INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Relative energy inequality for dissipative measure-valued solutions of compressible non-Newtonian fluids

Hind Al Baba
Matteo Caggio
Bernard Ducomet
Šárka Nečasová

Preprint No. 22-2017

PRAHA 2017

Relative energy inequality for dissipative measure-valued solutions of compressible non-Newtonian fluids

Hind Al Baba [1] [2], Matteo Caggio [1], Bernard Ducomet [3], Šárka Nečasová [1]

[1] Institute of Mathematics, Žitná 25, 115 67 Praha 1, Czech Republic,
[2] Laboratoire de Mathématiques et de leurs Applications, CNRS UMR 5142,
Université de Pau et des Pays de l'Adour
64013 Pau, France,
[3] CEA, DAM, DIF, F-91297 Arpajon, France

Abstract

The aim of the paper is to extend the result by Novotný and Nečasová [19] to the case of dissipative measure-valued solution and derive a relative energy inequality.

1 Formulation of the problem

We consider the compressible non-Newtonian system of power-law type. The aim of paper is to extend the result given by Novotný and Nečasová [19] to the more general case of measure-valued solution and derive relative energy inequality for this system.

Before stating the problem let us first explain the meaning of a measure-valued solution. It is a map which gives for every point in the domain a probability distribution of values and the equation is satisfied only in an average sense. In case that the probability distribution reduced to a point mass almost everywhere in the domain it means that measure valued solution is a weak solution of the problem, see e.g. the case of incompressible non-Newtonian case in work of Nečas et al. [13] or Bellout and Bloom [4].

The advantage of measure-valued solutions is the property that in many cases, the solutions can be obtained from weakly convergent sequences of approximate solutions.

Measure-valued solutions for systems of hyperbolic conservations laws were initially introduced by DiPerna [6]. He used Young measures to pass to limit in the artificial viscosity term. In the case of the incompressible Euler equations, DiPerna and Majda [7] also proved global existence of measure-valued solutions for any initial data with finite energy. They introduced generalized Young measures to take into account oscillation and concentration phenomena. Thereafter the existence of measure-valued solutions was finally shown for further models of fluids, e.g. compressible Euler and Navier-Stokes equations [18]. The measure-valued solution to the non-Newtonian case was proved by Novotný and Nečasová [19]. The generalization was given by Alibert and Bouchité [2]. More details can be found in [16], [17] and [21].

Recently, weak-strong uniqueness for generalized measure-valued solutions of isentropic Newtonian Euler equations were proved in [11]. Inspired by previous results, the concept of dissipative measure-valued solution was finally applied to the barotropic compressible Navier-Stokes system [12].

We will consider the motion of the fluid is governed by the following system of equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho u) = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{1.1}$$

$$\partial_t(\varrho u) + \mathrm{div}_x(\varrho u\otimes u) + \nabla_x p = \mathrm{div}_x S \quad \text{in} \quad (0,T)\times\Omega, \tag{1.2}$$

where ϱ is the mass density and u is the velocity field, functions of the spatial position $x\in\mathbb{R}^3$ and the time $t\in\mathbb{R}$. The scalar function p is termed pressure, given function of the density. In particular, we consider the isothermal case, namely $p=\lambda\varrho$, with $\lambda>0$ a constant. The stress tensor is given by

$$S_{ij} = \beta u_{l,l}\delta_{ij} + 2\omega e_{i,j}(u), \tag{1.3}$$

where

$$\beta = \beta\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right), \quad \omega = \omega\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right), \tag{1.4}$$

and

$$\widehat{u} = \sqrt{e_{i,j}(u)e_{i,j}(u)}, \quad e_{i,j}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

$$\beta \geq -\frac{2}{3}\omega, \quad \omega\geq 0.$$

We consider the Dirichlet boundary conditions

$$u = 0 \quad \text{in} \quad (0,T)\times\partial\Omega \tag{1.5}$$

and initial data

$$u(0) = u_0, \quad \varrho(0) = \varrho_0. \tag{1.6}$$

We consider the following hypothesis:

$$2\omega\left(\widehat{u},\mathrm{div}_x u,\det\left(\frac{\partial u_i}{\partial x_j}\right)\right)|\widehat{u}|^2$$
$$+\beta\left(\widehat{u},\mathrm{div}_x u,\det\left(\frac{\partial u_i}{\partial x_j}\right)\right)\mathrm{div}_x u\ \mathrm{div}_x u \geq k_2\,|\widehat{u}|^{\gamma}, \tag{1.7}$$
$$2\omega\left(\widehat{u},\mathrm{div}_x u,\det\left(\frac{\partial u_i}{\partial x_j}\right)\right)e_{ij}(u)$$
$$+\beta\left(\widehat{u},\mathrm{div}_x u,\det\left(\frac{\partial u_i}{\partial x_j}\right)\right)\mathrm{div}_x u\ \delta_{ij} \leq k_1\,|\widehat{u}|^{\overline{\gamma}-1}, \tag{1.8}$$

