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Abstract

We study the Cauchy problem for a system of equations corresponding to a singular limit of radiative hydrodynamics, namely the 3D radiative compressible Euler system coupled to an electromagnetic field. Assuming smallness hypotheses for the data, we prove that the problem admits a unique global smooth solution and study its asymptotics.

Keywords: compressible, Euler, radiation hydrodynamics.
AMS subject classification: 35Q30, 76N10

1 Introduction

In [3], after the studies of Lowrie, Morel and Hittinger [15] and Buet and Després [5] we considered a singular limit for a compressible inviscid radiative flow where the motion of the fluid is given by the Euler system for the evolution of the density $\varrho = \varrho(t, x)$, the velocity field $\vec{u} = \vec{u}(t, x)$, and the absolute temperature $\vartheta = \vartheta(t, x)$, and where radiation is described in the limit by an extra temperature $T_r = T_r(t, x)$. All of these quantities are functions of the time t and the Eulerian spatial coordinate $x \in \mathbb{R}^3$.

In [3] we proved that the associated Cauchy problem admits a unique global smooth solution, provided that the data are small enough perturbations of a constant state.

In [4] we coupled the previous model to the electromagnetic field through the so called magnetohydrodynamic (MHD) approximation, in presence of thermal and radiative dissipation. Hereafter we consider the perfect non-isentropic Euler-Maxwell's system and we also consider a radiative coupling through a pure convective transport equation for the radiation (without dissipation). Then we deal with a pure hyperbolic system with partial relaxation (damping on velocity).

More specifically the system of equations to be studied for the unknowns $(\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})$ reads

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x(p + p_r) = -\rho \left(\vec{E} + \vec{u} \times \vec{B}\right) - \nu \rho \vec{u}, \tag{1.2}$$

$$\partial_t (\varrho E) + \mathrm{div}_x ((\varrho E + p)\vec{u}) + \vec{u} \cdot \nabla_x p_r = -\sigma_a \left(a\vartheta^4 - E_r\right) - \rho \vec{E} \cdot \vec{u}, \tag{1.3}$$

$$\partial_t E_r + \mathrm{div}_x (E_r \vec{u}) + p_r \mathrm{div}_x \vec{u} = -\sigma_a \left(E_r - a\vartheta^4\right), \tag{1.4}$$

$$\partial_t \vec{B} + \mathrm{curl}_x \vec{E} = 0, \tag{1.5}$$

$$\partial_t \vec{E} - \mathrm{curl}_x \vec{B} = \varrho \vec{u}, \tag{1.6}$$

$$\mathrm{div}_x \vec{B} = 0, \tag{1.7}$$

$$\mathrm{div}_x \vec{E} = \overline{\varrho} - \varrho, \tag{1.8}$$

where ϱ is the density, $\vec{u}$ the velocity, ϑ the temperature of matter, $E = \frac{1}{2}|\vec{u}|^2 + e(\varrho, \vartheta)$ is the total mechanical energy, E_r is the radiative energy related to the temperature of radiation T_r by $E_r = aT_r^4$ and p_r is the radiative pressure given by $p_r = \frac{1}{3}aT_r^4 = \frac{1}{3}\ E_r$, with $a > 0$. Finally $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic induction,

We assume that the pressure $p(\varrho, \vartheta)$ and the internal energy $e(\varrho, \vartheta)$ are positive smooth functions of their arguments with

$$C_v := \frac{\partial e}{\partial \vartheta} > 0, \quad \frac{\partial p}{\partial \varrho} > 0,$$

and we also suppose for simplicity that $\nu = \frac{1}{\tau}$ (where $\tau > 0$ is a momentum-relaxation time), μ, σ_a and a are positive constants.

A simplification appears if one observes that, provided that equations (1.7) and (1.8) are satisfied at $t = 0$, they are satisfied for any time $t > 0$ and consequently they can be discarded from the analysis below.

Notice that the reduced system (1.1)-(1.4) is the non equilibrium regime of radiation hydrodynamics introduced by Lowrie, Morel and Hittinger [15] and more recently by Buet and Després [5], and studied mathematically by Blanc, Ducomet and Nečasová [3]. Extending this last analysis, our goal in this work is to prove global existence of solutions for the system (1.1) - (1.8) when data are sufficiently close to an equilibrium state, and study their large time behaviour.

Just mention for completeness that related non isentropic Euler-Maxwell systems have been the object of a number of studies in the recent past. Let us quote some recent works: Y. Feng, S. Wang, S. Kawashima [9], Y. Feng, S. Wang, X. Li [10], J.W. Jerome [12], C. Lin, T. Goudon [14], Z. Tan, Y. Wang [17] and J. Xiu, J. Xiong [21].

In the following we show that the ideas used by Y. Ueda, S. Wang and S. Kawashima in [19] [20] in the isentropic case can be extended to the (radiative) non isentropic system (1.1-1.6). To this purpose we follow the following plan: in Section 2 we present the main results, then (Section 3) we prove well-posedness of system (1.1-1.6). Finally in Section 4 we prove the large time asymptotics of the solution.

2 Main results

We are going to prove that system (1.1)-(1.8) has a global smooth solution close to any equilibrium state. Namely we have

Theorem 2.1. *Let* $\left(\overline{\varrho}, 0, \overline{\vartheta}, \overline{E_r}, \overline{\vec{B}}, 0\right)$ *be a constant state with* $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ *and* $\overline{E_r} > 0$ *with compatibility condition* $\overline{E_r} = a\overline{\vartheta}^4$ *and suppose that* $d \geq 3$.

There exists $\varepsilon > 0$ *such that, for any initial state* $\left(\varrho_0, \vec{u}_0, \vartheta_0, E_r^0, \vec{B}_0, \vec{E}_0\right)$ *satisfying*

$$\mathrm{div}_x \vec{B}_0 = \varrho_0 - \overline{\varrho}, \quad \mathrm{div}_x \vec{B}_0 = 0,$$

$$\left(\varrho_0 - \overline{\varrho}, \vec{u}_0, \vartheta_0 - \overline{\vartheta}, E_{r0} - \overline{E_r}, \vec{B}_0 - \overline{\vec{B}}, \vec{E}_0\right) \in H^d,$$

and

$$\left\| \left(\varrho_0, \vec{u}_0, \vartheta_0, E_r^0, \vec{B}_0, \vec{E}_0\right) - \left(\overline{\varrho}, 0, \overline{\vartheta}, \overline{E_r}, \overline{\vec{B}}, 0\right) \right\|_{H^d} \leq \varepsilon, \tag{2.1}$$

there exists a unique global solution $\left(\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E}\right)$ *to* (1.1)-(1.8)*, such that*

$$\left(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E_r}, \vec{B} - \overline{\vec{B}}, \vec{E}\right) \in C\left([0, +\infty); H^d\right) \cap C^1\left([0, +\infty); H^{d-1}\right).$$

In addition, this solution satisfies the following energy inequality:

$$\left\| (\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E_r}, \vec{B} - \overline{\vec{B}}, \vec{E})(t) \right\|_{H^d}$$

$$+\int_0^t \left(\left\| \left(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E_r}\right)(\tau)\right\|^2_{H^d} + \left\|\nabla_x\vec{B}(\tau)\right\|^2_{H^{d-2}} + \left\|\vec{E}(\tau)\right\|^2_{H^{d-1}} \right)\, d\tau$$

$$\leq C\left\| \left(\varrho_0-\overline{\varrho},0,\vartheta_0-\overline{\vartheta},E_r^0-\overline{E_r},\vec{B}_0-\overline{\vec{B}},\vec{E}_0\right)\right\|^2_{H^d}, \tag{2.2}$$

for some constant $C>0$ which does not depend on t.

