\{ -\partial_t \boldsymbol{\phi} \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \cdot \boldsymbol{\phi} + 2\nu \mathbb{D}(\mathbf{v}) : \nabla \boldsymbol{\phi} \big\} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_0^T \int_{\partial\Omega} (\mathbb{K} \cdot \mathbf{v}) \cdot \boldsymbol{\phi} \, \mathrm{d}S \, \mathrm{d}t$$
$$= \int_0^T \langle \mathbf{f}, \boldsymbol{\phi} \rangle_\Omega \, \mathrm{d}t + \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\phi}(\, . \, , 0) \, \mathrm{d}\mathbf{x} \tag{1.6}$$

for all infinitely differentiable divergence–free vector–functions $\boldsymbol{\phi}$ in $\overline{\Omega} \times [0,T]$, such that $\boldsymbol{\phi} \cdot \mathbf{n} = 0$ on $\partial\Omega \times [0,T]$ and $\boldsymbol{\phi}(\, . \, , T) = \mathbf{0}$. The existence of a weak solution of the problem (1.1)–(1.3), (1.5) with "classical" Navier's boundary condition $[2\nu \mathbb{D}(\mathbf{v}) \cdot \mathbf{n}]_\tau + \kappa \mathbf{v} = \mathbf{0}$ follows e.g. from papers [7] and [16]. (Note that the more general case of a time–varying domain $\Omega$ is considered in [16].) Applying the same methods, one can also extend the existential results from [7] and [16] to the problem (1.1)–(1.5), which includes the generalized Navier boundary condition (1.4). Moreover, by analogy with the Navier–Stokes equations with the no–slip boundary condition $\mathbf{v} = \mathbf{0}$ on $\partial\Omega \times (0,T)$, the weak solution can be constructed so that it satisfies the so called *strong energy inequality*

$$\|\mathbf{v}(t)\|_2^2 + 4\nu \int_s^t \int_\Omega |\mathbb{D}(\mathbf{v}(\vartheta))|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}\vartheta + 2 \int_s^t \int_{\partial\Omega} \mathbf{v}(\vartheta) \cdot \mathbb{K} \cdot \mathbf{v}(\vartheta) \, \mathrm{d}S \, \mathrm{d}\vartheta$$
$$\leq \|\mathbf{v}(s)\|_2^2 + \int_s^t \langle \mathbf{f}(\vartheta), \mathbf{v}(\vartheta) \rangle_\Omega \, \mathrm{d}\vartheta \tag{1.7}$$

for a.a $s \in (0,T)$ and all $t \in (s,T)$.

In contrast to the Navier–Stokes equations (1.1), (1.2) with the no–slip boundary condition, whose theory is relatively well elaborated, the equations with generalized Navier's boundary conditions (1.3), (1.4) have not yet been given so much attention. This is why a series of important results, well known from the theory of equations (1.1), (1.2) with the no–slip boundary condition, have not been explicitly proven in literature for equations with boundary conditions (1.3), (1.4), although many of them can be obtained in a similar or almost the same way. This concerns except others the local in time existence of a strong solution (here, however, one can cite the papers [7] and [12], where the local in time existence of a strong solution is proven in the case when $\mathbb{K} = \kappa \mathbb{I}$, $\kappa \geq 0$), the uniqueness of the weak solution and the so called "Theorem on structure". This theorem states that if the specific volume force $\mathbf{f}$ is at least in $L^2(0,T;\, \mathbf{L}^2(\Omega))$ and $\mathbf{v}$ is a weak solution of the Navier–Stokes problem problem with the no–slip boundary condition, satisfying the strong energy inequality, then $(0,T) = \bigcup_{\gamma\in\Gamma}(a_\gamma, b_\gamma) \cup G$, where set $\Gamma$ is at most countable, the intervals $(a_\gamma, b_\gamma)$ are pair–wise disjoint, the 1D Lebesgue measure of set $G$ is zero and solution $\mathbf{v}$ is coincides with a strong solution in the interior of each of the time intervals $(a_\gamma, b_\gamma)$. (See e.g. [10] for more details.) In this paper, we also use the Theorem on structure, but we apply it to the Navier–Stokes problem with boundary conditions (1.3), (1.4). (As is mentioned above, the validity of the theorem for the problem with boundary conditions (1.3), (1.4) can be proven by means of similar arguments as in the case of the no–slip boundary condition.)

## 2 Regularity up to the boundary in dependence on eigenvalues of tensor $\mathbb{D}(\mathbf{v})$

The main theorem of this section says:

**Theorem 1.** *Let $\mathbf{f} \in L^2(0,T;\, \mathbf{L}^2(\Omega))$, $\mathbb{K} \in W^{1,2}(\Omega)^{3\times 3}$ and $\mathbf{v}$ be a weak solution of the problem (1.1)–(1.5), satisfying the strong energy inequality. Suppose that $\zeta_1 \le \zeta_2 \le \zeta_3$ are the eigenvalues of tensor $\mathbb{D}(\mathbf{v})$ and*

(i) *one of the functions $\zeta_1$, $(\zeta_2)_+$, $\zeta_3$ belongs to $L^r(0,T;\, \mathbf{L}^s(\Omega))$ for some $r \in [1,\infty]$, $s \in (\frac{3}{2},\infty]$, satisfying $2/r + 3/s = 2$.*

*Then the norm $\|\nabla\mathbf{v}(t)\|_2$ is bounded for $t \in (\epsilon, T)$ for any $\epsilon > 0$. Moreover, if $\mathbf{v}_0 \in \mathbf{W}^{1,2}_\sigma(\Omega)$ then $\|\nabla\mathbf{v}(\,.\,,t)\|_2$ is bounded on the whole interval $(0,T)$.*