for $i,j \in 1,2,3$ with $k_1,k_2>0$, $\gamma \leq \overline{\gamma} < \gamma+1$, $\gamma \geq 2$. Further, we assume the existence of a positive function $\vartheta(e_{ij})$ such that

$$\frac{\partial\vartheta}{\partial e_{ij}} = 2\omega\left(\widehat{u},\mathrm{div}_x u,\det\left(\frac{\partial u_i}{\partial x_j}\right)\right)e_{ij}(u)+\delta_{ij}\beta\left(\widehat{u},\mathrm{div}_x u,\det\left(\frac{\partial u_i}{\partial x_j}\right)\right)\mathrm{div}_x u. \tag{1.9}$$

Remark 1. We consider power-law type of fluids. For more details see [13].

2 Mathematical preliminaries

We define $\phi(t) = e^t - t - 1$ and $\phi_2(t) = e^{t^2} - 1$ the Young functions and by $\psi(t) = (1+t)\ln(1+t) - t$, and $\psi_{1/2}(t)$ the complementary Young functions to them. The corresponding Orlicz spaces are $L_\phi(\Omega)$, $L_{\phi_2}(\Omega)$, $L_\psi(\Omega)$, $L_{\psi_{1/2}}(\Omega)$. These are Banach spaces equipped with a Luxembourg norm

$$\|u\|_{L_f(\Omega)} = \inf_h\left\{h>0; \int_\Omega f\left(\frac{|u(x)|}{h}\right)dx \leq 1\right\} < +\infty, \tag{2.1}$$

where f stands for ϕ_1, ϕ_2, ψ, $\psi_{1/2}$. Let $C(\Omega)$ be the set of bounded continuous functions which are defined in Ω. We denote C_ψ, C_ϕ, $C_{\psi_{1/2}}$ and C_{ϕ_2} the closure of $C(\Omega)$ in $L_\psi(\Omega)$, $L_\phi(\Omega)$, $L_{\psi_{1/2}}(\Omega)$, and $L_{\phi_2}(\Omega)$, respectively. We have $(C_\phi(\Omega))^* = L_\psi(\Omega)$, $(C_\psi(\Omega))^* = L_\phi(\Omega)$, $(C_{\phi_2}(\Omega))^* = L_{\psi_{1/2}}(\Omega)$, $\left(C_{\psi_{1/2}}(\Omega)\right)^* = L_{\phi_2}(\Omega)$, where C_ψ, C_ϕ, $C_{\psi_{1/2}}$ and C_{ϕ_2} are separable Banach spaces. Further, ψ, $\psi_{1/2}$ satisfy the $\triangle_2$-condition and we have $C_\psi(\Omega) = L_\psi(\Omega)$, $C_{\psi_{1/2}}(\Omega) = L_{\phi_2}(\Omega)$. $L^\infty_w\left(Q_T, M\left(\mathbb{R}^{N^2}\right)\right)$ denotes the spaces of all weakly measurable mappings from Q_T into $M\left(\mathbb{R}^{N^2}\right)$ with finite $L^\infty\left(Q_T, M\left(\mathbb{R}^{N^2}\right)\right)$ norm; $\nu \in L^\infty\left(Q_T, M\left(\mathbb{R}^{N^2}\right)\right)$ is a weakly measurable map if and only if $(x,t) \to (\nu_{x,t}, g(x,t))$ is Lebesgue measurable in Q_T for every $g \in L^1\left(Q_T, C_0\left(\mathbb{R}^{N^2}\right)\right)$; N is dimension. We define by $L^p(\Omega), W^{l,p}(\Omega)$ (resp. $W_0^{l,p}$), $0 \leq l,p < +\infty$, the usual Lebesque space, Sobolev spaces. We denote $V^k = W^{k,2} \cap W_0^{1,2}$, $Q_T = \Omega \times (0,T)$.

Remark 2. For more details about Orlicz spaces see [15].