The large time behaviour of the solution is described as follows

Theorem 2.2. *Let $d\geq 3$.*

The unique global solution $\left(\varrho,\vec{u},\vartheta,E_r,\vec{B},\vec{E}\right)$ to (1.1)-(1.8) defined in Theorem 2.1 converges to the constant state $\left(\overline{\varrho},\vec{0},\overline{\vartheta},\overline{E_r},\overline{\vec{B}},\vec{0}\right)$ uniformly in $x\in\mathbb{R}^3$ as $t\to\infty$. More precisely

$$\left\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E_r},\vec{E})(t)\right\|_{W^{d-2,\infty}} \to 0 \quad \text{as } t\to\infty. \tag{2.3}$$

Moreover if $d\geq 4$

$$\left\|(\vec{B}-\overline{\vec{B}})(t)\right\|_{W^{d-4,\infty}} \to 0 \quad \text{as } t\to\infty, \tag{2.4}$$

Remark 2.1. *Note that, due to lack of dissipation by viscous, thermal and radiative fluxes, the Kawashima-Shizuta stability criterion (see [18] and [1]) is not satisfied for the system under study and techniques of [13] relying on the existence of a compensating matrix do not apply. However we will check that radiative sources play the role of relaxation terms for temperature and radiative energy and will lead to global existence for the system.*

3 Global existence

3.1 A priori estimates

Multiplying (1.2) by $\vec{u}$, (1.5) by $\vec{B}$, (1.6) by $\vec{E}$ and adding the result to equations (1.3) and (1.4) we get the total energy conservation law

$$\partial_t\left(\frac{1}{2}\varrho\left|\vec{u}\right|^2+\varrho e+E_r+\frac{1}{2}\left(|\vec{B}|^2+|\vec{E}|^2\right)\right)+\operatorname{div}_x\left((\varrho E+E_r)\vec{u}+(p+p_r)\vec{u}+\vec{E}\times\vec{B}\right)=0. \tag{3.1}$$

Introducing the entropy s of the fluid by the Gibbs law $\vartheta ds = de + pd\left(\frac{1}{\varrho}\right)$ and denoting by $S_r := \frac{4}{3}aT_r^3$ the radiative entropy, equation (1.4) rewrites

$$\partial_t S_r + \operatorname{div}_x\left(S_r\vec{u}\right) = -\sigma_a\frac{E_r-a\vartheta^4}{T_r}. \tag{3.2}$$

The internal energy equation is

$$\partial_t(\varrho e)+\operatorname{div}_x(\varrho e\vec{u})+p\operatorname{div}_x\vec{u}-\nu\varrho\left|\vec{u}\right|^2=-\sigma_a\left(a\vartheta^4-E_r\right), \tag{3.3}$$

and dividing it by ϑ, we get the entropy equation for matter

$$\partial_t(\varrho s)+\operatorname{div}_x(\varrho s\vec{u})-\frac{\nu}{\vartheta}\left|\vec{u}\right|^2=-\sigma_a\frac{a\vartheta^4-E_r}{\vartheta}. \tag{3.4}$$

So adding (3.4) and (3.2) we obtain

$$\partial_t\left(\varrho s+S_r\right)+\operatorname{div}_x\left((\varrho s+S_r)\vec{u}\right)=\frac{a\sigma_a}{\vartheta T_r}\left(\vartheta-T_r\right)^2\left(\vartheta+T_r\right)\left(\vartheta^2+T_r^2\right)+\frac{\nu}{\vartheta}\left|\vec{u}\right|^2. \tag{3.5}$$

Subtracting (3.5) from (3.1) and using the conservation of mass, we get

$$\partial_t \left(\frac{1}{2}\varrho\,|\vec{u}|^2 + H_{\overline{\vartheta}}(\varrho,\vartheta) - (\varrho-\overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) + H_{r,\overline{\vartheta}}(T_r) + \frac{1}{2}\left(|\vec{B}-\overline{\vec{B}}|^2 + |\vec{E}|^2\right)\right)$$

$$= \mathrm{div}_x \left((\varrho E + E_r)\vec{u} + (p+p_r)\vec{u} + \overline{\vartheta}(\varrho s + S_r)\vec{u}\right) - \overline{\vartheta}\frac{a\sigma_a}{\vartheta T_r}\left(\vartheta - T_r\right)^2\left(\vartheta + T_r\right)\left(\vartheta^2 + T_r^2\right) - \frac{\nu}{\vartheta}\,|\vec{u}|^2\,. \quad (3.6)$$

Introducing the Helmholtz functions $H_{\overline{\vartheta}}(\varrho,\vartheta) := \varrho\left(e - \overline{\vartheta}s\right)$ and $H_{r,\overline{\vartheta}}(T_r) := E_r - \overline{\vartheta}S_r$, we check that the quantities $H_{\overline{\vartheta}}(\varrho,\vartheta) - (\varrho-\overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$ and $H_{r,\overline{\vartheta}}(T_r) - H_{r,\overline{\vartheta}}(\overline{T}_r)$ are non-negative and strictly coercive functions reaching zero minima at the equilibrium state $(\overline{\varrho},\overline{\vartheta},\overline{E}_r)$.

Lemma 1. *Let $\overline{\varrho}$ and $\overline{\vartheta} = \overline{T}_r$ be given positive constants. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the sets defined by*

$$\mathcal{O}_1 := \left\{ (\varrho,\vartheta)\in\mathbb{R}^2 \;:\; \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho},\; \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta}, \right\}. \quad (3.7)$$

$$\mathcal{O}_2 := \left\{ T_r \in \mathbb{R} \;:\; \frac{\overline{T}_r}{2} < T_r < 2\overline{T}_r, \right\}. \quad (3.8)$$

There exist positive constants $C_{1,2}(\overline{\varrho},\overline{\vartheta})$ and $C_{3,4}(\overline{T}_r)$ such that

1.
$$C_1\left(|\varrho-\overline{\varrho}|^2 + |\vartheta-\overline{\vartheta}|^2\right) \le H_{\overline{\vartheta}}(\varrho,\vartheta) - (\varrho-\overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$$
$$\le C_2\left(|\varrho-\overline{\varrho}|^2 + |\vartheta-\overline{\vartheta}|^2\right), \quad (3.9)$$

for all $(\varrho,\vartheta)\in\mathcal{O}_1$,

2.
$$C_3|T_r - \overline{T}_r|^2 \le H_{r,\overline{\vartheta}}(T_r) - H_{r,\overline{\vartheta}}(\overline{T}_r) \le C_4|T_r-\overline{T}_r|^2, \quad (3.10)$$

for all $T_r\in\mathcal{O}_2$.

Proof:

1. Point 1 is proved in [8] and we only sketch the proof for convenience. According to the decomposition

$$\varrho \to H_{\overline{\vartheta}}(\varrho,\vartheta) - (\varrho-\overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) = \mathcal{F}(\varrho) + \mathcal{G}(\varrho),$$

where $\mathcal{F}(\varrho) = H_{\overline{\vartheta}}(\varrho,\overline{\vartheta}) - (\varrho-\overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$ and $\mathcal{G}(\varrho) = H_{\overline{\vartheta}}(\varrho,\vartheta) - H_{\overline{\vartheta}}(\varrho,\overline{\vartheta})$, one checks that $\mathcal{F}$ is strictly convex and reaches a zero minimum at $\overline{\varrho}$, while $\mathcal{G}$ is strictly decreasing for $\vartheta < \overline{\vartheta}$ and strictly increasing for $\vartheta > \overline{\vartheta}$, according to the standard thermodynamic stability properties [8]. Computing the derivatives of $H_{\overline{\vartheta}}$ leads directly to the estimate (3.9).

2. Point 2 follows after properties of $x \to H_{r,\overline{\vartheta}}(x) - H_{r,\overline{\vartheta}}(T_r) = ax^3(x - \frac{4}{3}\overline{\vartheta}) + \frac{a}{3}\overline{\vartheta}^4$. □

Using the previous entropy properties, we have the energy estimate

Proposition 3.1. *Let the assumptions of Theorem 2.1 be satisfied with $V = \left(\rho,\vec{u},\vartheta,E_r,\vec{B},\vec{E}\right)$, $\overline{V} = \left(\overline{\rho},0,\overline{\vartheta},\overline{E_r},\overline{\vec{B}},0\right)$. Consider a solution $(\varrho,\vec{u},\vartheta,E_r,\vec{B},\vec{E})$ of system (1.1)-(1.2)-(1.3) on $[0,t]$, for some $t>0$. Then, one gets for a constant $C_0>0$*

$$\left\|V(t)-\overline{V}\right\|_{L^2}^2 + \int_0^t \|\; \vec{u}(\tau)\|_{L^2}^2\,d\tau \le C_0\left\|V_0-\overline{V}\right\|_{L^2}^2. \quad (3.11)$$

Proof: Defining

$$\eta(t,x) = H_{\overline{\vartheta}}(\varrho,\vartheta) - (\varrho - \overline{\varrho})\,\partial_\varrho H_{\overline{\vartheta}}\left(\overline{\varrho},\overline{\vartheta}\right) - H_{\overline{\vartheta}}\left(\overline{\varrho},\overline{\vartheta}\right) + H_{r,\overline{\vartheta}}\left(T_r\right), \tag{3.12}$$

we multiply (3.5) by $\overline{\vartheta}$, and subtract the result to (3.1). Integrating over $[0,t]\times\mathbb{R}^3$, we find

$$\int_{\mathbb{R}^3} \frac{1}{2}\varrho\,|\vec{u}|^2 + \eta(t,x) + \frac{1}{2}|\vec{B}-\overline{\vec{B}}|^2 + \frac{1}{2}|\vec{E}|^2 dx + \int_0^t \int_{\mathbb{R}^3} \frac{\overline{\vartheta}}{\vartheta}\nu\,|\vec{u}|^2$$
$$\leq \int_{\mathbb{R}^3} \frac{1}{2}\varrho_0\,|\vec{u}_0|^2\,(t) + \eta(0,x) + \frac{1}{2}|\vec{B}_0-\overline{\vec{B}}|^2 + \frac{1}{2}|\vec{E}_0|^2\;dx.$$