**2.1. Remark.** The eigenvalues $\zeta_1$, $\zeta_2$, $\zeta_3$ are all real, because tensor $\mathbb{D}(\mathbf{v})$ is symmetric. Since the dynamic stress tensor $\mathbb{T}_{\rm d}(\mathbf{v})$ equals $2\nu\mathbb{D}(\mathbf{v})$ in the Newtonian fluid, the eigenvalues of $\mathbb{D}(\mathbf{v})$ coincide, up to the factor $2\nu$, with the principal dynamic stresses. The eigenvalues are the roots of the characteristic equation of tensor $\mathbb{D}(\mathbf{v})$, i.e. the equation $F(\zeta) := \zeta^3 - E_1\zeta^2 + E_2\zeta - E_3 = 0$, where $E_1$, $E_2$, $E_3$ are the principal invariants of $\mathbb{D}(\mathbf{v})$. The invariant $E_1$ is equal to zero, because $\mathrm{Tr}\,\mathbb{D}(\mathbf{v}) = 0$. Furthermore,

$$\begin{aligned} E_2 \;&=\; \zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1 \;=\; -\frac{1}{2}(\zeta_1^2 + \zeta_2^2 + \zeta_3^2) \\ &=\; -\frac{1}{6}\left[(\zeta_1-\zeta_2)^2 + (\zeta_2-\zeta_3)^2 + (\zeta_3-\zeta_1)^2\right] \;\le\; 0 \end{aligned}$$

and $E_3 = \zeta_1\zeta_2\zeta_3$. Put $\zeta_0 := \sqrt{-\frac{1}{3}E_2}$. Number $\zeta_0$ is chosen so that $F'(\pm\zeta_0) = 0$ Obviously, $E_2 = 0$ implies $\zeta_1 = \zeta_2 = \zeta_3 = 0$. Thus, assume that $E_2 < 0$. Then $\mathrm{sgn}\,\zeta_2 = \mathrm{sgn}\,(-E_3)$. The rough estimate of $\zeta_2$ says that $-\zeta_0 < \zeta_2 < \zeta_0$. A more accurate estimate yields $\zeta_2$ between $\zeta^* := E_3/E_2$ (the point where the tangent line to the graph of $F$ at the point $\zeta = 0$ intersects the $\zeta$–axis) and $\zeta^{**} := \frac{3}{2}E_3/E_2$ (the point where the line connecting the points $(0, -E_3)$ and $((\zeta_0, F(\zeta_0))$ (if $E_3 < 0$) or $(-\zeta_0, F(-\zeta_0))$ (if $E_3 > 0$) intersects the $\zeta$–axis). Thus, we have

$$\begin{aligned} &a) \quad 0 < \frac{E_3}{E_2} < \zeta_2 < \frac{3E_3}{2E_2} < \sqrt{-\tfrac{1}{3}E_2} && \text{if } E_2 < 0 \text{ and } E_3 < 0, \\ &b) \quad -\sqrt{-\tfrac{1}{3}E_2} < \frac{3E_3}{2E_2} < \zeta_2 < \frac{E_3}{E_2} < 0 && \text{if } E_2 < 0 \text{ and } E_3 > 0, \\ &c) \quad \zeta_2 = 0 && \text{if } E_2 = 0 \text{ or } \big[E_2 < 0 \text{ and } E_3 = 0\big]. \end{aligned}$$

Since only the positive part of $\zeta_2$ plays the role in Theorem 1, we observe that the statement of the theorem is also valid if condition (i) is replaced by the condition

(ii) *the function* $f(E_2, E_3) := 0$ *(for* $E_2 = 0$*),* $f(E_2, E_3) := (E_3/E_2)_+$ *(for* $E_2 < 0$*) belongs to* $L^r(0, T;\ \mathbf{L}^s(\Omega))$ *for some* $r \in [1, \infty]$, $s \in (\frac{3}{2}, \infty]$, *satisfying* $2/r + 3/s = 2$.

**Proof of Theorem 1.** We assume that $t_0$ is in one of the intervals $(a_\gamma, b_\gamma)$ (see subsection 1.5) and $t_0 < t < b_\gamma$. We may assume without the loss of generality that $b_\gamma$ is the largest number $\leq T$ such that $\mathbf{v}$ is "smooth" on the time interval $(t_0, b_\gamma)$. Then there are two possibilities: a) the first singularity of solution $\mathbf{v}$ (after the time instant $t_0$) develops at the time $b_\gamma$, or b) no singularity of $\mathbf{v}$ develops at any time $t \in (t_0, T]$. Assume, by contradiction, that the the possibility a) takes place. In this case, $b_\gamma$ is called the epoch of irregularity.

There exists an associated pressure $p$ so that $\mathbf{v}$ and $p$ satisfy equations (1.1), (1.2) a.e. in $\Omega \times (a_\gamma, b_\gamma)$. Multiplying equation (1.1) by $P_\sigma \Delta \mathbf{v}$ and integrating in $\Omega$, we obtain

$$\int_\Omega \partial_t \mathbf{v} \cdot P_\sigma \Delta \mathbf{v}\; \mathrm{d}\mathbf{x} + \int_\Omega \mathbf{v} \cdot \nabla \mathbf{v} \cdot P_\sigma \Delta \mathbf{v}\; \mathrm{d}\mathbf{x} \;=\; \nu\, \| P_\sigma \Delta \mathbf{v} \|_2^2. \tag{2.1}$$

The first integral on the left hand side can be treated as follows:

$$\begin{aligned}
\int_\Omega \partial_t \mathbf{v} \cdot P_\sigma \Delta \mathbf{v}\; \mathrm{d}\mathbf{x} \;&=\; \int_\Omega \partial_t \mathbf{v} \cdot \Delta \mathbf{v}\; \mathrm{d}\mathbf{x} = \; 2 \int_\Omega \partial_t \mathbf{v} \cdot \operatorname{div} \mathbb{D}(\mathbf{v})\; \mathrm{d}\mathbf{x} \\
&=\; 2 \int_{\partial\Omega} \partial_t \mathbf{v} \cdot \big[ \mathbb{D}(\mathbf{v}) \cdot \mathbf{n} \big]\; \mathrm{d}S - 2 \int_\Omega \partial_t \nabla \mathbf{v} : \mathbb{D}(\mathbf{v})\; \mathrm{d}\mathbf{x} \\
&=\; -\frac{1}{\nu} \int_{\partial\Omega} \partial_t \mathbf{v} \cdot (\mathbb{K} \cdot \mathbf{v})\; \mathrm{d}S - \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\mathbb{D}(\mathbf{v})|^2\; \mathrm{d}\mathbf{x} \\
&=\; -\frac{1}{2\nu} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega} \mathbf{v} \cdot \mathbb{K} \cdot \mathbf{v}\; \mathrm{d}S - \frac{\mathrm{d}}{\mathrm{d}t}\, \| \mathbb{D}(\mathbf{v}) \|_2^2.
\end{aligned} \tag{2.2}$$

Before we estimate the second integral on the left hand side of (2.1), we recall some inequalities:

($\alpha$) the Friedrichs–type inequality $\|\mathbf{u}\|_2 \leq c_1\, \|\nabla \mathbf{u}\|_2$ (see e.g. [9, Exercise II.5.15]), satisfied for all functions $\mathbf{u} \in \mathbf{W}^{1,2}(\Omega)$ such that $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$,

($\beta$) the inequality $\|\nabla^2 \mathbf{u}\|_2 \leq c_2 \left( \|\Delta \mathbf{u}\|_2 + \|\mathbf{u}\|_2 \right)$, which holds for $\mathbf{u} \in \mathbf{W}^{2,2}(\Omega)$ that satisfy Navier's boundary conditions (1.3), (1.4) (follows from [7, Theorem 3.1]).