Definition 3. (Measure-valued solution) Let (ϱ, v, ν) be such that

$$\varrho \in L^\infty(I, L_\psi), \tag{2.2}$$

$$v \in L^2(I, V_k) \cap L^\gamma(I, W_0^{1,\gamma}), \tag{2.3}$$

$$\nu \in L_w^\infty\left(Q_T, M\left(\mathbb{R}^{N^2}\right)\right), \tag{2.4}$$

the functions

$$\sigma_{ij}, \quad \beta\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\left(\sigma\right)\right)\mathrm{Tr}\sigma, \quad \omega\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\left(\sigma\right)\right)\sigma_{ij} \tag{2.5}$$

are ν-integrable in $\mathbb{R}^{N^2}$ $(\mathrm{Tr}\sigma = \sigma_{ii})$ and

$$\int_{\mathbb{R}^{N^2}} \sigma_{ij} \mathrm{d}\nu_{t,x}(\sigma) = \frac{\partial u_i}{\partial x_j}, \quad a.e. \quad in \quad Q_T. \tag{2.6}$$

Then, we define a measured-valued solution for the system (1.1) - (1.9) in the sense of DiPerna [6], in the following way:

$$-\int_{Q_T} \varrho u_i \frac{\partial \varphi_i}{\partial t} \mathrm{d}x\mathrm{d}t - \int_{Q_T} \varrho u_i u_j \varphi_{i,j} \mathrm{d}x\mathrm{d}t - \int_{\Omega_0} \varrho_0 u_0 \varphi_i(0)\mathrm{d}x - k\int_{Q_T} \varrho \varphi_{i,i} \mathrm{d}x\mathrm{d}t$$

$$+\int_{Q_T} \mathrm{d}x\mathrm{d}t \left(\int_{\mathbb{R}^{N^2}} \beta\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\left(\sigma\right)\right)\mathrm{Tr}\sigma\delta_{ij} + 2\omega\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\left(\sigma\right)\right)\sigma_{ij}\mathrm{d}\nu_{t,x}\left(\sigma\right)\right)\varphi_{i,j} = 0, \tag{2.7}$$

for all $\varphi \in C^\infty\left(\overline{Q_t}\right)$, $\varphi(t) \in W_0^{1,\gamma}(\Omega)$ and for any $t \in I$, $\varphi(t) = 0$.

Remark 4. In the Definition 3 the Young measures are defined for the gradient of the velocity field. In the next Section the measures are considered for the density and the velocity field.

Theorem 5. *Let $u_0 \in V_k$, $\varrho_0 \in C^d(\overline{\Omega})$, $\varrho_0 > \varepsilon > 0$, $d = 1, 2, \ldots$. Let assumptions (1.7) - (1.9) be satisfied, $k > N$. Then, there exists (ϱ, u) and a family of a probability measure $\nu_{x,t}$ on $\mathbb{R}^{N^2}$ with properties such that*

$$(i)\ \nu \in L_w^\infty\left(Q_T, \mathbb{R}^{N^2}\right),\ \|\nu_{x,t}\| = 1,\ for\ a.e.\ (x,t) \in Q_T;$$

$$(ii)\ \mathrm{supp}\,\nu_{x,t} \subset \mathbb{R}^{N^2},\ for\ a.e.\ (x,t) \in Q_T;$$

$$(iii)\ u \in L^\gamma(I, W_0^{1,\gamma}) \cap L^\gamma\left(I, W_0^{1,\alpha}\right),\ \alpha\gamma > N,\ \alpha < 1;$$

$$(iv)\ \varrho \in L^\infty(I, L_\psi(\Omega)) \cap L^2(I, W^{-1,2});$$

$$(v)\ \varrho u \in L^{\gamma}(I, W^{-\alpha,\gamma}),\ \alpha\gamma > N,\ \alpha < 1,\ \gamma + \gamma^{-1} = 1;$$

$$(vi)\ \varrho u_i u_j \in L^{\gamma}(I, W^{-\alpha,\gamma})$$

and such that (ϱ, u, ν) satisfies (2.7).

Proof. To prove the existence of measure-valued solutions we introduce the following approximation scheme (multipolar fluid introduced by Nečas and Šilhavý, [20])

$$\tau_{ij} = -p\delta_{ij} + \sum_{s=0}^{k-1} \tau_{ij}^{(s,v)}, \tag{2.8}$$

where

$$\tau_{ij}^{(s,v)} = \tau_{ij}^{(s,v,lin)} + S_{ij}, \tag{2.9}$$

with

$$\tau_{ij}^{(s,v,lin)} = (-1)^s \left(\mu_1^s \triangle^s u_{l,l}\delta_{ij} + 2\mu_2^s \triangle^s e_{ij}(u)\right). \tag{2.10}$$

The second law of thermodynamics requires additional stress tensors with the power on an elementary surface

$$\mathrm{d}S \tau_{ii_1\ldots i_m j}^{\nu} \frac{\partial^m u_i}{\partial x_{i_1} \ldots \partial x_{i_m}} \nu_j.$$