Applying Lemma 1, we find (3.11). □

Defining for any $d \geq 3$ the auxiliary quantities

$$E(t) := \sup_{0\leq\tau\leq t} \|(\varrho-\overline{\varrho},\vec{u},\vec{B}-\overline{\vec{B}},\vec{E})(\tau)\|_{W^{1,\infty}},$$

$$F(t) := \sup_{0\leq\tau\leq t} \|(V-\overline{V})(\tau)\|_{H^d},$$

$$I^2(t) := \int_0^t \left\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\right\|_{L^\infty}^2 d\tau,$$

and

$$D^2(t) := \int_0^t \left(\left\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(\tau)\right\|_{H^d}^2 + \left\|\vec{E}(\tau)\right\|_{H^{d-1}}^2 + \left\|\partial_x\vec{B}(\tau)\right\|_{H^{d-2}}^2\right) d\tau,$$

we can bound the spatial derivatives as follows

Proposition 3.2. *Assume that the hypotheses of Theorem 2.1 are satisfied. Then, we have for a $C_0 > 0$*

$$\|\partial_x V(t)\|_{H^{d-1}}^2 + \int_0^t \|\partial_x \vec{u}(\tau)\|_{H^{d-1}}^2\;d\tau \leq C_0\,\|\partial_x V_0\|_{H^{d-1}}^2 + C_0\left(E(t)D(t)^2 + F(t)I(t)D(t)\right). \tag{3.13}$$

Proof: Rewriting the system (1.1)-(1.6) in the form

$$\begin{cases}
\partial_t\varrho + \vec{u}\cdot\nabla_x\varrho + \varrho\mathrm{div}_x\vec{u} = 0,\\
\partial_t\vec{u} + (\vec{u}\cdot\nabla_x)\vec{u} + \frac{p_\varrho}{\varrho}\nabla_x\varrho + \frac{p_\vartheta}{\varrho}\nabla_x\vartheta + \frac{1}{3a\varrho}\nabla_x E_r + \vec{E} + \vec{u}\times\overline{\vec{B}} + \nu\vec{u} = -\vec{u}\times\left(\vec{B}-\overline{\vec{B}}\right),\\
\partial_t\vartheta + (\vec{u}\cdot\nabla_x)\vartheta + \frac{\vartheta p_\vartheta}{\varrho C_v}\mathrm{div}_x\vec{u} = -\frac{\sigma_a}{\varrho C_v}\left(a\vartheta^4 - E_r\right),\\
\partial_t E_r + (\vec{u}\cdot\nabla_x)E_r + \frac{4}{3}E_r\mathrm{div}_x\vec{u} = -\sigma_a\left(E_r - a\vartheta^4\right),\\
\partial_t\vec{B} + \mathrm{curl}_x\vec{E} = 0,\\
\partial_t\vec{E} - \mathrm{curl}_x\vec{B} - \overline{\varrho}\vec{u} = (\varrho-\overline{\varrho})\vec{u},
\end{cases} \tag{3.14}$$

and applying ∂_x^ℓ to this system, we get

$$\partial_t(\partial_x^\ell\varrho) + (\vec{u}\cdot\nabla_x)\partial_x^\ell\varrho + \varrho\mathrm{div}_x\partial_x^\ell\vec{u} = F_1^\ell,$$
$$\partial_t(\partial_x^\ell\vec{u}) + (\vec{u}\cdot\nabla_x)\partial_x^\ell\vec{u} + \frac{p_\varrho}{\varrho}\nabla_x\partial_x^\ell\varrho + \frac{p_\vartheta}{\varrho}\nabla_x\partial_x^\ell\vartheta + \frac{1}{3a\varrho}\nabla_x\partial_x^\ell E_r + \partial_x^\ell\vec{E} + \partial_x^\ell\vec{u}\times\overline{\vec{B}} + \nu\partial_x^\ell\vec{u} = -\partial_x^\ell\left[\vec{u}\times\left(\vec{B}-\overline{\vec{B}}\right)\right] + F_2^\ell,$$

$$\partial_t(\partial_x^\ell \vartheta) + (\vec{u}\cdot\nabla_x)\partial_x^\ell\vartheta + \frac{\vartheta p_\vartheta}{\varrho C_v}\mathrm{div}_x\partial_x^\ell\vec{u} = -\partial_x^\ell\left[\frac{\sigma_a}{\varrho C_v}\left(a\vartheta^4 - E_r\right)\right] + F_3^\ell,$$

$$\partial_t(\partial_x^\ell E_r) + (\vec{u}\cdot\nabla_x)\partial_x^\ell E_r + \frac{4}{3}E_r\mathrm{div}_x\partial_x^\ell\vec{u} = -\partial_x^\ell\left[\sigma_a\left(E_r - a\vartheta^4\right)\right] + F_4^\ell,$$

$$\partial_t(\partial_x^\ell\vec{B}) + \mathrm{curl}_x\partial_x^\ell\vec{E} = 0,$$

$$\partial_t(\partial_x^\ell\vec{E}) - \mathrm{curl}_x\partial_x^\ell\vec{B} - \overline{\varrho}\partial_x^\ell\vec{u} = \partial_x^\ell[(\varrho-\overline{\varrho})\vec{u}],$$

where

$$F_1^\ell := -\left[\partial_x^\ell, \vec{u}\cdot\nabla_x\right]\vec{u} - \left[\partial_x^\ell, \varrho\mathrm{div}_x\right]\vec{u},$$

$$F_2^\ell := -\left[\partial_x^\ell, \vec{u}\cdot\nabla_x\right]\vec{u} - \left[\partial_x^\ell, \frac{p_\varrho}{\varrho}\nabla_x\right]\varrho - \left[\partial_x^\ell, \frac{p_\vartheta}{\varrho}\nabla_x\right]\vartheta - \left[\partial_x^\ell, \frac{1}{3a\varrho}\nabla_x\right]E_r,$$

$$F_3^\ell := -\left[\partial_x^\ell, \vec{u}\cdot\nabla_x\right]\vartheta - \left[\partial_x^\ell, \frac{\vartheta p_\vartheta}{\varrho C_v}\mathrm{div}_x\right]\vec{u},$$

and

$$F_4^\ell := -\left[\partial_x^\ell, \vec{u}\cdot\nabla_x\right]E_r - \left[\partial_x^\ell, \frac{4}{3}E_r\mathrm{div}_x\right]\vec{u}.$$

Then taking the scalar product of each of the previous equations respectively by $\frac{p_\varrho}{\varrho^2}\partial_x^\ell\varrho,\ \partial_x^\ell\vec{u},\ \frac{C_v}{\vartheta}\partial_x^\ell\vartheta,\ \frac{1}{4a\varrho E_r}\partial_x^\ell E_r,\ \partial_x^\ell\vec{B},$ and $\partial_x^\ell\vec{E}$ and adding the resulting equations, we get

$$\partial_t\mathcal{E}^\ell + \mathrm{div}_x\vec{\mathcal{F}}^\ell + \nu\left(\partial_x^\ell\vec{u}\right)^2 = \mathcal{R}^\ell + \mathcal{S}^\ell, \tag{3.15}$$

where

$$\mathcal{E}^\ell := \frac{1}{2}\ \left(\partial_x^\ell\vec{u}\right)^2 + \frac{1}{2}\ \frac{p_\varrho}{\varrho}\left(\partial_x^\ell\varrho\right)^2 + \frac{1}{2}\ \frac{C_v}{\vartheta}\left(\partial_x^\ell\vartheta\right)^2 + \frac{1}{2}\ \frac{1}{4a\varrho E_r}\left(\partial_x^\ell E_r\right)^2 + \frac{1}{2}\ \left(\partial_x^\ell\vec{E}\right)^2 + \frac{1}{2}\ \left(\partial_x^\ell\vec{B}\right)^2$$

$$\vec{\mathcal{F}}^\ell := \left(\frac{p_\varrho}{\varrho}\partial_x^\ell\varrho + \frac{p_\vartheta}{\varrho}\partial_x^\ell\vartheta + \frac{1}{3a\varrho}\partial_x^\ell E_r\right)\partial_x^\ell\vec{u} + \frac{1}{2}\ \left(\left(\partial_x^\ell\vec{u}\right)^2 + \frac{p_\varrho}{\varrho}\left(\partial_x^\ell\varrho\right)^2 + \frac{C_v}{\vartheta}\left(\partial_x^\ell\vartheta\right)^2 + \frac{1}{4a\varrho E_r}\left(\partial_x^\ell E_r\right)^2\right)\vec{u}$$

