Higher order lower bounds on eigenvalues of symmetric elliptic operators

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Supported by Neuron Fund for Support of Science, project no. 24/2016

Equadiff 2017, Bratislava, July 26

Lower bounds on eigenvalues

Laplace eigenvalue problem

$$\begin{aligned} -\Delta u_i &= \lambda_i u_i && \text{in } \Omega \\ u_i &= 0 && \text{on } \partial\Omega \end{aligned}$$

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$\lambda_i > 0, u_i \in V : \quad (\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V$

Notation:

$V = H_0^1(\Omega)$

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Finite element method

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Upper bound:

$$\lambda_i \leq \Lambda_{h,i}$$

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Can we compute lower bound?

$$\ell_i \le \lambda_i \le \Lambda_{h,i} \quad \Rightarrow \quad |\Lambda_{h,i} - \lambda_i| \le \Lambda_{h,i} - \ell_i$$

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Lower bounds on eigenvalues of differential operators

Old problem:
Temple 1928, Weinstein 1937, Kato 1949, Lehmann 1949, Goerisch 1985, . . .

Many results: M.G. Armentano, G. Barrenechea, H. Behnke, C. Carstensen, R.G. Duran, D. Galistl, J. Gedicke, L. Grubišić, Jun Hu, J.R. Kuttler, Y.A. Kuznetsov, Fubiao Lin, Qun Lin, Xuefeng Liu, M. Plum, S.I. Repin, V.G. Sigillito, M. Vohralík, Hehu Xie, Yidu Yang, Zhimin Zhang, . . . *many others*

FEM approaches

Nonconforming:

$$\frac{\lambda_{h,i}^{\mathrm{CR}}}{1 + 0.1893^2 h^2 \lambda_{h,i}^{\mathrm{CR}}} \leq \lambda_i \quad \text{(Crouzeix–Raviart, triangles)}$$

- no a priori information on spectrum needed
- first order only

[Carstensen, Gallistl 2013], [Carstensen, Gedicke 2014]
[Xuefeng Liu 2015]

Conforming:

- a priori information on spectrum needed
- higher order versions

[Behnke, Mertins, Plum, Wieners 2000]
[Cancès, Dusson, Maday, Stamm, Vohralík 2017], [Vejchodský, Šebestová 2017]

Flux reconstruction in $\mathbf{H}(\mathrm{div}, \Omega)$

$$\boldsymbol{\sigma}_{h,i} \approx \nabla u_i, \quad i = 1, 2, \ldots, n$$

(a) Global problem:

$\boldsymbol{\sigma}_{h,i} \in \mathbf{W}_h$ minimizes $\|\nabla u_{h,i} - \boldsymbol{\sigma}_{h,i}\|^2_{L^2(\Omega)}$

under constraint: $-\operatorname{div} \boldsymbol{\sigma}_{h,i} = \Lambda_{h,i} u_{h,i}$

Spaces:

$\mathbf{W}_h = \{\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega) : \boldsymbol{\sigma}_h|_K \in \mathbf{RT}_p(K)\ \forall K \in \mathcal{T}_h\}$

Flux reconstruction in $\mathbf{H}(\mathrm{div},\Omega)$

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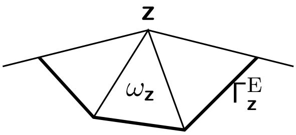

(b) Local problems:

$\boldsymbol{\sigma}_{h,i} = \sum_{\mathbf{z}\in\mathcal{N}_h} \boldsymbol{\sigma}_{\mathbf{z},i}$

$\boldsymbol{\sigma}_{\mathbf{z},i} \in \mathbf{W}_{\mathbf{z}}$ minimize $\|\psi_{\mathbf{z}}\nabla u_{h,i} - \boldsymbol{\sigma}_{\mathbf{z},i}\|^2_{L^2(\omega_{\mathbf{z}})}$

under constraint: $-\mathrm{div}\,\boldsymbol{\sigma}_{\mathbf{z},i} = \Lambda_{h,i}\Pi(\psi_{\mathbf{z}} u_{h,i}) - \nabla\psi_{\mathbf{z}} \cdot \nabla u_{h,i}$

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$\mathbf{W}_{\mathbf{z}} = \{\boldsymbol{\sigma}_{\mathbf{z}} \in \mathbf{H}(\mathrm{div},\omega_{\mathbf{z}}) : \boldsymbol{\sigma}_{\mathbf{z}}|_K \in \mathbf{RT}_p(K)\ \forall K \in \mathcal{T}_{\mathbf{z}}$

and $\boldsymbol{\sigma}_{\mathbf{z}} \cdot \mathbf{n}_{\mathbf{z}} = 0$ on $\Gamma^{\mathrm{E}}_{\mathbf{z}}\}$