The higher stress tensors are defined as follows

$$\tau_{ii_1\ldots i_m j}^{\nu} = \mathrm{Sym}\left(\sum_{r=m}^{k-1} (-1)^{r+m} \triangle^{r-m} \frac{\partial^m q_{ii_m}^r}{\partial x_{i_1} \ldots \partial x_{i_{m-1}} \partial x_j}\right), \tag{2.11}$$

where

$$q_{ij}^s = \mu_1^s \left(\frac{\partial u_l}{\partial x_l}\right) \delta_{ij} + 2\mu_2^s e_{ij}(u) \tag{2.12}$$

and symmetrization is taken with respect to $(i_1, ..., i_m)$. We assume that μ_1^s and μ_2^s are constants and

$$\mu_1^s \geq -\frac{2}{3}\mu_2^s, \quad \mu_2^s > 0, \quad 0 \leq s \leq k-2,$$

$$\mu_1^{k-1} > -\frac{2}{3}\mu_2^{k-1}, \quad \mu_2^{k-1} > 0. \tag{2.13}$$

We denote

$$((v,w)) = \int_\Omega \left(\sum_{s=0}^{k-1} \left(2\mu_2^s \frac{\partial^s e_{ij}(v)}{\partial x_{i_1} ... \partial x_{i_s}} \frac{\partial^s e_{ij}(w)}{\partial x_{i_1} ... \partial x_{i_s}} + \right.\right.$$

$$\left.\left. + \mu_1^s \frac{\partial^s e_{rr}(w)}{\partial x_{i_1} ... \partial x_{i_s}} \frac{\partial^s e_{ll}(v)}{\partial x_{i_1} ... \partial x_{i_s}} \right)\right) \mathrm{d}x. \tag{2.14}$$

Moreover, we consider

$$\mu_1^s > -\frac{2}{3}\mu_2^s, \quad (s = 0, ..., k-2). \tag{2.15}$$

The system is defined by the following equations:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho u_i)}{\partial x_i} = 0, \tag{2.16}$$

$$\frac{\partial (\varrho u_i)}{\partial t} + \frac{\partial (\varrho u_i u_j)}{\partial x_j} - \frac{\partial \tau_{ij}^v}{\partial x_j} = -k \frac{\partial \varrho}{\partial x_i}, \tag{2.17}$$

with the initial data

$$u(0) = u_0, \quad \varrho(0) = \varrho_0 \tag{2.18}$$

and boundary conditions

$$u = 0 \quad \text{on } \partial\Omega \times I, \quad [[v,w]] = 0 \quad \text{on } \partial\Omega \times I, \tag{2.19}$$

where

$$[[v,w]] = \sum_{m=1}^{k-1} \int_{\partial\Omega} \tau_{ii_1 ... i_m j}^\nu \frac{\partial w_i^m}{\partial x_{i_1} ... \partial x_{i_m}} \nu_j \mathrm{d}S. \tag{2.20}$$

Weak formulation of (2.17) reads

$$\int_{Q_T} \frac{\partial (\varrho u_i)}{\partial t} \varphi_i - \int_{Q_T} \varrho u_i u_j \varphi_{i,j} + \int_0^T ((u,\varphi)) +$$

$$+ \int_{Q_T} \beta \left(\widehat{u}, \mathrm{div}_x u, \det \left(\frac{\partial u_i}{\partial x_j} \right) \right) \mathrm{div}_x u \frac{\partial \varphi_i}{\partial x_i} +$$

$$+2 \int_{Q_T} \omega \left(\widehat{u}, \mathrm{div}_x u, \det \left(\frac{\partial u_i}{\partial x_j} \right) \right) e_{ij}(u) \frac{\partial \varphi_i}{\partial x_j} - k \int_{Q_T} \varrho \frac{\partial \varphi_i}{\partial x_i}$$

$$= \int_{Q_T} \varrho b_i \varphi_i, \quad \forall \varphi \in L^2 \left(I, V_k \cap W_0^{1,\gamma} \right). \tag{2.21}$$

Let us formulate the existence and uniqueness results for the approximation scheme:

Lemma 6. Assume that $u_0 \in V_k$ and $\varrho_0 \in C^d\left(\overline{\Omega}\right)$, where $\varrho_0 > \varepsilon > 0$ and $d = 1, 2, ...$ Let assumptions (1.7) - (1.9) be satisfied, $k > N$. Then, there exists at least one solution (ϱ, u) of (2.16) - (2.17) satisfying (2.21) such that

$$\varrho \in L^{\infty}\left(I, W^{p,q}\right), \tag{2.22}$$

where

$$p = \min\left(d, k-2\right), \quad 1 \le q \le 6\left(N=3\right), \quad 1 \le q < \infty\left(N=2\right), \tag{2.23}$$

$$\frac{\partial \varrho}{\partial t} \in L^2\left(I, W^{p-1,q}\right), \tag{2.24}$$

$$u \in L^2\left(I, V_k\right) \cap L^{\infty}\left(I, W^{k,2}\left(\Omega\right)\right), \tag{2.25}$$

$$\frac{\partial u}{\partial t} \in L^2\left(Q_T\right), \tag{2.26}$$

$$u \in L^{\gamma}\left(I, W_0^{1,\gamma}\left(\Omega\right)\right). \tag{2.27}$$

Moreover, assuming that $\theta(e_{ij})$ satisfying (1.9) is continuously differentiable in $\mathbb{R}^{N^2}$. Then, in the class of solutions satisfying (2.22) - (2.27), there exists at most one solution of the problem (2.16) - (2.21).

