On a certain generalization of first-countable spaces

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19 June 2017

Motivation - products of Baire spaces

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A topological space X is a *Baire space* if every intersection of countably many dense open sets is dense.

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If X is separable then $\{X\}$ is a rich family in X \bigcirc

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Player I

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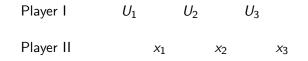
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Definition (G. Gruenhage, 1976)

We say that $x \in X$ is a *W*-point in X if I has a winning strategy in G(x).

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(Fréchet space \equiv if $x \in \overline{A}$ then there is a sequence (x_n) in A s.t. $x_n \to x$)

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Example (G. Gruenhage, 1976)

• A *W*-space which is not first-countable: $\Sigma_{\alpha \in A} \{0, 1\} = \{(x_{\alpha})_{\alpha \in A} : x_{\alpha} \neq 0 \text{ for at most countably many } \alpha\},$ where *A* is uncountable

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- A Fréchet space which is not a W-space

Theorem (P. Lin, W. B. Moors, 2008)

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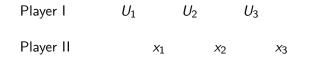
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W-spaces \widetilde{W} -spaces

Let X be a topological space, and let $x \in X$ be fixed. Consider the following game $G(x) \ \widetilde{G}(x)$:



- U_n are open sets containing x which are nonempty
- $x_n \in U_n$

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Definition

We say that $x \in X$ is a W-point \widetilde{W} -point in X if I has a winning strategy in G(x) $\widetilde{G}(x)$. We say that X is a W-space \widetilde{W} -space if each $x \in X$ is a W-point \widetilde{W} -point in X.

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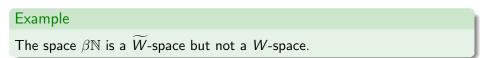
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The space $\beta \mathbb{N}$ is a *W*-space but not a *W*-space.

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The space $\beta \mathbb{N}$ is not a Fréchet space \Rightarrow it is not a *W*-space.

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The point x is an accumulation point of this sequence.

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Proof.

A certain Σ -product of uncountably many copies of the space $\beta \mathbb{N}$ works.

Theorem

Let $f : X \times Y \to Z$ be separately continuous. Suppose that X is a Baire space, Z is regular, and $y_0 \in Y$ is a \widetilde{W} -point. Then f is quasi-continuous at each point of $X \times \{y_0\}$.

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(separately continuous \equiv separately continuous in each coordinate,

f is quasi-continuous at $p \equiv$ for every open sets $U \ni p$ and $W \ni f(p)$ there is an open set $\emptyset \neq V \subseteq U$ such that $f(V) \subseteq W$

Corollary

Let G be a semitopological group. Suppose that G is a regular Baire \widetilde{W} -space and a Δ -Baire space. Then G is a topological group.

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Let $f : X \times Y \to Z$ be separately continuous. Suppose that X is a Baire space, Y is a \widetilde{W} -space which possesses a rich family of Baire spaces, and Z is a regular space that is fragmented by some metric whose topology contains the topology of Z. Then f is continuous at the points of a dense G_{δ} -subset of $X \times Y$.

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