Weinstein and Kato bounds

Set $\eta_i = \|\nabla u_{h,i} - \boldsymbol{\sigma}_{h,i}\|_{L^2(\Omega)} \qquad i = 1, 2, \ldots, n$

Weinstein bound: $\ell_i^{\mathrm{W}} = \frac{1}{4}\left(-\eta_i + \sqrt{\eta_i^2 + 4\Lambda_{h,i}}\right)^2$

Kato bound: $\ell_i^{\mathrm{K}} = \Lambda_{h,i}\left(1 + \nu\Lambda_{h,i}\sum_{j=i}^{n}\frac{\eta_j^2}{\Lambda_{h,j}^2(\nu - \Lambda_{h,j})}\right)^{-1}$

where $\Lambda_{h,n} < \nu$

Theorem 1.
If $\sqrt{\lambda_{i-1}\lambda_i} \leq \Lambda_{h,i} \leq \sqrt{\lambda_i\lambda_{i+1}}$ then $\ell_i^{\mathrm{W}} \leq \lambda_i$.

Theorem 2.
If $\nu \leq \lambda_{n+1}$ then $\ell_i^{\mathrm{K}} \leq \lambda_i$ for all $i = 1, 2, \ldots, n$.

[Vejchodský, Šebestová 2017]

Lehmann–Goerisch method

Let $\gamma > 0$, $\nu \le \lambda_{n+1}$, and $\tilde{\boldsymbol{\sigma}}_{h,i} = (\Lambda_{h,i} + \gamma)^{-1} \boldsymbol{\sigma}_{h,i}$

For $m = n, n-1, \dots, 2, 1$ do

- $\rho = \nu + \gamma$
- $\mathbf{M}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - \rho)(u_{h,i}, u_{h,j})$
- $\mathbf{N}_{ij} = (\nabla u_{h,i}, \nabla u_{h,j}) + (\gamma - 2\rho)(u_{h,i}, u_{h,j}) + \rho^2 (\tilde{\boldsymbol{\sigma}}_{h,i}, \tilde{\boldsymbol{\sigma}}_{h,j})$
  $+ (\rho^2/\gamma)(u_{h,i} + \operatorname{div} \tilde{\boldsymbol{\sigma}}_{h,i}, u_{h,j} + \operatorname{div} \tilde{\boldsymbol{\sigma}}_{h,j})$
- $\mu_1 \le \dots \le \mu_m: \quad \mathbf{M}\mathbf{y}_i = \mu_i \mathbf{N}\mathbf{y}_i, \quad i = 1, 2, \dots, m$
- If $\mathbf{N}$ is s.p.d. and if $\mu_{m+1-j} < 0$ then
  $$\ell^*_{j,m} = \rho - \gamma - \rho / (1 - \mu_{m+1-j}) \le \lambda_j, \quad j = 1, 2, \dots, m$$
- $\ell^{\mathrm{LG}}_m = \max\{\ell^*_{m,i}, \ i = m, m+1, \dots, n\} \le \lambda_m$
- $\nu = \ell^{\mathrm{LG}}_m$

end for

Theorem

If $\nu \le \lambda_{n+1}$ then $\ell^{\mathrm{LG}}_i \le \lambda_i$ for all $i = 1, 2, \dots, n$.

[Behnke, Mertins, Plum, Wieners 2000]

Adaptive mesh refinement

Residual

$$w_i \in V : \quad (\nabla w_i, \nabla v) = (\nabla u_{h,i}, \nabla v) - \Lambda_{h,i}(u_{h,i}, v) \quad \forall v \in V$$

Theorem

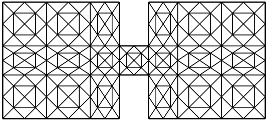
$$\|\nabla w_i\|_{L^2(\Omega)} \leq \eta_i, \quad \text{where } \eta_i = \|\nabla u_{h,i} - \boldsymbol{\sigma}_{h,i}\|_{L^2(\Omega)}.$$

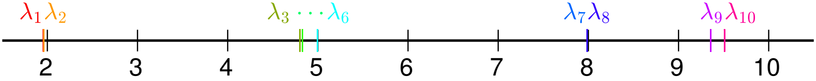
Local error indicators for mesh refinement

$$\eta_{i,K} = \|\nabla u_{h,i} - \boldsymbol{\sigma}_{h,i}\|_{L^2(K)} \quad \forall K \in \mathcal{T}_h$$