The Helmholtz decomposition of $\Delta \mathbf{u}$ is $\Delta \mathbf{u} = P_\sigma \Delta \mathbf{u} + \nabla \varphi$, where

$$\text{a)} \quad \Delta \varphi = 0 \quad \text{in } \Omega, \qquad \text{b)} \quad \frac{\partial \varphi}{\partial \mathbf{n}} = \Delta \mathbf{u} \cdot \mathbf{n} \quad \text{on } \partial\Omega. \tag{2.3}$$

The next lemma brings the crucial estimates of $\|\nabla \varphi\|_2$ and $\|\mathbf{v}\|_{2,2}$.

**Lemma 1.** *There exist* $c_3,\ c_4,\ c_5\, c_6 > 0$ *such that if* $\mathbf{u}$ *is a divergence–free function from* $\mathbf{W}^{2,2}(\Omega)$ *that satisfies boundary conditions (1.3), (1.4) and* $\varphi$ *is a solution of the Neumann problem (2.3) then*

$$\|\nabla \varphi\|_2 \;\leq\; c_3\, \|\nabla (\mathbb{K} \cdot \mathbf{u})\|_2 + c_4\, \|\mathbf{u}\|_{1,2}, \tag{2.4}$$

$$\|\mathbf{u}\|_{2,2} \;\leq\; c_5\, \|P_\sigma \Delta \mathbf{u}\|_2 + c_6\, \|\mathbf{u}\|_2. \tag{2.5}$$

**Proof.** The right hand side $\Delta\mathbf{u}\cdot\mathbf{n}$ in the boundary condition (2.3b) equals

$$-\mathbf{curl}^2\mathbf{u}\cdot\mathbf{n} \;=\; -\mathbf{curl}\,[(\mathbf{curl}\,\mathbf{u})_\tau]\cdot\mathbf{n}-\mathbf{curl}\,[(\mathbf{curl}\,\mathbf{u})_n]\cdot\mathbf{n} \;=\; -\mathbf{curl}\,[(\mathbf{curl}\,\mathbf{u})_\tau]\cdot\mathbf{n}.$$

(The vector field $\mathbf{curl}\,[(\mathbf{curl}\,\mathbf{u})_n]$ is tangential because $(\mathbf{curl}\,\mathbf{u})_n$ is normal. Hence the term $\mathbf{curl}\,[(\mathbf{curl}\,\mathbf{u})_n]\cdot\mathbf{n}$ equals zero on $\partial\Omega$.) The tangential component of $\mathbf{curl}\,\mathbf{u}$, i.e. $(\mathbf{curl}\,\mathbf{u})_\tau$, equals $\mathbf{n}\times\mathbf{curl}\,\mathbf{u}\times\mathbf{n}$. In order to express $\mathbf{curl}\,\mathbf{u}\times\mathbf{n}$, we apply the formula $[2\mathbb{D}(\mathbf{u})\cdot\mathbf{n}]_\tau = \mathbf{curl}\,\mathbf{u}\times\mathbf{n}-2\mathbf{u}\cdot\nabla\mathbf{n}$ (see e.g. [7]). Hence, using also the boundary condition (1.4), we obtain:

$$\begin{aligned}(\mathbf{curl}\,\mathbf{u})_\tau \;&=\; \mathbf{n}\times(\mathbf{curl}\,\mathbf{u}\times\mathbf{n}) \;=\; \mathbf{n}\times\big([2\mathbb{D}(\mathbf{u})\cdot\mathbf{n}]_\tau+2\mathbf{u}\cdot\nabla\mathbf{n}\big)\\ &=\; \mathbf{n}\times\Big(-\frac{1}{\nu}\,\mathbb{K}\cdot\mathbf{u}+2\mathbf{u}\cdot\nabla\mathbf{n}\Big).\end{aligned}$$

Thus, the boundary condition (2.3b) takes the form

$$\frac{\partial\varphi}{\partial\mathbf{n}} \;=\; -\mathbf{curl}\,\Big[\mathbf{n}\times\Big(-\frac{1}{\nu}\,\mathbb{K}\cdot\mathbf{u}+2\mathbf{u}\cdot\nabla\mathbf{n}\Big)\Big]\cdot\mathbf{n}. \tag{2.6}$$

In comparison to (2.3b), the right hand side of (2.6) contains only the first order derivatives of $\mathbf{u}$. The classical theory of solution of the Neumann problem now implies that

$$\|\nabla\varphi\|_2 \;\le\; C\,\Big\|\mathbf{curl}\,\Big[\mathbf{n}\times\Big(-\frac{1}{\nu}\,\mathbb{K}\cdot\mathbf{u}+2\mathbf{u}\cdot\nabla\mathbf{n}\Big)\Big]\cdot\mathbf{n}\Big\|_{-1/2,2;\,\partial\Omega}.$$

(We use $C$ as a generic constant.) The right hand side can be estimated by means of continuity of the linear operator, acting from the space $\mathbf{L}^2_{\mathrm{div}}(\Omega)$ (which is the space functions $\mathbf{w}\in\mathbf{L}^2(\Omega)$, whose divergence in the sense of distributions is in $L^2(\Omega)$, with the norm $\|\mathbf{w}\|_2+\|\mathrm{div}\,\mathbf{w}\|_2$) to $W^{-1/2,2}(\partial\Omega)$, that assigns to "smooth" functions $\mathbf{w}\in\mathbf{L}^2_{\mathrm{div}}(\Omega)$ the normal component $\mathbf{w}\cdot\mathbf{n}$. Thus, we obtain the estimate

$$\|\nabla\varphi\|_2 \;\le\; C\,\Big\|\mathbf{curl}\,\Big[\mathbf{n}\times\Big(-\frac{1}{\nu}\,\mathbb{K}\cdot\mathbf{u}+2\mathbf{u}\cdot\nabla\mathbf{n}\Big)\Big]\cdot\mathbf{n}\Big\|_2,$$

