Stationary solutions to problems involving compressible fluids

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Compressible Navier–Stokes system

Field equations

$$\mathrm{div}_x(\varrho\mathbf{u}) = 0,\ \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) + \varrho\mathbf{f}$$

$$\mathbb{S}(\nabla_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \lambda\mathrm{div}_x\mathbf{u}\mathbb{I},\ \mu > 0$$

Boundary conditions, I

$$\Omega = \left\{(x_1, x_2, z)\ \middle|\ 0 < z < F(x_1, x_2)\right\}\ \varrho, \mathbf{u} \text{ periodic in } (x_1, x_2)$$

$$\mathbf{u} = \mathbf{u}_B \text{ on } \partial\Omega,\ \mathbf{u}_B\cdot\mathbf{n} = 0$$

Boundary conditions, II

$$\Omega \subset R^N\ C^{2+\nu} - \text{domain } \mathbf{u} = \mathbf{u}_B \text{ on } \partial\Omega$$

$$\varrho = \varrho_B \text{ on } \Gamma_{\mathrm{in}} = \left\{x \in \partial\Omega\ \middle|\ \mathbf{u}_B\cdot\mathbf{n} < 0\right\}$$

Known results

Small data, smooth solutions

Inhomogeneous boundary conditions, small perturbations of an equilibrium state

- Plotnikov, Ruban, Sokolowski [2008]
- Mucha, Piasecki [2014]
- Piasecki [2010]
- Piasecki, Pokorný [2014]

Large force, homogeneous (periodic) boundary conditions

$$p(\varrho) = a\varrho^{\gamma}$$

- Lions [1998] $\gamma > \frac{5}{3}$
- Březina, Novotný [2008] $\gamma >> 3/2$
- Frehse, Steinhauer, Weigant [2012] $\gamma > 4/3$
- Plotnikov, Sokolowski [2007] $\gamma > 4/3$
- Jiang, Zhou [2011] $\gamma > 1$ periodic BC

Principal hypotheses

Pressure term

- **Molecular hypothesis (hard sphere model).** The specific volume of the fluid is bounded below away from zero. Equivalently, the fluid density cannot exceed a limit value $\overline{\varrho} > 0$. Accordingly, the pressure $p = p(\varrho)$ satisfies

$$\lim_{\varrho \to \overline{\varrho}} p(\varrho) = \infty$$

- **Positive compressibility.** The pressure $p = p(\varrho)$ is a non-decreasing function of the density, more precisely

$$p \in C[0, \overline{\varrho}) \cap C^1(0, \overline{\varrho}), \ p(0) = 0, \ p'(\varrho) \geq 0 \text{ for } \varrho \geq 0.$$

Main result, general boundary conditions

Theorem EF, A.Novotný [2017]

Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded simply connected domain of class $C^{2,1}$. Let the boundary data $\mathbf{u}_B$, ϱ_B satisfy

$$\mathbf{u}_B \in C^2(\partial\Omega; R^N), \ \varrho_B \in C(\partial\Omega)$$

and

$$0 < \min \varrho_B \leq \max \varrho_B < \overline{\varrho}, \ \int_{\partial\Omega} \mathbf{u}_B \cdot \mathbf{n} \, \mathrm{d}S_x \geq 0.$$

Then the problem possesses at least one weak solution $[\varrho, \mathbf{u}]$.

Main result, infinite strip

Theorem I.S.Ciuperca, EF, M.Jai, A.Petrov

Let

$$\Omega = \left\{ (x_1, x_2, z) \ \middle| 0 < z < F(x_1, x_2) \right\}$$

and the velocity satisfies

$$\mathbf{u} = \mathbf{u}_B \text{ in } \partial\Omega, \ \mathbf{u}_B \cdot \mathbf{n} = 0$$

Let

$$M = \int_\Omega \varrho \, \mathrm{d}x > 0$$

be given.
Then the problem admits a weak solution $[\varrho, \mathbf{u}]$.

Approximate problem

Regularization

$$-\delta\Delta_x\varrho + \delta\varrho + \mathrm{div}_x(T(\varrho)\mathbf{u}) = 0$$

$$(-\delta\nabla_x\varrho + T(\varrho)\mathbf{u})\cdot\mathbf{n}|_{\partial\Omega} = \left\{ \begin{array}{l} T(\varrho_B)\mathbf{u}_B\cdot\mathbf{n} \text{ if } \mathbf{u}_B\cdot\mathbf{n}\leq 0, \\ T(\varrho)\mathbf{u}_B\cdot\mathbf{n} \text{ if } \mathbf{u}_B\cdot\mathbf{n} > 0 \end{array} \right.$$

$$\mathrm{div}_x(T(\varrho)\mathbf{u}\otimes\mathbf{u}) + \nabla_x p_{\varepsilon,\delta}(\varrho) = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) + \delta\Delta_x(\varrho\mathbf{u}) - \delta\varrho\mathbf{u}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B$$