Example: Dumbbell shaped domain

$$-\Delta u_i = \lambda_i u_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega$$





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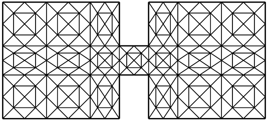

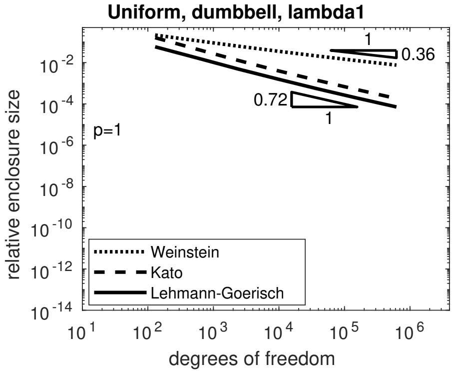

- relative enclosure size: $(\Lambda_{h,i} - \ell_i)/\ell_i$
- $\gamma = 10^{-6}, \quad \nu := \ell_{11}^{\mathrm{W}} \leq \lambda_{11} \approx 10.0017$

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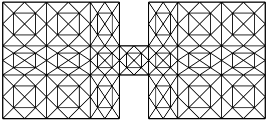

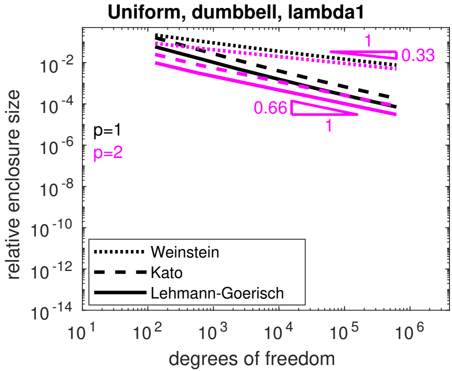

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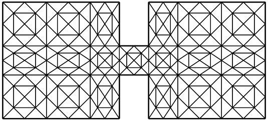

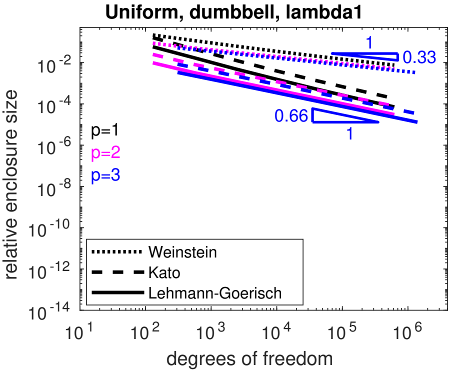

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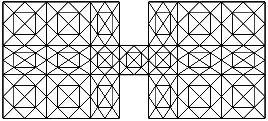

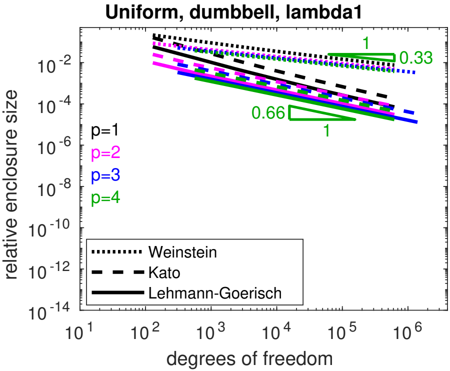

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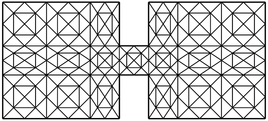

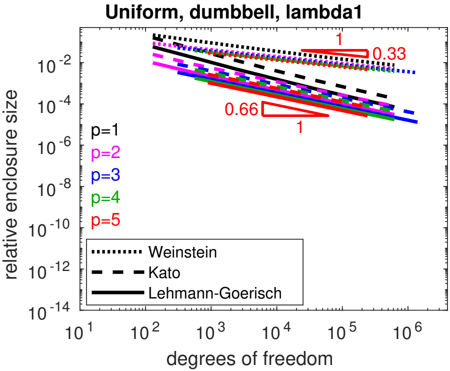

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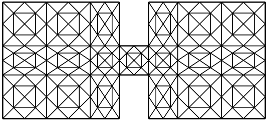