$$\begin{aligned}\mathcal{R}^\ell :=\ & \frac{1}{2}\left[\frac{p_\varrho}{\varrho^2}\right]_t\left(\partial_x^\ell\varrho\right)^2 + \frac{1}{2}\left[\frac{C_v}{\vartheta}\right]_t\left(\partial_x^\ell\vartheta\right)^2 + \frac{1}{2}\left[\frac{1}{4a\varrho E_r}\right]_t\left(\partial_x^\ell E_r\right)^2\\
&+\frac{1}{2}\mathrm{div}_x\left(\frac{p_\varrho}{\varrho^2}\vec{u}\right)\left(\partial_x^\ell\varrho\right)^2 + \frac{1}{2}\mathrm{div}_x\vec{u}\left(\partial_x^\ell\vec{u}\right)^2 + \frac{1}{2}\mathrm{div}_x\left(\frac{C_v}{\vartheta}\vec{u}\right)\left(\partial_x^\ell\vartheta\right)^2 + \frac{1}{2}\mathrm{div}_x\left(\frac{1}{4a\varrho E_r}\vec{u}\right)\left(\partial_x^\ell E_r\right)^2\\
&+\nabla_x\left(\frac{p_\varrho}{\varrho}\right)\partial_x^\ell\varrho\ \partial_x^\ell\vec{u} + \nabla_x\left(\frac{p_\vartheta}{\varrho}\right)\partial_x^\ell\vartheta\ \partial_x^\ell\vec{u} + \nabla_x\left(\frac{1}{3a\varrho}\right)\partial_x^\ell E_r\ \partial_x^\ell\vec{u}\\
&+\frac{p_\varrho}{\varrho^2}\partial_x^\ell\varrho\ F_1^\ell + \partial_x^\ell\vec{u}\ F_2^\ell + \frac{C_v}{\vartheta}\partial_x^\ell\vartheta\ F_3^\ell + \partial_x^\ell E_r\ F_4^\ell + \overline{\varrho}\partial_x^\ell\vec{E}\cdot\partial_x^\ell\vec{u},\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}^\ell :=\ & -\partial_x^\ell\vec{u}\cdot\partial_x^\ell\left[\vec{u}\times\left(\vec{B}-\overline{\vec{B}}\right)\right] - \frac{C_v}{\vartheta}\partial_x^\ell\vartheta\ \partial_x^\ell\left[\frac{\sigma_a}{\varrho C_v}\left(a\vartheta^4 - E_r\right)\right]\\
&-\frac{1}{4a\varrho E_r}\partial_x^\ell E_r\ \partial_x^\ell\left[\sigma_a\left(E_r - a\vartheta^4\right)\right] + \partial_x^\ell\vec{E}\ \partial_x^\ell[(\varrho-\overline{\varrho})\vec{u}]\end{aligned}$$

Integrating (3.15) on space, one gets

$$\partial_t\int_{\mathbb{R}^3}\mathcal{E}^\ell dx + \left\|\partial_x^\ell\vec{u}\right\|_{L^2}^2 \le \int_{\mathbb{R}^3}\left(|\mathcal{R}^\ell| + |\mathcal{S}^\ell|\right)\,dx.$$

Integrating now with respect to t and summing on ℓ with $|\ell| \leq d$, we get

$$\|\partial_x V(t)\|_{H^{d-1}}^2 + \int_0^t \|\partial_x \vec{u}(\tau)\|_{H^{d-1}}^2 \, d\tau \leq C_0 \|\partial_x V_0\|_{H^{d-1}}^2 + C_0 \sum_{|\ell|=1}^{d} \int_{\mathbb{R}^3} (|\mathcal{R}^\ell| + |\mathcal{S}^\ell|) \, dx.$$

Observing that $|\partial_t \varrho| \leq C|\partial_x \varrho|$, $|\partial_t \vartheta| \leq C(|\partial_x \varrho| + |\partial_x \vartheta| + |\partial_x E_r||\Delta\vartheta|)$ and $|\partial_t E_r| \leq C(|\partial_x \varrho| + |\partial_x \vartheta| + |\partial_x E_r|)$, and that, using commutator estimates (see Moser-type calculus inequalities in [16])

$$\|(F_1^\ell, F_2^\ell, F_3^\ell, F_4^\ell)\|_{L^2} \leq \|\partial_x(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)\|_{L^\infty} \|\partial_x^\ell(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)\|_{L^2}^2,$$

we see that

$$|\mathcal{R}^\ell| \leq C \left(\|\partial_x \varrho\|_{L^\infty} + \|\partial_x \vec{u}\|_{L^\infty} + \|\partial_x \vartheta\|_{L^\infty} + \|\partial_x E_r\|_{L^\infty}\right) \|\partial_x^\ell(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)\|_{L^2}^2.$$

Then integrating with respect to time

$$\begin{aligned} &\int_0^t |\mathcal{R}^\ell(\tau)| \, d\tau \\ &\leq C \sup_{0 \leq \tau \leq t} \{\|\partial_x \varrho\|_{L^\infty} + \|\partial_x \vec{u}\|_{L^\infty} + \|\partial_x \vartheta\|_{L^\infty} + \|\partial_x E_r\|_{L^\infty}\} \int_0^t \|\partial_x^\ell(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)\|_{L^2}^2 d\tau \\ &\leq C E(t) D^2(t), \end{aligned}$$

for any $|\ell| \leq d$. In the same stroke, we estimate

$$\begin{aligned} |\mathcal{S}^\ell| \leq{}& C\|\partial_x^\ell \vec{u}\|_{L^2}^2 \left\|\partial_x^\ell \left[\vec{u} \times \left(\vec{B} - \overline{\vec{B}}\right)\right]\right\|_{L^2}^2 + C\|\partial_x^\ell \vartheta\|_{L^2}^2 \left\|\partial_x^\ell \left[\frac{\sigma_a}{\varrho C_v}\left(a\vartheta^4 - E_r\right)\right]\right\|_{L^2}^2 \\ &+ C\|\partial_x^\ell E_r\|_{L^2}^2 \left\|\partial_x^\ell \left[\sigma_a\left(E_r - a\vartheta^4\right)\right]\right\|_{L^2}^2 + C\|\partial_x^\ell \vec{E}\|_{L^2}^2 \left\|\partial_x^\ell [(\varrho - \overline{\varrho})\vec{u}]\right\|_{L^2}^2. \end{aligned}$$

Then we get

$$\begin{aligned} |\mathcal{S}^\ell| \leq{}& C\|\vec{B} - \overline{\vec{B}}\|_{L^\infty} \|\partial_x^\ell \vec{u}\|_{L^2}^2 \\ &+ C\|\left(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r\right)\|_{L^\infty} \|\partial_x^\ell\left(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r\right)\|_{L^2} \|\partial_x^\ell(\vec{B}, \vec{E})\|_{L^\infty} \\ &+ C\left(\|\partial_x \varrho\|_{L^\infty} + \|\partial_x \vec{u}\|_{L^\infty} + \|\partial_x \vartheta\|_{L^\infty} + \|\partial_x E_r\|_{L^\infty}\right) \|\partial_x^\ell(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)\|_{L^2}^2. \end{aligned}$$

Then integrating with respect to time

$$\begin{aligned} \int_0^t |\mathcal{S}^\ell(\tau)| \, d\tau \leq{}& C \sup_{0 \leq \tau \leq t} \|(\vec{B} - \overline{\vec{B}})(\tau)\|_{L^\infty} \int_0^t \|\partial_x^\ell \vec{u}(\tau)\|_{L^2}^2 d\tau \\ &+ C \sup_{0 \leq \tau \leq t} \|\partial_x^\ell(\vec{B}, \vec{E})(\tau)\|_{L^2} \int_0^t \|\left(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r\right)(\tau)\|_{L^\infty} \|\partial_x^\ell\left(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r\right)(\tau)\|_{L^2} d\tau \\ &+ C \sup_{0 \leq \tau \leq t} \{\|\partial_x \varrho\|_{L^\infty} + \|\partial_x \vec{u}\|_{L^\infty} + \|\partial_x \vartheta\|_{L^\infty} + \|\partial_x E_r\|_{L^\infty}(\tau)) \\ &\times \int_0^t \|\partial_x^\ell(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)\|_{L^2}^2 d\tau \\ \leq{}& C\left(E(t) D^2(t) + F(t) I(t) D(t)\right), \end{aligned}$$

for any $|\ell| \leq d$. ∎

The above results, together with (3.11), allow to derive the following energy bound:

Corollary 3.1. *Assume that the assumptions of Proposition 3.1 are satisfied. Then*

$$\left\|(V-\overline{V})(t)\right\|_{H^d}^2+\int_0^t\|\vec{u}(\tau)\|_{H^d}^2\;d\tau\le C\left\|(V-\overline{V})(0)\right\|_{H^d}^2+C\left(E(t)D(t)^2+F(t)I(t)D(t)\right). \tag{3.16}$$

Our goal is now to derive bounds for the integrals in the right-hand and left-hand sides of equation (3.16). For that purpose we adapt the results of Ueda, Wang and Kawashima [19].