Proof. The methods of characteristic applying to the continuity equations together with Galerkin approach on the momentum equation we get existence of solution. For more details on the proof see [19]. □

Passing with higher viscosity in the limit the most problematic point is to find a representation in terms of

$$\int_{Q_T} \beta\left(\widehat{u}^{\mu}, \mathrm{div}_x \widehat{u}^{\mu}, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right) u^{\mu}_{i,i} \varphi_{i,i} +$$
$$+2\int_{Q_T} \omega\left(\widehat{u}^{\mu}, \mathrm{div}_x \widehat{u}^{\mu}, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right) u^{\mu}_{i,j} \varphi_{i,j} \tag{2.28}$$

We follow the classical theory introduced by Ball [3]. We define for each $(x,t) \in Q_T$ a sequence

$$\nu^j_{x,t} \equiv \delta_{\nabla v^j(x,t)}, \tag{2.29}$$

where δ_x is the Dirac measure which lives in the point $x \in \mathbb{R}^{N^2}$ $\left(\nabla v^{\mu}\left(x,t\right) \in \mathbb{R}^{N^2}\right)$ and let us put

$$\nu^j \,:\, (x,t) \in Q_T \to \nu^j_{x,t}. \tag{2.30}$$

Since $\{\nu^j\}$ is uniformly bounded in $L_w^\infty\left(Q_T; M\left(\mathbb{R}^{N^2}\right)\right)$, thanks to the representation theorem

$$\left(\left[L^1\left(Q_T; C_0\left(\mathbb{R}^{N^2}\right)\right)\right]^* \approx L_w\left(Q_T; M\left(\mathbb{R}^{N^2}\right)\right)\right) \tag{2.31}$$

and the separability of ν^j, we have $\nu \in L_w^\infty\left(Q_T; M\left(\mathbb{R}^{N^2}\right)\right)$ such that

$$\nu^j \to \nu, \quad \text{weakly} - * \text{ in } L_w^\infty\left(Q_T; M\left(\mathbb{R}^{N^2}\right)\right). \tag{2.32}$$

Let us recall the special case of the Ball theorem (see [3]).

Lemma 7. Let $\nabla v^j : Q_T \to \mathbb{R}^{N^2}$ be uniformly bounded in $L^\gamma(Q_T)$ and let the continuous function $\tau : \mathbb{R}^{N^2} \to \mathbb{R}$ satisfy

$$c\,|\widehat{\sigma}^\gamma| \le \tau\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\sigma\right) \le c\left(1 + |\widehat{\sigma}|\right)^{\overline{\gamma}-1}, \tag{2.33}$$

where $\gamma > \overline{\gamma} - 1$ and

$$\sup_{j=1,2,\dots} \int_{Q_T} \eta\left(\left|\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\sigma\right)\right|\right) \mathrm{d}x\mathrm{d}t < \infty \tag{2.34}$$

with η being Young function. Then,

$$\|\nu_{x,t}\| = 1, \quad \textit{a.e. in } \mathbb{R}^{N^2} \tag{2.35}$$

and

$$\tau\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\sigma\right) \to \left(\tau, \nu_{x,t}\right) = \int_{\mathbb{R}^{N^2}} \tau\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\sigma\right) \mathrm{d}\nu_{x,t}(\sigma) \tag{2.36}$$

weakly - * in $L_\eta(Q_T)$.

Applying Lemma 7 with $\eta(\xi) = \xi^{\gamma/(\overline{\gamma}-1)}$, we get

$$\int_{Q_T} \left[\beta\left(\widehat{u}, \mathrm{div}\sigma, \det\sigma\right)\right]^{\gamma/(\overline{\gamma}-1)} \mathrm{d}x\mathrm{d}t \le \int_{Q_T} |\widehat{\sigma}^\gamma|\, \mathrm{d}x\mathrm{d}t \le \text{const.}, \tag{2.37}$$

which give us the measure-valued solution in the sense of DiPerna. □

3 Dissipative measure-valued solutions to the compressible isothermal system

We introduce the concept of dissipative measure-valued solution to the system (1.1) - (1.2) in the spirit of [11] and [12].

Definition 8. We say that a parameterized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\nu \in L^{\infty}_{w}\left((0,T)\times\Omega;\mathcal{P}\left([0,\infty)\times\mathbb{R}^N\right)\right), \quad \langle\nu_{t,x};s\rangle \equiv \varrho, \quad \langle\nu_{t,x};v\rangle \equiv u,$$

is a dissipative measure-valued solution of the compressible Navier-Stokes system (1.1) - (1.2) in $(0,T)\times\Omega$, with the initial conditions ν_0 and dissipation defect $\mathcal{D}$,

$$\mathcal{D} \in L^{\infty}(0,T), \quad \mathcal{D} \geq 0,$$

if the following holds.