(where $C=C(\Omega,\nu)$), which yields (2.4). Furthermore, $\|\Delta\mathbf{u}\|_2\le\|P_\sigma\Delta\mathbf{u}\|_2+\|\nabla\varphi\|_2$. Estimating the norm $\|\nabla\varphi\|_2$ by means of (2.4), we get

$$\|\mathbf{u}\|_{2,2} \;\le\; C\,\|\Delta\mathbf{u}\|_2 \;\le\; C\,\big(\|P_\sigma\Delta\mathbf{u}\|_2+\|\nabla(\mathbb{K}\cdot\mathbf{u})\|_2+\|\mathbf{u}\|_2\big).$$

The norm of $\nabla(\mathbb{K}\cdot\mathbf{v})$ satisfies

$$\|\nabla(\mathbb{K}\cdot\mathbf{v})\|_2 \;\le\; \|\nabla\mathbb{K}\|_2\,\|\mathbf{u}\|_\infty+\|\mathbb{K}\|_6\,\|\nabla\mathbf{u}\|_3 \;\le\; C\,\|\mathbf{u}\|_{1,q} \;\le\; \epsilon\,\|\mathbf{u}\|_{2,2}+C(\epsilon)\,\|\mathbf{u}\|_2 \tag{2.7}$$

for any $q\in(3,6)$ and $\epsilon>0$ due to the imbedding $\mathbf{W}^{2,2}(\Omega)\hookrightarrow\hookrightarrow\mathbf{W}^{1,q}(\Omega)\hookrightarrow\mathbf{L}^\infty(\Omega)$. Hence

$$\|\mathbf{u}\|_{2,2} \;\le\; C\,\|P_\sigma\Delta\mathbf{u}\|_2+C\epsilon\,\|\mathbf{u}\|_{2,2}+C(\epsilon)\,\|\mathbf{u}\|_2.$$

Choosing $\epsilon$ sufficiently small, we obtain (2.5). □

The second integral in (2.1) satisfies

$$\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot P_\sigma\Delta\mathbf{v}\;\mathrm{d}\mathbf{x} \;=\; \int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{v}\;\mathrm{d}\mathbf{x}-\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\nabla\varphi\;\mathrm{d}\mathbf{x}. \tag{2.8}$$

The second term on the right hand side can be estimated by means of Lemma 1, (2.7) and (2.5):

$$\begin{aligned}\left|\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\right| &\leq \|\mathbf{v}\|_\infty\,\|\nabla\mathbf{v}\|_2\,\|\nabla\varphi\|_2 \leq C\,\|\mathbf{v}\|_\infty\,\|\nabla\mathbf{v}\|_2\left(\|\nabla(\mathbb{K}\cdot\mathbf{v})\|_2+\|\mathbf{v}\|_{1,2}\right)\\ &\leq C\,\|\mathbf{v}\|_{2,2}\,\|\nabla\mathbf{v}\|_2\left(\epsilon\,\|\mathbf{v}\|_{2,2}+C(\epsilon)\,\|\mathbf{v}\|_{1,2}\right)\\ &\leq \delta\,\|P_\sigma\Delta\mathbf{v}\|_2^2+C(\delta)\,\|\nabla\mathbf{v}\|_2^4, \qquad (2.9)\end{aligned}$$

where $\delta>0$ can be chosen arbitrarily small. The first term on the right hand side of (2.8) equals

$$\int_{\partial\Omega}(\mathbf{v}\cdot\nabla\mathbf{v})\cdot(\mathbf{n}\cdot\nabla\mathbf{v})\,\mathrm{d}S-\int_\Omega\nabla(\mathbf{v}\cdot\nabla\mathbf{v}):\nabla\mathbf{v}\,\mathrm{d}S \equiv I_1+I_2-I_3,$$

where $I_3$ denotes the last integral on the left hand side and

$$I_1 := \int_{\partial\Omega}[\mathbf{v}\cdot\nabla\mathbf{v}]_n\cdot(\mathbf{n}\cdot\nabla\mathbf{v})\,\mathrm{d}S, \qquad I_2 := \int_{\partial\Omega}[\mathbf{v}\cdot\nabla\mathbf{v}]_\tau\cdot(\mathbf{n}\cdot\nabla\mathbf{v})\,\mathrm{d}S.$$

(Subscripts $n$ and $\tau$ denote the normal and tangential components, respectively.) Applying the inequalities in ($\alpha$) and ($\beta$), Lemma 1 and the boundary conditions (1.3), (1.4), the integrals $I_1$, $I_2$ and $I_3$ can be treated as follows:

$$\begin{aligned}I_1 &= \int_{\partial\Omega}[\mathbf{v}\cdot\nabla\mathbf{v}]_n\cdot(\mathbf{n}\cdot\nabla\mathbf{v})_n\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega}[v_j\,(\partial_j v_l)\,n_l]\,[n_k\,(\partial_k v_m)\,n_m]\,\mathrm{d}S\\ &= \int_{\partial\Omega}[v_j\,\partial_j(v_l\,n_l)-v_j\,v_l\,(\partial_j n_l)]\,[n_k\,(\partial_k v_m)\,n_m]\,\mathrm{d}S\\ &= -\int_{\partial\Omega}[v_j\,v_l\,(\partial_j n_l)]\,[n_k\,(\partial_k v_m)\,n_m]\,\mathrm{d}S = -\int_\Omega\partial_m\big\{[v_j\,v_l\,(\partial_j n_l)]\,[n_k\,(\partial_k v_m)]\big\}\,\mathrm{d}\mathbf{x}\\ &\leq C\,\|\mathbf{v}\|_\infty\,\|\nabla\mathbf{v}\|_2^2 \leq C\,\|\mathbf{v}\|_{2,2}\,\|\nabla\mathbf{v}\|_2^2 \leq C\left(\|P_\sigma\Delta\mathbf{v}\|_2+\|\mathbf{v}\|_2\right)\|\nabla\mathbf{v}\|_2^2\\ &\leq \delta\,\|P_\sigma\Delta\mathbf{v}\|_2^2+C(\delta)\,\|\nabla\mathbf{v}\|_2^4+C, \qquad (2.10)\\ I_2 &= \int_{\partial\Omega}(\mathbf{v}\cdot\nabla\mathbf{v})_\tau\cdot(\mathbf{n}\cdot\nabla\mathbf{v})\,\mathrm{d}S\\ &= \int_{\partial\Omega}(\mathbf{v}\cdot\nabla\mathbf{v})_\tau\cdot\big(\mathbf{n}\cdot[\nabla\mathbf{v}+(\nabla\mathbf{v})^T]\big)\,\mathrm{d}S-\int_{\partial\Omega}(\mathbf{v}\cdot\nabla\mathbf{v})_\tau\cdot[\mathbf{n}\cdot(\nabla\mathbf{v})^T]\,\mathrm{d}S\\ &= \int_{\partial\Omega}(\mathbf{v}\cdot\nabla\mathbf{v})_\tau\cdot\big[2\mathbb{D}(\mathbf{v})\cdot\mathbf{n}\big]_\tau\,\mathrm{d}S-\int_{\partial\Omega}(\mathbf{v}\cdot\nabla\mathbf{v})_\tau\cdot\big[\nabla(\mathbf{n}\cdot\mathbf{v})-\nabla\mathbf{n}\cdot\mathbf{v}\big]\,\mathrm{d}S.\end{aligned}$$

Since $(\mathbf{v}\cdot\nabla\mathbf{v})_\tau$ is tangential and $\mathbf{n}\cdot\mathbf{v}=0$ on $\partial\Omega$, the scalar product $(\mathbf{v}\cdot\nabla\mathbf{v})_\tau\cdot\nabla(\mathbf{n}\cdot\mathbf{v})$ is equal to zero. Thus, if we also use the boundary condition (1.4), the inequalities in ($\alpha$) and ($\beta$) and Lemma 1, we get

$$\begin{aligned}|I_2| &= \left|-\frac{1}{\nu}\int_{\partial\Omega}(\mathbf{v}\cdot\nabla\mathbf{v})_\tau\cdot(\mathbb{K}\cdot\mathbf{v})\,\mathrm{d}S+\int_{\partial\Omega}(\mathbf{v}\cdot\nabla\mathbf{v})_\tau\,(\nabla\mathbf{n}\cdot\mathbf{v})\,\mathrm{d}S\right|\\ &\leq C\int_{\partial\Omega}|\mathbf{v}|^2\,|\nabla\mathbf{v}|\left(|\mathbb{K}|+1\right)\mathrm{d}S \leq C\,\|\mathbf{v}\|_{4;\,\partial\Omega}^2\,\|\nabla\mathbf{v}\|_{4;\,\partial\Omega}\left(\|\mathbb{K}\|_{4;\,\partial\Omega}+1\right)\\ &\leq C\,\|\mathbf{v}\|_{1,2}^2\,\|\mathbf{v}\|_{2,2}\left(\|\mathbb{K}\|_{1,2}+1\right) \leq C\,\|\nabla\mathbf{v}\|_2^2\left(\|P_\sigma\Delta\mathbf{v}\|_2+\|\mathbf{v}\|_2\right)\\ &\leq \delta\,\|P_\sigma\Delta\mathbf{v}\|_2^2+C(\delta)\,\|\nabla\mathbf{v}\|_2^4+C, \qquad (2.11)\end{aligned}$$

$$I_3 \;=\; \int_\Omega \big[(\partial_k v_j)\,(\partial_j v_i)\,\partial_k v_i) + v_j\,(\partial^2_{jk} v_i)\,(\partial_k v_i)\big]\;\mathrm{d}\mathbf{x} \;=\; \int_\Omega (\partial_k v_j)\,(\partial_j v_i)\,\partial_k v_i)\;\mathrm{d}\mathbf{x}.$$

If we denote (for $i,j=1,2,3$) $d_{ij} := \frac{1}{2}[(\partial_i v_j)+(\partial_j v_i)]$ (the entries of tensor $\mathbb{D}$) and $s_{ij} := \frac{1}{2}[\partial_i v_j) - (\partial_j v_i)]$ (the entries of the skew–symmetric part of $\nabla\mathbf{v}$), we obtain

$$\begin{aligned} I_3 \;&=\; \int_\Omega (d_{kj}+s_{kj})(d_{ji}+s_{ji})(d_{ki}+s_{ki})\;\mathrm{d}S \\ &=\; \int_\Omega [d_{kj}\,d_{ji}\,d_{ki} + d_{kj}\,s_{ji}\,s_{ki} + d_{ji}\,s_{kj}\,s_{ki} + d_{ki}\,s_{kj}\,s_{ji}]\;\mathrm{d}\mathbf{x}. \end{aligned}$$

As $s_{ji}=-s_{ij}$, we have $d_{kj}\,s_{ji}\,s_{ki} + d_{ki}\,s_{kj}\,s_{ji} = d_{kj}\,s_{ji}\,s_{ki} + d_{kj}\,s_{ki}\,s_{ij} = 0$. Hence

$$I_3 \;=\; \int_\Omega [d_{kj}\,d_{ji}\,d_{ik} + d_{ij}\,s_{ki}\,s_{kj}]\;\mathrm{d}\mathbf{x} \;=\; \int_\Omega d_{kj}\,d_{ji}\,d_{ik}\;\mathrm{d}\mathbf{x} - \frac{1}{4}\int_\Omega d_{ij}\omega_i\omega_j\;\mathrm{d}\mathbf{x}, \tag{2.12}$$

where $\omega_i$ and $\omega_j$ are the components of $\boldsymbol{\omega} := \mathbf{curl}\,\mathbf{v}$. The estimates (2.10), (2.11) and the identity (2.12) yield

$$\begin{aligned} &\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{v}\;\mathrm{d}\mathbf{x} \\ &\qquad\le\; 2\delta\,\|P_\sigma\Delta\mathbf{v}\|_2^2 + C(\delta)\,\|\nabla\mathbf{v}\|_2^4 + C - \int_\Omega d_{kj}\,d_{ji}\,d_{ik}\;\mathrm{d}\mathbf{x} + \frac{1}{4}\int_\Omega d_{ij}\omega_i\omega_j\;\mathrm{d}\mathbf{x}. \end{aligned} \tag{2.13}$$