Lemma 2. *Under the same assumptions as in Theorem 2.1, and supposing that $d\ge 3$, we have the following estimate for any $\varepsilon>0$*

$$\int_0^t\left(\|\left(\varrho-\overline{\varrho},\vartheta-\overline{\vartheta},E_r-\overline{E}_r\right)(\tau)\|_{H^d}^2+\left\|\vec{E}(\tau)\right\|_{H^{d-1}}^2\right)\;d\tau$$

$$\le\varepsilon\int_0^t\left\|\partial_x\vec{B}(\tau)\right\|_{H^{d-2}}^2\;d\tau+C_\varepsilon\left\{\|V_0-\overline{V}\|_{H^{d-1}}^2+E(t)D(t)^2+F(t)I(t)D(t)\right\}. \tag{3.17}$$

Proof: We linearize the principal part of the system (1.1)-(1.2)-(1.3) as follows

$$\partial_t\varrho+\overline{\varrho}\;\mathrm{div}_x\vec{u}=g_1, \tag{3.18}$$

$$\partial_t\vec{u}+\overline{a}_1\;\nabla_x\varrho+\overline{a}_2\;\nabla_x\vartheta+\overline{a}_3\;\nabla_xE_r+\vec{E}+\vec{u}\times\overline{\vec{B}}+\nu\vec{u}=g_2, \tag{3.19}$$

$$\partial_t\vartheta+\overline{b}_1\;\mathrm{div}_x\vec{u}+\overline{b}_2(\vartheta-\overline{\vartheta})=g_3, \tag{3.20}$$

$$\partial_tE_r+\overline{c}_1\;\mathrm{div}_x\vec{u}+\overline{c}_3(E_r-\overline{E}_r)=g_4, \tag{3.21}$$

$$\partial_t\vec{B}+\mathrm{curl}_x\vec{E}=0, \tag{3.22}$$

$$\partial_t\vec{E}-\mathrm{curl}_x\vec{B}-\overline{\varrho}\;\vec{u}=g_5, \tag{3.23}$$

with coefficients

$$a_1(\varrho,\vartheta)=\frac{p_\varrho}{\varrho},\quad a_2(\varrho,\vartheta)=\frac{p_\vartheta}{\varrho},\quad a_3(\varrho,\vartheta)=\frac{1}{3\varrho},\quad \overline{a}_j=a_j(\overline{\varrho},\overline{\vartheta}),$$

$$b_1(\varrho,\vartheta)=\frac{\vartheta p_\vartheta}{\varrho C_v},\quad b_2(\varrho,\vartheta,E_r)=\frac{a\sigma_a}{\varrho C_v}(\vartheta^2+\overline{\vartheta}^2)(\vartheta+\overline{\vartheta}),\quad b_3(\varrho,\vartheta,E_r)=\frac{a\sigma_a}{\varrho C_v},\quad \overline{b}_j=b_j(\overline{\varrho},\overline{\vartheta}),$$

$$c_1(\varrho,\vartheta,E_r)=\frac{4}{3}E_r,\quad c_2(\varrho,\vartheta,E_r)=a\sigma_a(\vartheta^2+\overline{\vartheta}^2)(\vartheta+\overline{\vartheta}),\quad c_3(\varrho,\vartheta,E_r)=\sigma_a,\quad \overline{c}_j=c_j(\overline{\varrho},\overline{\vartheta}),$$

and sources

$$g_1:=-\{\vec{u}\cdot\nabla_x\varrho+(\varrho-\overline{\varrho})\mathrm{div}_x\vec{u}\},$$

$$g_2:=-\left\{(\vec{u}\cdot\nabla_x)\vec{u}+(a_1-\overline{a}_1)\nabla_x\varrho+(a_2-\overline{a}_2)\nabla_x\vartheta+(a_3-\overline{a}_3)\nabla_xE_r+\vec{u}\times(\vec{B}-\overline{\vec{B}})\right\},$$

$$g_3:=-\left\{(\vec{u}\cdot\nabla_x)\vartheta+(b_1-\overline{b}_1)\mathrm{div}_x\vec{u}+(b_2-\overline{b}_2)(\vartheta-\overline{\vartheta})+b_3(E_r-\overline{E}_r)\right\},$$

$$g_4:=-\left\{(\vec{u}\cdot\nabla_x)E_r+(\overline{c}_1-c_1)\mathrm{div}_x\vec{u}+c_2(\vartheta-\overline{\vartheta})+(c_3-\overline{c}_3)(E_r-\overline{E}_r)\right\},$$

and

$$g_5=(\varrho-\overline{\varrho})\vec{u}.$$

Multiplying (3.18) by $-\overline{a}_1\mathrm{div}_x\vec{u}$, (3.19) by $\overline{a}_1\;\nabla_x\varrho+\overline{a}_2\;\nabla_x\vartheta+\overline{a}_3\;\nabla_xE_r+\vec{E}$, (3.20) by $-\overline{a}_2\mathrm{div}_x\vec{u}+\vartheta-\overline{\vartheta}$, (3.21) by $-\overline{a}_3\mathrm{div}_x\vec{u}+E_r-\overline{E}_r$, (3.22) by 1, (3.23) by $\vec{u}$ and summing up, we get

$$\overline{a}_1(\nabla_x\varrho\;\vec{u}_t-\varrho_t\;\mathrm{div}_x\vec{u})+\overline{a}_2(\nabla_x\vartheta\;\vec{u}_t-\vartheta_t\;\mathrm{div}_x\vec{u})+\overline{a}_3(\nabla_xE_r\;\vec{u}_t-(E_r)_t\;\mathrm{div}_x\vec{u})+\vec{E}\vec{u}_t+\vec{E}_t\vec{u}$$

$$+\left\{\frac{1}{2}\left[(\vartheta-\overline{\vartheta})^2+(E_r-\overline{E}_r)^2\right]\right\}_t$$

$$+(\overline{a}_1\nabla_x\varrho+\overline{a}_2\nabla_x\vartheta+\overline{a}_3\nabla_xE_r+\vec{E})^2+(\overline{a}_1\ \nabla_x\varrho+\overline{a}_2\ \nabla_x\vartheta+\overline{a}_3\ \nabla_xE_r+\vec{E})(\vec{u}\times\overline{\vec{B}}+\nu\vec{u})$$
$$+\overline{b}_2(\vartheta-\overline{\vartheta})^2+\overline{c}_3(E_r-\overline{E}_r)^2$$
$$+\overline{b}_1(\vartheta-\overline{\vartheta})\mathrm{div}_x\vec{u}+\overline{c}_1(E_r-\overline{E}_r)\mathrm{div}_x\vec{u}$$
$$+(\overline{a}_3\overline{c}_2-\overline{a}_2\overline{b}_2)(\vartheta-\overline{\vartheta})\mathrm{div}_x\vec{u}+(\overline{a}_2\overline{b}_3-\overline{a}_3\overline{c}_3)(E_r-\overline{E}_r)\mathrm{div}_x\vec{u}$$
$$-\vec{u}\ \mathrm{curl}_x\vec{B}-\overline{\varrho}\vec{u}^2-(\mathrm{div}_x\vec{u})^2\left[\overline{a}_1+\overline{a}_2+\overline{a}_3\right]=G_1^0, \tag{3.24}$$

where

$$G_1^0:=-\overline{a}_1g_1\mathrm{div}_x\vec{u}+[\overline{a}_1\nabla_x\varrho+\overline{a}_2\nabla_x\vartheta+\overline{a}_3\nabla_xE_r+\vec{E}]g_2-[\overline{a}_2+\vartheta-\overline{\vartheta}]\mathrm{div}_x\vec{u}g_3-[\overline{a}_3+E_r-\overline{E}_r]\mathrm{div}_x\vec{u}g_4+g_5\vec{u}.$$