Cut–off

$$T(\varrho) = \left\{ \begin{array}{l} 0 \text{ if } \varrho\leq 0, \\ \varrho \text{ if } 0\leq\varrho\leq\overline{\varrho} \\ \overline{\varrho} \text{ if } \varrho\geq\overline{\varrho} \end{array} \right\}, \ p_{\varepsilon,\delta}(\varrho) = p_\varepsilon(\varrho) + \sqrt{\delta}\varrho,$$

$$p_\varepsilon(\varrho) = \left\{ \begin{array}{l} p(\varrho) \text{ if } 0\leq\varrho\leq\overline{\varrho}-\varepsilon, \\ p(\overline{\varrho}-\varepsilon) + p'(\overline{\varrho}-\varepsilon)(\varrho-\overline{\varrho}+\varepsilon). \end{array} \right\}$$

Auxiliary result

Lemma - variant of Leray's inequality

Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded simply connected domain of class $C^{2+\nu}$. Let $K > 0$ and $\varepsilon > 0$ be given.
Then there exists a vector field $\mathbf{V} \in C^2(\overline{\Omega}; R^N)$ enjoying the following properties:

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}S_x = K$$

$$\mathrm{div}_x \mathbf{V} \geq 0 \text{ in } \Omega$$

$$\|\mathbf{V}\|_{L^4(\Omega)} < \varepsilon$$

Singular limit

Thin domain

$$\Omega_\varepsilon = \left\{ x = (x_h, z) \ \middle| \ x_h \in \mathcal{T}^D,\ 0 < z < \varepsilon h(x_h),\ h \in C^{2+\nu}(\mathcal{T}^D) \right\}$$

Rescaled system

$$\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho_\varepsilon) = \mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon))$$

Boundary conditions

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = \bar{\mathbf{u}} \begin{cases} [\mathbf{s}, 0] \text{ if } z = 0, \\ 0 \text{ if } z = \varepsilon h(x_h) \end{cases}$$

Justification of the Reynolds system

Theorem I.S.Ciuperca, EF, M.Jai, A.Petrov [2017]

Let

$$Q = \left\{ x = (x_h, z) \ \middle| \ x_h \in \mathcal{T}^D,\ 0 < z < h(x_h),\ h \in C^{2+\nu}(\mathcal{T}^D) \right\}$$

$$\varepsilon \int_Q \varrho \ \mathrm{d}x = M_\varepsilon$$

$$0 < \inf_{\varepsilon > 0} \frac{M_\varepsilon}{|Q|\varepsilon} \leq \sup_{\varepsilon > 0} \frac{M_\varepsilon}{|Q|\varepsilon} < \bar{\varrho}.$$

Let $(\varrho_\varepsilon, \mathbf{u}_{h,\varepsilon}, V_\varepsilon)$ be a family of weak solutions to the rescaled problem.

Reynolds system

Conclusion

Then, up to a subsequence, we have

$$\frac{M_\varepsilon}{\varepsilon} \to M,\ 0 \le \varrho_\varepsilon \le \bar{\varrho},\ \varrho_\varepsilon \to \varrho \quad \text{in} \quad L^1(Q)$$

$$p(\varrho_\varepsilon) \to p(\varrho) \quad \text{in} \quad L^2_{\mathrm{loc}}(Q)$$

$\mathbf{u}_{h,\varepsilon} \to \mathbf{u}_h,\ \partial_Z \mathbf{u}_{h,\varepsilon} \to \partial_Z \mathbf{u}_h,\ V_\varepsilon \to V,\ \partial_Z V_\varepsilon \to \partial_Z V$ weakly in $L^2(Q)$

where the limit satisfies

$$\varrho = \varrho(x_h),\ 0 \le \varrho \le \bar{\varrho},\ p = p(\varrho) \in L^2(\mathcal{T}^D)$$

$$\int_Q \varrho \,\mathrm{d}x = \int_{\mathcal{T}^D} h\varrho \mathrm{d}x_h = M$$

$$\mathbf{u}_h|_{Z=0} = \mathbf{s},\ \mathbf{u}_h|_{Z=h(x_h)} = 0$$

$$\mathrm{div}_h\Big(\int_0^{h(x_h)} \varrho \mathbf{u}_h \,\mathrm{d}Z\Big) = 0,\ -\mu\partial_Z^2 \mathbf{u}_h + \nabla_h p(\varrho) = 0 \text{ in } \mathcal{D}'(Q)$$