On a certain generalization of first-countable spaces

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Motivation - products of Baire spaces

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- P. E. Cohen (1976); W. G. Fleissner, K. Kunen (1978); J. van Mill, R. Pol (1986) $[\mathrm{ZFC}]$: A Baire space X such that X^2 is not Baire.

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☺ If X is separable then $\{X\}$ is a rich family in X ☺

W-spaces

Let X be a topological space, and let $x \in X$ be fixed. Consider the following game $G(x)$:

Player I

Player II

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Example (G. Gruenhage, 1976)

- A W-space which is not first-countable: $\Sigma_{\alpha\in A}\{0,1\} = \{(x_\alpha)_{\alpha\in A} \colon x_\alpha \neq 0 \text{ for at most countably many } \alpha\}$, where A is uncountable

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~~W-spaces~~ $\widetilde{W}$-spaces

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Player I	U_1		U_2		U_3		$\dots$
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- U_n are open sets ~~containing x~~ which are nonempty
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The point x is an accumulation point of this sequence.

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- X is a Baire space which possesses a rich family of Baire spaces.

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Proof.

A certain Σ-product of uncountably many copies of the space $\beta\mathbb{N}$ works. □

Further applications of $\widetilde{W}$-spaces

Theorem

Let $f : X \times Y \to Z$ be separately continuous. Suppose that X is a Baire space, Z is regular, and $y_0 \in Y$ is a $\widetilde{W}$-point. Then f is quasi-continuous at each point of $X \times \{y_0\}$.

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(separately continuous $\equiv$ separately continuous in each coordinate,

f is quasi-continuous at p $\equiv$ for every open sets $U \ni p$ and $W \ni f(p)$ there is an open set $\emptyset \neq V \subseteq U$ such that $f(V) \subseteq W$)

Further applications of $\widetilde{W}$-spaces

Corollary

Let G be a semitopological group. Suppose that G is a regular Baire $\widetilde{W}$-space and a Δ-Baire space. Then G is a topological group.

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Corollary

Let $f : X \times Y \to Z$ be separately continuous. Suppose that X is a Baire space, Y is a $\widetilde{W}$-space which possesses a rich family of Baire spaces, and Z is a regular space that is fragmented by some metric whose topology contains the topology of Z. Then f is continuous at the points of a dense G_δ-subset of $X \times Y$.

The End