Rearranging the left hand side of (3.24) we get

$$\{H_1^0\}_t+\mathrm{div}_x\vec{F}_1^0+D_1^0=M_1^0+G_1^0, \tag{3.25}$$

where

$$H_1^0=-\left[\overline{a}_1(\varrho-\overline{\varrho})+\overline{a}_2(\vartheta-\overline{\vartheta})+\overline{a}_3(E_r-\overline{E}_r)\right]\mathrm{div}_x\vec{u}+\vec{E}\cdot\vec{u}+\frac{1}{2}\left[(\vartheta-\overline{\vartheta})^2+(E_r-\overline{E}_r)^2\right],$$

$$\vec{F}_1^0=\left[\overline{a}_1(\varrho-\overline{\varrho})+\overline{a}_2(\vartheta-\overline{\vartheta})+\overline{a}_3(E_r-\overline{E}_r)\right]\vec{u}_t-2\left[\overline{a}_1(\varrho-\overline{\varrho})+\overline{a}_2(\vartheta-\overline{\vartheta})+\overline{a}_3(E_r-\overline{E}_r)\right]\vec{E}$$
$$+(\overline{a}_3\overline{c}_2-\overline{a}_2\overline{b}_2+\overline{b}_1)(\vartheta-\overline{\vartheta})\vec{u}+(\overline{a}_2\overline{b}_3-\overline{a}_3\overline{c}_3+\overline{c}_1)(E_r-\overline{E}_r)\vec{u},$$

$$D_1^0=\overline{a}_1^2|\nabla_x\varrho|^2+\overline{a}_2^2|\nabla_x\vartheta|^2+\overline{a}_3^2|\nabla_xE_r|^2+|\vec{E}|^2+2\overline{a}_1(\varrho-\overline{\varrho})^2+\overline{b}_2(\vartheta-\overline{\vartheta})^2+\overline{c}_3(E_r-\overline{E}_r)^2,$$

and

$$M_1^0=-\left\{2\overline{a}_1\overline{a}_2\nabla_x\varrho\cdot\nabla_x\vartheta+2\overline{a}_1\overline{a}_3\nabla_x\varrho\cdot\nabla_xE_r+2\overline{a}_2\overline{a}_3\nabla_x\vartheta\cdot\nabla_xE_r\right.$$
$$+2\overline{a}_2(\varrho-\overline{\varrho})(\vartheta-\overline{\vartheta})+2\overline{a}_2(\varrho-\overline{\varrho})(E_r-\overline{E}_r)$$
$$+(\overline{a}_1\ \nabla_x\varrho+\overline{a}_2\ \nabla_x\vartheta+\overline{a}_3\ \nabla_xE_r+\vec{E})(\vec{u}\times\overline{\vec{B}}+\nu\vec{u})-\vec{u}\ \mathrm{curl}_x\vec{B}-\overline{\varrho}\vec{u}^2-(\mathrm{div}_x\vec{u})^2\left[\overline{a}_1+\overline{a}_2+\overline{a}_3\right]$$
$$\left.-(\overline{a}_3\overline{c}_2-\overline{a}_2\overline{b}_2+\overline{b}_1)\nabla_x\vartheta\cdot\vec{u}-(\overline{a}_2\overline{b}_3-\overline{a}_3\overline{c}_3+\overline{c}_1)\nabla_xE_r\cdot\vec{u}\right\}.$$

Integrating (3.25) over space and using Young's inequality, we find

$$\frac{d}{dt}\int_{\mathbb{R}^3}H_1^0\ dx+C\left(\|\varrho\|_{L^2}^2+\|\nabla_x\vartheta\|_{L^2}^2+\|\nabla_xE_r\|_{L^2}^2+\|\vec{E}\|_{L^2}^2+\|\varrho-\overline{\varrho}\|_{L^2}^2\right)$$
$$\leq\varepsilon\|\partial_x\vec{B}\|_{L^2}^2+C_\varepsilon\left(\|\vec{u}\|_{H^1}^2+\|\vartheta-\overline{\vartheta}\|_{H^1}^2+\|E_r-\overline{E}_r\|_{H^1}^2\right)+\int_{\mathbb{R}^3}|G_1^0|\ dx. \tag{3.26}$$

In fact one obtains in the same way estimates for the derivatives of V.

Namely, applying ∂_x^ℓ to the system (3.18-3.23), we get

$$\{H_1^\ell\}_t+\mathrm{div}_x\vec{F}_1^\ell+D_1^\ell=M_1^\ell+G_1^\ell, \tag{3.27}$$

where

$$H_1^\ell=-\left[\overline{a}_1\partial_x^\ell(\varrho-\overline{\varrho})+\overline{a}_2\partial_x^\ell(\vartheta-\overline{\vartheta})+\overline{a}_3\partial_x^\ell(E_r-\overline{E}_r)\right]\mathrm{div}_x\partial_x^\ell\vec{u}+\partial_x^\ell\vec{E}\cdot\partial_x^\ell\vec{u}$$
$$+\frac{1}{2}[(\partial_x^\ell\vartheta)^2+(\partial_x^\ell E_r)^2],$$

$$\vec{F}_1^\ell=\left[\overline{a}_1\partial_x^\ell(\varrho-\overline{\varrho})+\overline{a}_2\partial_x^\ell(\vartheta-\overline{\vartheta})+\overline{a}_3\partial_x^\ell(E_r-\overline{E}_r)\right]\vec{u}_t$$
$$+(\overline{a}_3\overline{c}_2-\overline{a}_2\overline{b}_2+\overline{b}_1)\partial_x^\ell\vartheta\ \partial_x^\ell\vec{u}+(\overline{a}_2\overline{b}_3-\overline{a}_3\overline{c}_3+\overline{c}_1)\partial_x^\ell E_r\ \partial_x^\ell\vec{u},$$
$$-2\left[\overline{a}_1\partial_x^\ell(\varrho-\overline{\varrho})+\overline{a}_2\partial_x^\ell(\vartheta-\overline{\vartheta})+\overline{a}_3\partial_x^\ell(E_r-\overline{E}_r)\right]\partial_x^\ell\vec{E}+\partial_x^\ell\vec{u}\times\partial_x^\ell(\vec{B}-\overline{\vec{B}}),$$

$$D_1^\ell = \overline{a}_1^2|\nabla_x\partial_x^\ell\varrho|^2 + \overline{a}_2^2|\partial_x^\ell\nabla_x\vartheta|^2 + \overline{a}_3^2|\partial_x^\ell\nabla_x E_r|^2 + |\partial_x^\ell\vec{E}|^2 + 2\overline{a}_1\left(\partial_x^\ell(\varrho-\overline{\varrho})\right)^2 + \overline{b}_2(\partial_x^\ell\vartheta)^2 + \overline{c}_3(\partial_x^\ell E_r)^2,$$

$$M_1^\ell = -\left\{2\overline{a}_1\overline{a}_2\nabla_x\partial_x^\ell\varrho\cdot\nabla_x\partial_x^\ell\vartheta + 2\overline{a}_1\overline{a}_3\nabla_x\partial_x^\ell\varrho\cdot\nabla_x\partial_x^\ell E_r + 2\overline{a}_2\overline{a}_3\nabla_x\partial_x^\ell\vartheta\cdot\nabla_x\partial_x^\ell E_r\right.$$

$$+2\overline{a}_2\partial_x^\ell(\varrho-\overline{\varrho})\partial_x^\ell(\vartheta-\overline{\vartheta})+2\overline{a}_2\partial_x^\ell(\varrho-\overline{\varrho})\partial_x^\ell(E_r-\overline{E}_r)+(\overline{a}_1\,\nabla_x\partial_x^\ell\varrho+\overline{a}_2\,\nabla_x\partial_x^\ell\vartheta+\overline{a}_3\,\nabla_x\partial_x^\ell E_r+\partial_x^\ell\vec{E})(\partial_x^\ell\vec{u}\times\overline{\vec{B}}+\nu\partial_x^\ell\vec{u})$$

$$-(\overline{a}_3\overline{c}_2-\overline{a}_2\overline{b}_2+\overline{b}_1)\nabla_x\partial_x^\ell\vartheta\cdot\partial_x^\ell\vec{u} - (\overline{a}_2\overline{b}_3-\overline{a}_3\overline{c}_3+\overline{c}_1)\nabla_x\partial_x^\ell E_r\cdot\partial_x^\ell\vec{u}$$

$$\left.-\mathrm{curl}_x\partial_x^\ell\vec{u}\;\partial_x^\ell(\vec{B}-\overline{\vec{B}}) - \overline{\varrho}\left(\partial_x^\ell\vec{u}\right)^2 - (\mathrm{div}_x\partial_x^\ell\vec{u})^2\left[\overline{a}_1+\overline{a}_2+\overline{a}_3\right]\right\},$$

and

$$G_1^\ell = -\overline{a}_1\partial_x^\ell g_1\mathrm{div}_x\partial_x^\ell\vec{u} + [\overline{a}_1\nabla_x\partial_x^\ell\varrho + \overline{a}_2\nabla_x\partial_x^\ell\vartheta + \overline{a}_3\nabla_x\partial_x^\ell E_r + \partial_x^\ell\vec{E}]\partial_x^\ell g_2$$

$$-\overline{a}_2\partial_x^\ell g_3\mathrm{div}_x\partial_x^\ell\vec{u} - \overline{a}_3\partial_x^\ell g_4\mathrm{div}_x\partial_x^\ell\vec{u} + \partial_x^\ell g_5\partial_x^\ell\vec{u} + \partial_x^\ell g_3\partial_x^\ell\vartheta + \partial_x^\ell g_4\partial_x^\ell E_r.$$