The integral on the left hand side of (2.13) can also be treated in another way:

$$\begin{aligned} \int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{v}\;\mathrm{d}\mathbf{x} \;&=\; -\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\mathbf{curl}^2\mathbf{v}\;\mathrm{d}\mathbf{x} \\ &=\; -\int_{\partial\Omega} \mathbf{v}\cdot\nabla\mathbf{v}\cdot(\mathbf{n}\times\mathbf{curl}\,\mathbf{v})\;\mathrm{d}S - \int_\Omega \mathbf{curl}\,(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{curl}\,\mathbf{v}\;\mathrm{d}\mathbf{x}. \end{aligned} \tag{2.14}$$

The integrals on the right hand side can be estimated or modified as follows:

$$\begin{aligned} \left|\int_{\partial\Omega} \mathbf{v}\cdot\nabla\mathbf{v}\cdot(\mathbf{n}\times\mathbf{curl}\,\mathbf{v})\;\mathrm{d}S\right| \;&=\; \left|\int_{\partial\Omega} (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\big([2\mathbb{D}\cdot\mathbf{n}]_\tau + 2\mathbf{v}\cdot\nabla\mathbf{n}\big)\;\mathrm{d}S\right| \\ &=\; \left|\frac{1}{\nu}\int_{\partial\Omega} (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\big(-\mathbb{K}\cdot\mathbf{v} + 2\mathbf{v}\cdot\nabla\mathbf{n}\big)\;\mathrm{d}S\right| \\ &\le\; C\int_{\partial\Omega} |\mathbf{v}|^2\,|\nabla\mathbf{v}|\,\big(|\mathbb{K}|+1\big)\;\mathrm{d}S \;\le\; C\,\|\mathbf{v}\|_{4;\,\partial\Omega}^2\,\|\nabla\mathbf{v}\|_{2;\,\partial\Omega}\,\big(\|\mathbb{K}\|_{4;\,\partial\Omega}+1\big) \\ &\le\; \delta\,\|P_\sigma\Delta\mathbf{v}\|_2^2 + C(\delta)\,\|\nabla\mathbf{v}\|_2^4 + C \end{aligned} \tag{2.15}$$

(by analogy with (2.11),

$$\begin{aligned} \int_\Omega \mathbf{curl}\,(\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{curl}\,\mathbf{v}\;\mathrm{d}\mathbf{x} \;&=\; \int_\Omega [\mathbf{v}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{v}]\cdot\boldsymbol{\omega}\;\mathrm{d}\mathbf{x} \;=\; -\int_\Omega \boldsymbol{\omega}\cdot\nabla\mathbf{v}\cdot\boldsymbol{\omega}\;\mathrm{d}\mathbf{x} \\ &=\; -\int_\Omega \boldsymbol{\omega}\cdot\mathbb{D}(\mathbf{v})\cdot\boldsymbol{\omega}\;\mathrm{d}\mathbf{x} \;=\; -\int_\Omega d_{ij}\omega_i\omega_j\;\mathrm{d}\mathbf{x}. \end{aligned} \tag{2.16}$$

Multiplying (2.14)–(2.16) by $\frac{1}{4}$, we get

$$\frac{1}{4}\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{v}\;\mathrm{d}\mathbf{x} \;\le\; \frac{\delta}{4}\,\|P_\sigma\Delta\mathbf{v}\|_2^2 + C(\delta)\,\|\nabla\mathbf{v}\|_2^4 + C - \frac{1}{4}\int_\Omega d_{ij}\omega_i\omega_j\;\mathrm{d}\mathbf{x}. \tag{2.17}$$

Summing (2.13) and (2.17), we obtain

$$\frac{5}{4}\int_\Omega \mathbf{v}\cdot\nabla\mathbf{v}\cdot\Delta\mathbf{v}\,\mathrm{d}\mathbf{x} \;\leq\; \frac{9\delta}{4}\,\|P_\sigma\Delta\mathbf{v}\|_2^2 + C(\delta)\,\|\nabla\mathbf{v}\|_2^4 + C - \int_\Omega d_{kj}\,d_{ji}\,d_{ik}\,\mathrm{d}\mathbf{x}.$$

Dividing this inequality by $\frac{5}{4}$, choosing $\delta = \frac{5}{18}\nu$, substituting to (2.1) and expressing the first integral in (2.1) by means of (2.2), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbb{D}(\mathbf{v})\|_2^2 + \frac{1}{2\nu}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\mathbf{v}\cdot\mathbb{K}\cdot\mathbf{v}\,\mathrm{d}S + \frac{\nu}{2}\,\|P_\sigma\Delta\|_2^2 \\ \leq\; -\frac{4}{5}\int_\Omega d_{kj}\,d_{ji}\,d_{ik}\,\mathrm{d}\mathbf{x} + c_7\,\|\nabla\mathbf{v}\|_2^2\,\|\mathbb{D}(\mathbf{v})\|_2^2 + c_8.\end{aligned} \tag{2.18}$$

The product $d_{jk}\,d_{ki}\,d_{ij}$ equals the trace of the tensor $\mathbb{D}(\mathbf{v})^3$. It is invariant with respect to rotation of the coordinate system. Hence it can be represented in the system in which $\mathbb{D}(\mathbf{v})$ has the diagonal representation $\mathbb{D} = (d_{ij})$ with $d_{ij} = 0$ for $i \neq j$ and $d_{11} = \zeta_1$, $d_{22} = \zeta_2$, $d_{33} = \zeta_3$, where $\zeta_1,\ \zeta_3\,\zeta_3$ are the eigenvalues of tensor $\mathbb{D}(\mathbf{v})$. The eigenvalues are real because $\mathbb{D}(\mathbf{v})$ is symmetric and their sum is zero because the trace if $\mathbb{D}(\mathbf{v})$ is equal to zero. Then $\mathrm{Tr}\,\mathbb{D}(\mathbf{v})^3 = d_{jk}\,d_{ki}\,d_{ij} = \zeta_1^3 + \zeta_2^3 + \zeta_3^3 = 3\zeta_1\zeta_2\zeta_3$. We may assume that the eigenvalues are ordered so that $\zeta_1 \leq \zeta_2\zeta_3$, which implies that $\zeta_1 \leq 0$ and $\zeta_3 \geq 0$. Then inequality (2.18) takes the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\mathbb{D}(\mathbf{v})\|_2^2 + \frac{1}{2\nu}\,\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\mathbf{v}\cdot\mathbb{K}\cdot\mathbf{v}\,\mathrm{d}S + \frac{\nu}{2}\,\|P_\sigma\Delta\|_2^2 \\ \leq\; -\frac{12}{5}\int_\Omega(-\zeta_1)\,(\zeta_2)_+\,\zeta_3\,\mathrm{d}\mathbf{x} + c_7\,\|\nabla\mathbf{v}\|_2^2\,\|\mathbb{D}(\mathbf{v})\|_2^2 + c_8.\end{aligned} \tag{2.19}$$