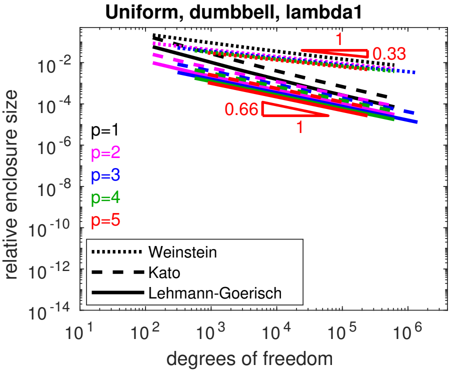

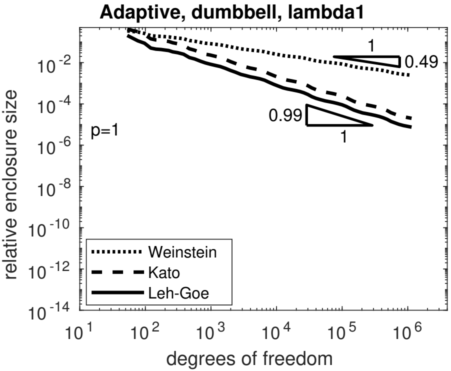

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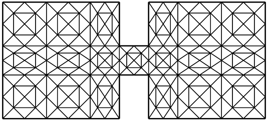

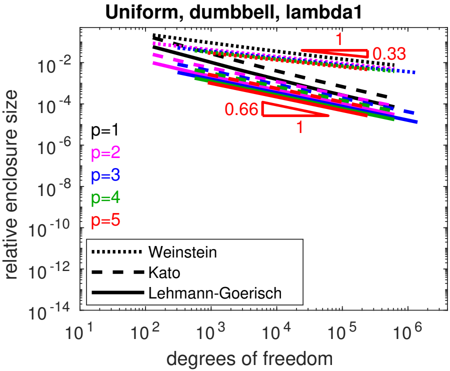

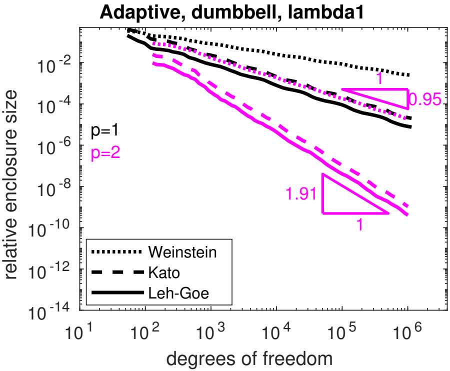

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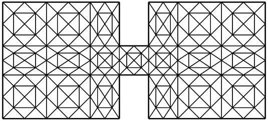

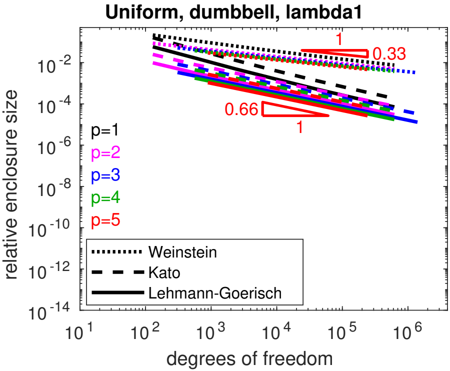

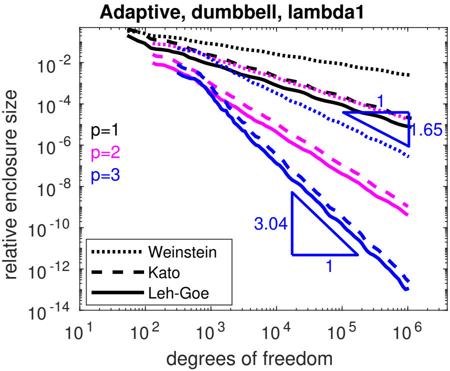

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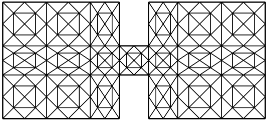