(i) Continuity equation. There exist a measure $r^C \in L^1\left([0,T],\mathcal{M}\left(\overline{\Omega}\right)\right)$ and $\chi \in L^1(0,T)$ such that for a.a. $\tau \in (0,T)$ and every $\psi \in C^1\left([0,T]\times\overline{\Omega}\right)$,

$$\left|\left\langle r^C(\tau);\nabla_x\psi\right\rangle\right| \leq \chi(\tau)\,\mathcal{D}(\tau)\,\|\psi\|_{C^1(\overline{\Omega})} \tag{3.1}$$

and

$$\int_\Omega \langle\nu_{t,x};s\rangle\,\psi(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \langle\nu_0;s\rangle\,\psi(0,\cdot)\,\mathrm{d}x$$

$$= \int_0^\tau\int_\Omega \left[\langle\nu_{t,x};s\rangle\,\partial_t\psi + \langle\nu_{t,x};sv\rangle\cdot\nabla_x\psi\right]\mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^C;\nabla_x\psi\right\rangle\mathrm{d}t. \tag{3.2}$$

(ii) Momentum equation.

$$u = \langle\nu_{t,x};v\rangle \in L^2\left(0,T;W_0^{1,2}\left(\Omega;\mathbb{R}^N\right)\right)$$

and there exists a measure $r^M \in L^1\left([0,T],\mathcal{M}\left(\overline{\Omega}\right)\right)$ and $\xi \in L^1(0,T)$ such that for a.a. $\tau \in (0,T)$ and every $\varphi \in C^1\left([0,T]\times\overline{\Omega};\mathbb{R}^N\right)$, $\varphi|_{\partial\Omega} = 0$,

$$\left|\left\langle r^M(\tau);\nabla_x\varphi\right\rangle\right| \leq \xi(\tau)\,\mathcal{D}(\tau)\,\|\varphi\|_{C^1(\overline{\Omega})} \tag{3.3}$$

and

$$\int_\Omega \langle\nu_{t,x};sv\rangle\,\varphi(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \langle\nu_0;sv\rangle\,\varphi(0,\cdot)\,\mathrm{d}x$$

$$= \int_0^\tau\int_\Omega \left[\langle\nu_{t,x};sv\rangle\,\partial_t\varphi + \langle\nu_{t,x};s\,(v\otimes v)\rangle : \nabla_x\varphi + \langle\nu_{t,x};p(s)\rangle\,\mathrm{div}_x\varphi\right]\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega S\left(\widehat{u},\mathrm{div}_x u,\det\left(\frac{\partial u_i}{\partial x_j}\right)\right) : \nabla_x\varphi\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^M;\nabla_x\varphi\right\rangle\mathrm{d}t. \tag{3.4}$$

(iii) Energy inequality.

$$\int_\Omega \left\langle \nu_{t,x}; \left(\frac{1}{2}s\,|u|^2 + P(s)\right)\right\rangle \mathrm{d}x + \int_0^\tau \int_\Omega S\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right) : \nabla_x u \,\mathrm{d}x\mathrm{d}t$$

$$+\mathcal{D}(\tau) \le \int_\Omega \left\langle \nu_0; \left(\frac{1}{2}s\,|u|^2 + P(s)\right)\right\rangle \mathrm{d}x, \ \textit{for a.e. } \tau \in (0,T),$$

where $P(s) = (1+s)\ln(1+s) - s$. Moreover, the following version of Poincare's inequality holds

$$\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; |v-u|^2 \right\rangle \mathrm{d}x\mathrm{d}t \le c\mathcal{D}(\tau).$$

We introduce the relative energy functional

$$\mathcal{E}\left(\varrho, u \mid r, U\right) = \int_\Omega \left[\frac{1}{2}\varrho\,|u-U|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)\right] \mathrm{d}x, \tag{3.5}$$

$$P(\varrho) = (1+\varrho)\ln(1+\varrho) - \varrho.$$

In fact it is shown in [8] that any finite energy weak solution (ϱ, u) to the compressible newtonian barotropic Navier-Stokes system satisfies the relative energy inequality for any pair (r, U) of sufficiently smooth test functions such that $r > 0$ and $U|_{\partial\Omega} = 0$ and this inequality is an essential tool in order to prove the convergence to a target system. For other details see [9].