Integrating this inequality on the time interval $(t_0, t_1)$, where $t_0 < t_1 \leq b_\gamma$, we deduce that

$$\begin{aligned}|\!|\!|\mathbb{D}(\mathbf{v})|\!|\!|^2_{\infty,2;\,(t_0,t_1)} + \frac{\nu}{2}\,|\!|\!|P_\sigma\Delta\mathbf{v}|\!|\!|^2_{2,2;\,(t_0,t_1)} \\ \leq\; c_9\,\|\mathbb{D}(\mathbf{v}(t_0))\|_2^2 + c_{10}\int_{t_0}^{t_1}\int_\Omega(-\zeta_1)\,(\zeta_2)_+\,\zeta_3\,\mathrm{d}\mathbf{x}\,\mathrm{d}\vartheta + c_{11},\end{aligned} \tag{2.20}$$

where constants $c_9,\ c_{10},\ c_{11}$ depend on $\nu$, $\Omega$, $c_7$, $c_8$ and also on the norm $|\!|\!|\nabla\mathbf{v}|\!|\!|_{2,2;\,(0,T)}$. Let us further estimate the integral of $(-\zeta_1)\,(\zeta_2)_+\,\zeta_3$ on the right hand side of (2.18). Assume e.g. that $(\zeta_2)_+ \in L^r(0,T;\,L^s(\Omega))$, where $2/r + 3/s \leq 1$. Since $|\zeta_i| \leq C\,|\nabla\mathbf{v}|$ $(i = 1,2,3)$, we have

$$\begin{aligned}\int_{t_0}^{t_1}\int_\Omega(-\zeta_1)\,(\zeta_2)_+\,\zeta_3\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;&\leq\; |\!|\!|(\zeta_2)_+|\!|\!|_{r,s;\,(t_0,t_1)}\;|\!|\!|\zeta_1\zeta_3|\!|\!|_{\frac{r}{r-1},\frac{s}{s-1};\,(t_0,t_1)} \\ &\leq\; c_{12}\,|\!|\!|(\zeta_2)_+|\!|\!|_{r,s;\,(t_0,t_1)}\;|\!|\!|\nabla\mathbf{v}|\!|\!|^2_{\frac{2r}{r-1},\frac{2s}{s-1};\,(t_0,t_1)}.\end{aligned}$$

Estimating the norm of $\nabla\mathbf{v}$ by means of the inequality

$$|\!|\!|g|\!|\!|_{\alpha,\beta;\,(t_0,t_1)} \;\leq\; |\!|\!|g|\!|\!|_{2,2;\,(t_0,t_1)}^{\frac{2}{\alpha}+\frac{3}{\beta}-\frac{3}{2}}\left(|\!|\!|g|\!|\!|_{\infty,2;\,(t_0,t_1)} + |\!|\!|g|\!|\!|_{2,6;\,(t_0,t_1)}\right)^{\frac{5}{2}-(\frac{2}{\alpha}+\frac{3}{\beta})},$$

which can be proven by means of Hölder's inequality and which is valid for $2 \leq \alpha \leq \infty$, $2 \leq \beta \leq 6$ and $\frac{3}{2} \leq 2/\alpha + 3/\beta \leq \frac{5}{2}$, with $\alpha = 2r/(r-1)$ and $\beta = 2s/(s-1)$, we obtain:

$$\int_{t_0}^{t_1}\int_\Omega(-\zeta_1)\,(\zeta_2)_+\,\zeta_3\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;\leq\; |\!|\!|(\zeta_2)_+|\!|\!|_{r,s;\,(t_0,t_1)}\left(|\!|\!|\nabla\mathbf{v}|\!|\!|_{\infty,2;\,(t_0,t_1)} + |\!|\!|\nabla\mathbf{v}|\!|\!|_{2,6;\,(t_0,t_1)}\right)^{\frac{2}{r}+\frac{3}{s}}$$

$$\leq\ c_{13}\, \| (\zeta_2)_+ \|_{r,s;\, (t_0,t_1)} \, \Big( \| \mathbb{D}(\mathbf{v}) \|_{\infty,2;\, (t_0,t_1)}^2 + \frac{\nu}{2}\, \| P_\sigma \Delta \mathbf{v} \|_{2,2;\, (t_0,t_1)}^2 + c_{14} \Big).$$

(The norm $\|\nabla\mathbf{v}\|_2$ inside $\|\nabla\mathbf{v}\|_{\infty,2;\,(t_0,t_1)}$ has been estimated by Korn's inequality and the norm $\|\nabla\mathbf{v}\|_6$ inside $\|\nabla\mathbf{v}\|_{2,6;\,(t_0,t_1)}$ is estimated by the norm $\|\mathbf{v}\|_{2,2}$, which is less than or equal to $c_5\, \|P_\sigma\Delta\mathbf{v}\|_2 + c_6\, \|\mathbf{v}\|_2$ due to Lemma 1.) Using this inequality in (2.20), we get

$$\begin{aligned} \| \mathbb{D}(\mathbf{v}) \|_{\infty,2;\, (t_0,t_1)}^2 + \frac{\nu}{2}\, \| P_\sigma \Delta \mathbf{v} \|_{2,2;\, (t_0,t_1)}^2 \ &\leq\ c_9\, \| \mathbb{D}(\mathbf{v}(\,.\,,t_0)) \|_2^2 \\ + c_{10}\, c_{13}\, \| (\zeta_2)_+ \|_{r,s;\, (t_0,t_1)} \, &\Big( \| \nabla \mathbf{v} \|_{\infty,2;\, (t_0,t_1)}^2 + \frac{\nu}{2}\, \| P_\sigma \Delta \mathbf{v} \|_{2,2;\, (t_0,t_1)}^2 + c_{14} \Big). \end{aligned}$$