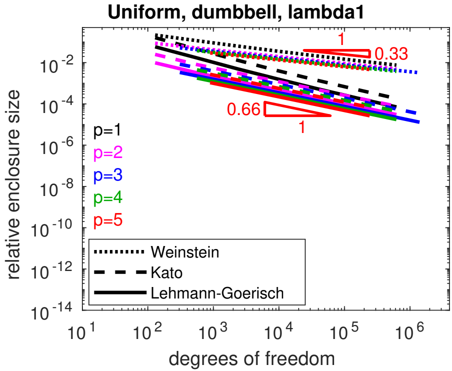

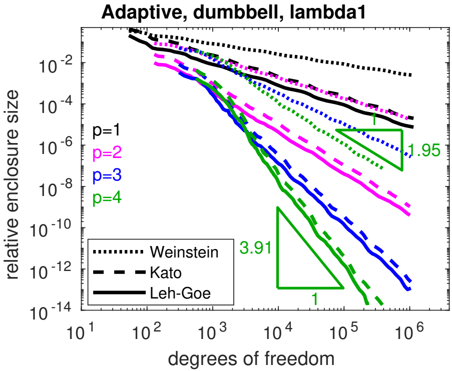

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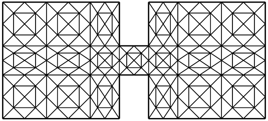

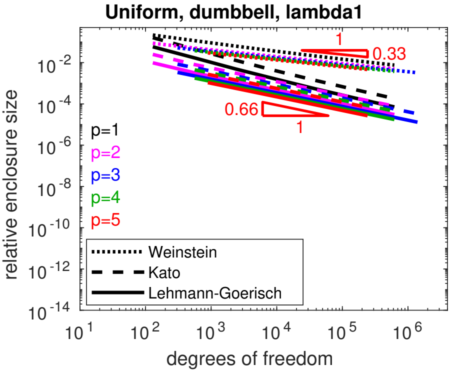

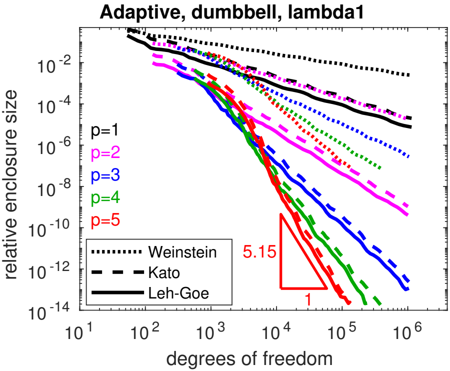

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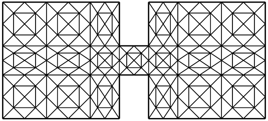

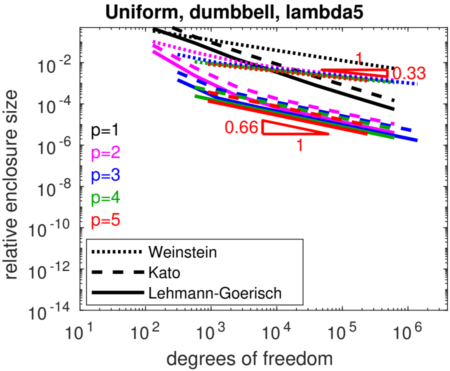

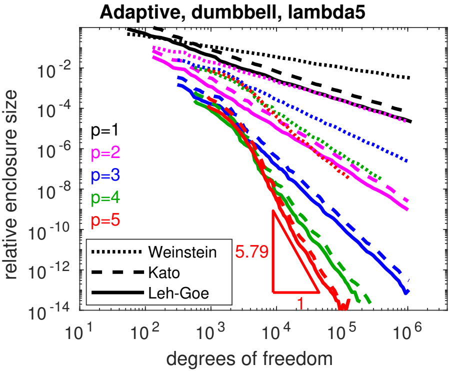

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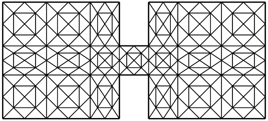

Tight pairs of eigenvalues:

$$4.996837097248\mathbf{9} \le \lambda_5 \le 4.996837097249\mathbf{0}$$
$$4.996850904101\mathbf{5} \le \lambda_6 \le 4.996850904101\mathbf{6}$$

$$7.98696729210\mathbf{28} \le \lambda_7 \le 7.98696729210\mathbf{38}$$
$$7.98703430682\mathbf{16} \le \lambda_8 \le 7.98703430682\mathbf{27}$$

Conclusions

- Kato and Lehmann–Goerisch methods provide lower bounds with optimal rates of convergence even for higher-order approximations
- The same flux reconstruction can be used in all methods and it can be used for adaptive mesh refinement
- Weinstein bound has suboptimal convergence rate, but it is usefull for a priori lower bounds

Thank you for your attention

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Supported by Neuron Fund for Support of Science, project no. 24/2016

Equadiff 2017, Bratislava, July 26