Integrating (3.27) over space and time, we find

$$\int_{\mathbb{R}^3} H_1^\ell(t)\;dx - \int_{\mathbb{R}^3} H_1^\ell(0)\;dx$$

$$+C\int_0^t \left(\|\nabla_x\partial_x^\ell\varrho\|_{L^2}^2 + \|\nabla_x\partial_x^\ell\vartheta\|_{L^2}^2 + \|\nabla_x\partial_x^\ell E_r\|_{L^2}^2 + \|\partial_x^\ell\vec{E}\|_{L^2}^2\right)\;d\tau$$

$$+C\int_0^t \left(+\|\partial_x^\ell(\varrho-\overline{\varrho})\|_{L^2}^2 + \|\partial_x^\ell(\vartheta-\overline{\vartheta})\|_{L^2}^2 + \|\partial_x^\ell(E_r-\overline{E}_r)\|_{L^2}^2\right)\;d\tau$$

$$\leq \varepsilon\int_0^t \|\partial_x^\ell(\vec{B}-\overline{\vec{B}})\|_{L^2}^2\;d\tau + C_\varepsilon\int_0^t \left(\|\partial_x^\ell\vec{u}\|_{H^1}^2 + \|\partial_x^\ell(\vartheta-\overline{\vartheta})\|_{H^1}^2 + \|\partial_x^\ell(E_r-\overline{E}_r)\|_{H^1}^2\right)\;d\tau$$

$$+\int_0^t\int_{\mathbb{R}^3} |G_1^\ell|\;dx\;d\tau. \tag{3.28}$$

Observing that

$$\left|\int_{\mathbb{R}^3} H_1^\ell(t)\;dx\right| \leq C\left(\|\partial_x^\ell(\varrho-\overline{\varrho})\|_{L^2}^2 + \|\partial_x^\ell(\vartheta-\overline{\vartheta})\|_{L^2}^2 + \|\partial_x^\ell(E_r-\overline{E}_r)\|_{L^2}^2 + \|\partial_x^\ell\vec{u}\|_{H^1}^2\right),$$

and summing (3.28) on ℓ for $1 \leq \ell \leq d-1$, we get

$$\int_0^t \left(\|\left(\varrho-\overline{\varrho},\vartheta-\overline{\vartheta},E_r-\overline{E}_r\right)(\tau)\|_{H^d}^2 + \|\vec{E}(\tau)\|_{H^{d-1}}^2\right)\;d\tau \leq C_\varepsilon\|(V-\overline{V})(0)\|_{H^d}^2$$

$$+\varepsilon\int_0^t \|\partial_x\vec{B}(\tau)\|_{H^{d-2}}^2\;d\tau + C_\varepsilon\left(E(t)D^2(t) + F(t)I(t)D(t)\right) + \sum_{|\ell|=1}^{d-1}\int_0^t\int_{\mathbb{R}^3} |G_1^\ell(\tau)|\;dx\;d\tau, \tag{3.29}$$

where we used Corollary 3.1.

Let us estimate the last integral in (3.28). we have

$$\begin{cases} \|\partial_x^\ell g_1\|_{L^2} \le C\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+1}(\varrho,\vec{u})\|_{L^2}, \\ \|\partial_x^\ell g_2\|_{L^2} \le C\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+1}(\varrho,\vec{u})\|_{L^2} \\ \qquad +C\|\vec{B}-\overline{\vec{B}}\|_{L^\infty}\|\partial_x^\ell\vec{u}\|_{L^2}+C\|\partial_x^\ell(\vec{B}-\overline{\vec{B}})\|_{L^2}\|\vec{u}\|_{L^\infty}, \\ \|\partial_x^\ell g_3\|_{L^2} \le C\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+1}(\varrho,\vec{u})\|_{L^2} \\ \qquad +C\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+2}(\vartheta,E_r)\|_{L^2}, \\ \|\partial_x^\ell g_4\|_{L^2} \le C\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+1}(\varrho,\vec{u})\|_{L^2} \\ \qquad +C\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell+2}(\vartheta,E_r)\|_{L^2}, \\ \|\partial_x^\ell g_5\|_{L^2} \le C\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)\|_{L^\infty}\|\partial_x^{\ell}(\varrho,\vec{u})\|_{L^2}, \end{cases} \tag{3.30}$$

for $1 \le |\ell| \le d-1$. Then

$$\int_0^t\int_{\mathbb{R}^3}|G_1^\ell(\tau)|\ dx\ d\tau \le C\|\partial_x^{\ell+1}\vec{u}\|_{L^2}\|\partial_x^\ell g_1\|_{L^2}$$

$$+C\left(\|\partial_x^{\ell+1}\varrho\|_{L^2}+\|\partial_x^{\ell+1}\vartheta\|_{L^2}+\|\partial_x^{\ell+1}E_r\|_{L^2}+\|\partial_x^{\ell}\vec{E}\|_{L^2}\right)\|\partial_x^\ell g_2\|_{L^2}$$

$$+C\|\partial_x^{\ell+1}\vec{u}\|_{L^2}\|\partial_x^\ell g_3\|_{L^2}+C\|\partial_x^{\ell+1}\vec{u}\|_{L^2}\|\partial_x^\ell g_4\|_{L^2}+C\|\partial_x^{\ell}\vec{u}\|_{L^2}\|\partial_x^\ell g_5\|_{L^2}.$$

Plugging bounds (3.30) into this last inequality gives

$$\sum_{|\ell|=1}^{d-1}\int_0^t\int_{\mathbb{R}^3}|G_1^\ell(\tau)|\ dx\ d\tau \le CE(t)D^2(t),$$

which ends the proof of Lemma 2. □

Finally we check from [19] (see Lemma 4.4) that the following result for the Maxwell's system holds true for our system with a similar proof

Lemma 3. *Under the same assumptions as in Theorem 2.1, and supposing that $d \ge 3$, for any $\varepsilon > 0$ the following estimate (here, we set $V = (\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})^T$) holds*

$$\int_0^t\left\|\partial_x\vec{B}(\tau)\right\|_{H^{s-2}}^2\ d\tau \le C\|V_0-\overline{V}\|_{H^{s-1}}^2+C\int_0^t\left\|\partial_x\vec{E}(\tau)\right\|_{H^{s-2}}^2\ d\tau+C(E(t)D(t)^2+F(t)I(t)D(t)). \tag{3.31}$$

Proof: Applying ∂_x^ℓ to (1.5) and (1.6), multiplying respectively by $-\mathrm{curl}_x\partial_x^\ell\vec{B}$, (1.6) by $\mathrm{curl}_x\partial_x^\ell\vec{E}$ and adding the resulting equations, we get

$$-\left(\partial_x^\ell\vec{E}\cdot\mathrm{curl}_x\partial_x^\ell\vec{B}\right)_t+|\mathrm{curl}_x\partial_x^\ell\vec{B}|^2-\mathrm{div}_x\left(\partial_x^\ell\vec{E}\times\partial_x^\ell\vec{B}_t\right)=M_2^\ell+G_2^\ell,$$

where

$$M_2^\ell = -\overline{\varrho}\partial_x^\ell\vec{u}\cdot\mathrm{curl}_x\partial_x^\ell\vec{B}+|\mathrm{curl}_x\partial_x^\ell\vec{E}|^2,$$

and

$$G_2^\ell = -\partial_x^\ell\left((\varrho-\overline{\varrho})\vec{u}\right)\cdot\mathrm{curl}_x\partial_x^\ell\vec{B}.$$

Integrating in space we get

$$-\frac{d}{dt}\int_{\mathbb{R}^3}\partial_x^\ell\vec{E}\cdot\text{curl}_x\partial_x^\ell\vec{B}\;dx+C\|\text{curl}_x\partial_x^\ell\vec{B}\|_{L^2}^2\le\|\text{curl}_x\partial_x^\ell\vec{E}\|_{L^2}^2+\|\partial_x^\ell\vec{u}\|_{L^2}^2+\int_{\mathbb{R}^3}|G_2^\ell|\;dx.$$

Integrating on time and summing for $1\le|\ell|\le d-2$, we have

$$\int_0^t\|\partial_x\vec{B}\|_{H^{d-2}}^2dt\le C\|(V-\overline{V})(t)\|_{H^{d-1}}+C\|(V-\overline{V})(0)\|_{H^{d-1}}+C\int_0^t\|\partial_x\vec{E}\|_{H^{d-2}}^2dt$$

$$+C\int_0^t\|\vec{u}\|_{H^{d-2}}^2dt+C\sum_{|\ell|=0}^{d-2}\int_{\mathbb{R}^3}|G_2^\ell(\tau)|\;dx\;d\tau$$

$$\le C\|(V-\overline{V})(0)\|_{H^{d-1}}+C\int_0^t\|\partial_x\vec{E}|_{H^{d-2}}^2dt+C(E(t)D(t)^2+F(t)I(t)D(t)),$$

where we used the bound $\sum_{|\ell|=1}^{d-1}\int_0^t\int_{\mathbb{R}^3}|G_2^\ell(\tau)|\;dx\;d\tau\le CE(t)D^2(t)$, obtained in the same way as in the proof of Lemma 2, which ends the proof of Lemma 3. □

We are now in position to conclude with the proofs of Theorems 2.1 and 2.2.