Assume that $t_1 = b_\gamma$ and $t_1 - t_0 < \xi$, where $\xi$ is so small that $c_{10}\, c_{13}\, \| (\zeta_2)_+ \|_{r,s;\, (t',t'')} < \frac{1}{2}$ for any $t'$, $t'' \in (0,T)$ such that $0 \leq t' < t'' \leq T$, $t'' - t' \leq \xi$. Then

$$\| \mathbb{D}(\mathbf{v}) \|_{\infty,2;\, (t_0,b_\gamma)}^2 + \frac{\nu}{2}\, \| P_\sigma \Delta \mathbf{v} \|_{2,2;\, (t_0,b_\gamma)}^2 \ \leq\ 2c_9\, \| \mathbb{D}(\mathbf{v}(\,.\,,t_0)) \|_2^2 + c_{10} c_{13} c_{14}.$$

From this, we observe that $b_\gamma$ cannot be an epoch of irregularity of the weak solution $\mathbf{v}$. The proof of Theorem 1 is completed. □

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# References

[1] Beirão da Veiga, H. (1995). Concerning the regularity problem for the solutions of the Navier–Stokes equations. *C. R. Acad. Sci. Paris* 321(Série I):405-408.

[2] Beirão da Veiga, H., Berselli, L. (2002). On the regularizing effect of the vorticity direction in incompressible viscous flows. *Diff. Int. Equations* 15:345356.

[3] Beirão da Veiga, H. (2012). Direction of vorticity and regularity up to the boundary: On the Lipschitz-continuous case. *J. Math. Fluid Mech.* 15:5563.

[4] Berselli, L., Galdi, G. P. (2002). Regularity criterions involving the pressure for the weak solutions to the Navier–Stokes equations. *Proc. Am. Math. Soc.* 130(12):35853595 (electronic).

[5] Caffarelli, L., Kohn, R. V., Nirenberg, L. (1982). Partial regularity of suitable weak solutions of the Navier–Stokes equations. *Comm. Pure Appl. Math.* 35:771831.

[6] Chae D., Choe H. J. (1999). Regularity of solutions to the Navier–Stokes equation. *Electronic J. of Diff. Equations* 5:1–7.

[7] Chen, G. Q., Qian, Z. (2010). A study of the NavierStokes equations with the kinematic and Navier boundary conditions. *Indiana Univ. Math. J.* 59(2):721-760.

[8] Choe, H. J. (1998). Boundary regularity of weak solutions of the Navier-Stokes equations. *J. of Diff. Equations* 149:211-247.

[9] Galdi, G. P. (2011). *An Introduction to the Mathematical Theory of the Navier–Stokes Equations.* 2nd edition, Springer–Verlag, New York–Berlin–Heidelberg.

[10] Galdi, G. P. (2000) An Introduction to the Navier–Stokes initial–boundary value problem. In *Fundamental Directions in Mathematical Fluid Mechanics*, editors G. P. Galdi, J. Heywood, R. Rannacher, series “Advances in Mathematical Fluid Mechanics”, Birkhauser–Verlag, Basel, 1–98.

[11] Gustafson, S., Kang, K., Tsai, T.-P. (2006). Regularity criteria for suitable weak solutions of the Navier–Stokes equations near the boundary. *J. Diff. Equations* 226(2):594618.

[12] Kučera, P., Neustupa, J. (2017). On robustness of a strong solution to the Navier–Stokes equations with Navier's boundary conditions in the $L^3$–norm. *Nonlinearity* 30:15641583.

[13] Nečas, J., Neustupa, J. (2002). New conditions for local regularity of a suitable weak solution to the Navier–Stokes equations. *J. Math. Fluid Mech.* 4:237-256.

[14] Neustupa, J., Penel, P. (2001) Anisotropic and geometric criteria for interior regularity of weak solutions to the 3D Navier–Stokes equations. In *Mathematical Fluid Mechanics, Recent Results and Open Questions,* editors J. Neustupa and P. Penel, Birkhauser Verlag, Basel–Boston–Berlin, 237-268.

[15] Neustupa, J., Penel, P. (2004). Regularity of weak solutions to the Navier–Stokes equations in dependence on eigenvalues and eigenvectors of the rate of deformation tensor. In *Progress in Nonlinear Differential Equations and their Applications,* Vol. 61, editors J. F. Rodriguez, G. Seregin and J. M. Urbano, Birkhauser, Basel, 197-213.

[16] Neustupa, J., Penel, P. (2016). A weak solution to the Navier–Stokes system with Navier's boundary condition in a time–varying domain. In *Recent Developments of Mathematical Fluid Mechanics,* Series: *Advances in Mathematical Fluid Mechanics,* Birhaeser–Verlag, editors H. Amann, Y. Giga, H. Kozono, H. Okamoto, M. Yamazaki, Springer, Basel, 375-400.

[17] Penel, P., Pokorný, M. (2004). Some new regularity criteria for the Navier-Stokes equations containing gradient of the velocity. *Appl. Math* 49(5):483-493.

[18] Pokorný, M., Zhou, Y. (2010). On the regularity of the solutions of the Navier-Stokes equations via one velocity component. *Nonlinearity* 23(5):1097-1107.

[19] Seregin, G. A. (2002). Local regularity of suitable weak solutions to the Navier–Stokes equations near the boundary. *J. Math. Fluid Mech.* 4:1-29.

[20] Seregin, G. A., Šverák, V. (2002). Navier–Stokes equations with lower bounds on the pressure. *Arch. Rat. Mech. Anal.* 163:65-86.

[21] Seregin, G. A., Shilkin, T. N., Solonnikov V. A. (2006). Boundary partial regularity for the Navier–Stokes equations. *J. Math. Sci. (N. Y.)* 132(3):339358.

[22] Serrin, J. (1962). On the interior regularity of weak solutions of the Navier–Stokes equations. *Arch. Rat. Mech. Anal.* 9:187-195.

[23] Skalák, Z. (2005). Regularity of weak solutions of the Navier–Stokes equations near the smooth boundary. *El. J. Diff. Equations* 45:11 pp.

[24] Wolf, J. (2010). On the boundary regularity of suitable weak solutions to the Navier–Stokes equations. *Ann. Univ. Ferrara Ser. VII Sci. Mat.* 56(1):97139.