In the framework of dissipative measure-valued solution (in the spirit of [11] and [12]) we define the functional

$$\mathcal{E}_{mv}\left(\varrho, u, \mid r, U\right) \equiv \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2}s\,|v-U|^2 + P(s) - P'(r)(\varrho - r) - P(r) \right\rangle \mathrm{d}x.$$

Theorem 9. *Let the parameterized measure* $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ *with*

$$\nu \in L^\infty_w\left((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times\mathbb{R}^N\right)\right), \quad \langle \nu_{t,x}; s\rangle \equiv \varrho, \quad \langle \nu_{t,x}; v\rangle \equiv u,$$

be a dissipative measure-valued solution to the compressible non-Newtonian system (1.1) - (1.2) with the initial condition ν_0 *and dissipation defect* $\mathcal{D}$*. Then,* (s, ν) *satisfies the following relative energy inequality*

$$\mathcal{E}_{mv} + \int_0^\tau \int_\Omega S\left(\widehat{u}, \mathit{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right)(e(u) - e(U)) + \mathcal{D}(\tau)$$

$$\le \int_\Omega \left\langle \nu_{0,x}; \left(\frac{1}{2}s\,|v - U(0,\cdot)|^2\right) + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle dx$$

$$+ \int_0^\tau \mathcal{R}(s, v, r, U)(t)\,dt \tag{3.6}$$

for a.a. $\tau \in (0,T)$ *and any pair of test functions* (r,U) *such that* $U \in C^1([0,T]\times \overline{\Omega}, R^n)$, $U|_{\partial\Omega} = 0$, $r \in C_c^\infty(\overline{Q_T})$, $r > 0$, *where*

$$\mathcal{R}(s,v,r,U)(t) = -\int_\Omega \left(\langle \nu_{t,x}; sv\rangle \partial_t U + \langle \nu_{t,x}; sv\otimes v\rangle \cdot \nabla_x U\right) dx$$

$$\int_0^\tau \int_\Omega \left(\langle \nu_{t,x}; -p(s)\rangle\, div_x U\right)\ dx$$

$$\int_\Omega \left(\langle \nu_{t,x}; s\rangle\, U\partial_t U + \langle \nu_{t,x}; sv\rangle \cdot U\cdot \nabla_x U\right) dx$$

$$-\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; (1-\frac{s}{r})\right\rangle p'(r)\partial_t r - \langle \nu_{t,x}: sv\rangle \cdot \frac{p'(r)}{r}\nabla_x r\, dx$$

$$+\int_0^\tau \left\langle r^M; \nabla_x U\right\rangle dt + \int_0^\tau \int_\Omega \left\langle r^C; \frac{1}{2}\nabla_x |U|^2 - \nabla_x P'(r)\right\rangle dx. \tag{3.7}$$

Proof. Using the continuity equation (3.2) with test function $\frac{1}{2}|U|^2$, we get

$$\int_\Omega \frac{1}{2}\langle \nu_{t,x}; s\rangle |U|^2 (\tau,\cdot)\,\mathrm{d}x - \int_\Omega \frac{1}{2}\langle \nu_0; s\rangle |U|^2(0,\cdot)\,\mathrm{d}x$$

$$= \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; s\rangle\, U\partial_t U + \langle \nu_{t,x}; sv\rangle \cdot U\cdot \nabla_x U\right]\mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^C; \frac{1}{2}\nabla_x U\right\rangle \mathrm{d}t, \tag{3.8}$$

provided $U \in C^1\left([0,T]\times\overline{\Omega};\mathbb{R}^N\right)$. Testing (3.2) by $P'(r)$

$$\int_\Omega \langle \nu_{t,x}; s\rangle\, P'(r)\,(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \langle \nu_0; s\rangle\, P'(r)\,(0,\cdot)\,\mathrm{d}x$$

$$= \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; s\rangle\, P''(r)\partial_t r + \langle \nu_{t,x}; sv\rangle\, P''(r)\cdot \nabla_x r\right]\mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^C; \nabla_x P'(r)\right\rangle \mathrm{d}t$$

$$= \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; s\rangle\, \frac{p'(r)}{r}\partial_t r + \langle \nu_{t,x}; sv\rangle\, \frac{p'(r)}{r}\cdot \nabla_x r\right]\mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^C; \nabla_x P'(r)\right\rangle \mathrm{d}t, \tag{3.9}$$

provided $r \in C^1\left([0,T]\times\overline{\Omega};\mathbb{R}^N\right)$. Moreover, we use (3.4) tested by U

$$\int_\Omega \langle \nu_{t,x}; sv\rangle\, U\,(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \langle \nu_0; sv\rangle\, U\,(0,\cdot)\,\mathrm{d}x$$

$$= \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; sv\rangle\, \partial_t U + \langle \nu_{t,x}; s\,(v\otimes v)\rangle : \nabla_x U + \langle \nu_{t,x}; p(s)\rangle\, \mathrm{div}_x U\right]\mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_\Omega S \left(\widehat{u}, \mathrm{div}_x u, \det \left(\frac{\partial u_i}{\partial x_j} \right) \right) : \nabla_x U \mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^M ; \nabla_x U \right\rangle \mathrm{d}t, \quad (3.10)$$

for any $U \in C^1 \left([0,T] \times \overline{\Omega}; \mathbb{R}^N \right),\ U|_{\partial\Omega} = 0$. Summing up (3.8) - (3.10), we get (3.6) - (3.7). □