3.2 Proof of Theorem 2.1:

We first point out that local existence for the hyperbolic system (1.1)-(1.6) may be proved using standard fixed-point methods. We refer to [16] for the proof.

Now plugging (3.31) into (3.17) with ε small enough, we get

$$\int_0^t\left(\|\left(\varrho-\overline{\varrho},\vartheta-\overline{\vartheta},E_r-\overline{E}_r\right)\|_{H^d}^2+\left\|\vec{E}(\tau)\right\|_{H^{d-1}}^2\right)\;d\tau$$

$$\le C\left\{\|V_0-\overline{V}\|_{H^{d-1}}^2+E(t)D(t)^2+F(t)I(t)D(t)\right\}. \tag{3.32}$$

Putting this last estimate into (3.31) we find

$$\int_0^t\left\|\partial_x\vec{B}(\tau)\right\|_{H^{s-2}}^2\;d\tau\le C\|V_0-\overline{V}\|_{H^s}^2+C(E(t)D(t)^2+F(t)I(t)D(t)). \tag{3.33}$$

Then from (3.17), (3.32) and (3.33) we get

$$\|(V-\overline{V})(t)\|_{H^d}^2+\int_0^t\left(\|\left(\varrho-\overline{\varrho},\vartheta-\overline{\vartheta},E_r-\overline{E}_r\right)(\tau)\|_{H^d}^2+\left\|\vec{E}(\tau)\right\|_{H^{d-1}}^2+\left\|\partial_x\vec{B}(\tau)\right\|_{H^{d-2}}^2\right)\;d\tau$$

$$\le C\|V_0-\overline{V}\|_{H^d}^2+C(E(t)D(t)^2+F(t)I(t)D(t)),$$

or equivalently

$$F(t)^2+D(t)^2\le C\|V_0-\overline{V}\|_{H^d}^2+C(E(t)D(t)^2+F(t)I(t)D(t)).$$

Now observing that, provided that $d\ge3$ one has $\|(V-\overline{V})(t)\|_{H^d}\le E(t)\le CF(t)$, and that, provided that $d\ge2$ one has $I(t)\le CD(t)$, for some positive constant C, we see that

$$F(t)^2+D(t)^2\le C\|V_0-\overline{V}\|_{H^d}^2+CF(t)D(t)^2.$$

In order to prove global existence, we argue by contradiction, and assume that $T_c > 0$ is the maximum time existence. Then, we necessarily have

$$\lim_{t \to T_c} N(t) = +\infty,$$

where $N(t)$ is defined by

$$N(t) := (F(t)^2 + D(t)^2)^{1/2}.$$

We are thus reduced to prove that N is bounded. For this purpose, we use the argument used in [3]. After the previous calculation, we have

$$\forall T \in [0, T_c], \quad N(t)^2 \le C \left(\|V_0 - \overline{V}\|^2_{H^d} + N(t)^3 \right). \tag{3.34}$$

Hence, setting $\|V_0 - \overline{V}\|_{H^d} = \varepsilon$, we have

$$\frac{N(t)^2}{\varepsilon^2 + N(t)^3} \le C. \tag{3.35}$$

Studying the variation of $\phi(N) = N^2/\left(\varepsilon^2 + N^3\right)$, we see that $\phi'(0) = 0$, that ϕ is increasing on the interval $\left[0, \left(2\varepsilon^2\right)^{1/3}\right]$ and decreasing on the interval $\left[\left(2\varepsilon^2\right)^{1/3}, +\infty\right)$. Hence,

$$\max \phi = \phi\left(\left(2\varepsilon^2\right)^{1/3}\right) = \frac{1}{3}\left(\frac{2}{\varepsilon}\right)^{2/3}.$$

Hence we can choose ε small enough to have $\phi(N) \le C$ for all $N \in [0, N^*]$, where $N^* > 0$, we see that $N \le N^*$, which contradicts (3.34). □

4 Large time behaviour

We have the following analogous of Proposition 3.1 for time derivatives

Corollary 4.1. *Let the assumptions of Theorem 2.1 be satisfied and consider the solution* $V := (\varrho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})$ *of system* (1.1)-(1.2)-(1.3) *on* $[0, t]$, *for some* $t > 0$. *Then, one gets for a constant* $C_0 > 0$

$$\|\partial_t V(t)\|^2_{H^{d-1}} + \int_0^t \left(\| \partial_t \left(\rho, \vec{u}, \vartheta, E_r\right)(\tau)\|^2_{H^{d-1}} + \|\partial_t(\vec{B}, \vec{E})(\tau)\|^2_{H^{d-2}} \right) \, d\tau \le C_0 \left\|V_0 - \overline{V}\right\|^2_{H^d}. \tag{4.1}$$

Proof: Using System (3.14) we see that

$$\|\partial_t V\|_{H^{d-1}} \le C \left\|V - \overline{V}\right\|_{H^d},$$

$$\| \partial_t \left(\rho, \vec{u}, \vartheta, E_r\right)\|_{H^{d-1}} \le \|\partial_x(\rho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})\|_{H^{d-1}} + C\|(\rho, \vec{u}, \vartheta, E_r, \vec{B}, \vec{E})\|_{H^{d-1}},$$

and

$$\|\partial_t(\vec{B}, \vec{E})\|_{H^{d-2}} \le \|\partial_x(\vec{B}, \vec{E})\|_{H^{d-2}} + C \| \vec{u}\|_{H^{d-1}}.$$

Then for $d \ge 3$, using the uniform estimate $\left\|V - \overline{V}\right\|^2_{H^d} \le C$ of Theorem 2.1, we get estimate (4.1). □

4.1 Proof of Theorem 2.2:

Using Corollary 4.1, we get

$$\int_0^\infty \left| \frac{d}{dt} \|(\varrho - \overline{\varrho}, \vec{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E}_r)(t)\|_{H^{d-1}} \right| \, dt$$

$$\le 2\int_0^\infty \|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(t)\|_{H^{d-1}}\|\partial_t(\varrho,\vec{u},\vartheta,E_r)(t)\|_{H^{d-1}}\ dt$$

$$\le \int_0^\infty \|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(t)\|^2_{H^{d-1}}+\|\partial_t(\varrho,\vec{u},\vartheta,E_r)(t)\|^2_{H^{d-1}}\ dt \le C_0\left\|V_0-\overline{V}\right\|^2_{H^d}.$$

This implies that

$$t\to\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(t)\|^2_{H^{d-1}}\in L^1(0,\infty)\ \text{ and }\ t\to\frac{d}{dt}\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(t)\|_{H^{d-1}}\in L^1(0,\infty),$$

and then

$$\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(t)\|_{H^{d-1}}\to 0,$$

when $t\to\infty$.

Now applying Gagliardo-Nirenberg's inequality, and (2.2) we get

$$\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(t)\|_{W^{d-2,\infty}}\le\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(t)\|^{1/4}_{H^{d-2}}\|\partial_x^2(\varrho,\vec{u},\vartheta,E_r)(t)\|^{3/4}_{H^{d-2}}.$$

So

$$\|(\varrho-\overline{\varrho},\vec{u},\vartheta-\overline{\vartheta},E_r-\overline{E}_r)(t)\|_{W^{d-2,\infty}}\to 0 \text{ when } t\to\infty.$$

Now in the same stroke

$$t\to\|\vec{E}(t)\|^2_{H^{d-1}}\in L^1(0,\infty)\ \text{ and }\ t\to\frac{d}{dt}\|\vec{E}(t)\|_{H^{d-1}}\in L^1(0,\infty),$$

and then

$$\|\vec{E}(t)\|_{W^{d-1,\infty}}\to 0 \text{ when } t\to\infty.$$

Finally

$$t\to\|\partial_x\vec{B}(t)\|^2_{H^{d-3}}\in L^1(0,\infty)\ \text{ and }\ t\to\frac{d}{dt}\|\partial_x\vec{B}(t)\|_{H^{d-3}}\in L^1(0,\infty),$$

then arguing as before

$$\|(\vec{B}-\overline{\vec{B}})(t)\|_{W^{d-4,\infty}}\le\|(\vec{B}-\overline{\vec{B}})(t)\|^{1/4}_{H^{d-4}}\|\partial_x^2\vec{B}(t)\|^{3/4}_{H^{d-3}}.$$

So

$$\|(\vec{B}-\overline{\vec{B}})\|_{W^{d-4,\infty}}\to 0 \text{ when } t\to\infty,$$

which ends the proof □

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