Acknowledgment: *Š. N., H. Al B., M. C. acknowledge the support of the GAČR (Czech Science Foundation) project 16-03230S in the framework of RVO: 67985840. B. D. is partially supported by the ANR project INFAMIE (ANR-15-CE40-0011).*

References

[1] Al Baba H., M. Caggio, B. Ducomet, Š. Nečasová, Dissipative measure-valued solutions to the compressible non-Newtonian system, in preparation.

[2] Alibert J. J., G. Bouchitté, Non-uniform integrability and generalized Young measures, *J. Convex Anal.*, **4(1)**, 129-147, 1997.

[3] Ball J. M., A version of the fundamental theorem for Young measures, *Lect. Notes in Physics*, **344**, 207-215, 1989.

[4] Bellout, H., F. Bloom, Incompressible bipolar and non-Newtonian viscous fluid flow, *Advances in Mathematical Fluid Mechanics.*, Birkhäuser/Springer, Cham, 2014.

[5] Dafermos C. M., The second law of thermodynamics and stability, *Arch. Rational Mech. Anal.*, **70**, 167-179, 1979.

[6] DiPerna R. J., Measure-valued solutions to conservation laws, *Arch. Rational Mech. Anal.*, **88(3)**, 223-270, 1985.

[7] DiPerna R. J., A. Majda, Oscillations and concentrations in weak solutions of the incompressible fluid equations, *Comm. Math. Phys.*, **108(4)**, 667-689, 1987.

[8] Feireisl E., B. J. Jin, A. Novotný, Relative entropies, suitable weak solutions and weak-strong uniqueness for the compressible Navier-Stokes system, *J. Math. Fluid Mech.*, **14(4)**, 717-730, 2012.

[9] Feireisl E., A. Novotný, Inviscid Incompressible Limits of the Full Navier-Stokes-Fourier System, *Commun. Math. Phys.*, **321**, 605-628, 2013.

[10] Germain P., Weak-strong uniqueness for the isentropic compressible Navier-Stokes system, *J. Math. Fluid Mech.*, **13**, 137-146, 2011.

[11] Gwiazda P., A. Swierczewska-Gwiazda, E. Wiedemann, Weak-strong uniqueness for measure-valued solutions of some compressible fluid models, *Nonlinearity*, **28(11)**, 3873-3890, 2015.

[12] Feireisl E., P. Gwiazda, A. Swierczewska-Gwiazda, E. Wiedemann, Dissipative measure-valued solutions to the compressible Navier-Stokes system, *Calc. Var. Partial Differential Equations*, **55(6)**, 141, 2016.

[13] Málek J., J. Nečas, M. Rokyta, M. Ružička, Weak and Measure-valued Solutions to Evolutionary PDEs, *Applied Mathematics and Mathematical Computation,*, **13**, Chapman & Hall, London, 1996.

[14] Mellet A., A. Vasseur, Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations, *SIAM J. Math. Anal.*, **39**, 1344-1365, 2007.

[15] Kufner A., O. John, S. Fučik, Functional Spaces, *Academia, Prague*, 1977.

[16] Kristensen J., F. Rindler, Characterization of generalized gradient Young measures generated by sequences in W1,1 and BV. *Arch. Ration. Mech. Anal.* **197(2)**, 539-598, 2010.

[17] Kristensen, J., F. Rindler, Erratum to: Characterization of generalized gradient Young measures generated by sequences in W1,1 and BV [MR2660519]. *Arch. Ration. Mech. Anal.* **203(2)**, 693-700, 2012.

[18] Neustupa J., Measure-valued solutions of the Euler and Navier-Stokes equations for compressible barotropic fluids, *Math. Nachr.*, **163**, 217-227, 1993.

[19] Matušu - Nečasová Š., A. Novotný, Measure-valued solution for non-Newtonian compressible isothermal monopolar fluid, Mathematical problems for Navier-Stokes equations (Centro, 1993), *Acta Appl. Math.*, **37(1-2)**, 109-128, 1994.

[20] Nečas J., M. Šilhavý, Multipolar viscous fluids, *Quart. Appl. Math.*, **49(2)**, 247-265, 1991.

[21] Székelyhidi L., E. Wiedemann, Young measures generated by ideal incompressible fluid flows, *Arch. Ration. Mech. Anal.* **206(1)**, 333-366, 2012.