

EXISTENCE AND NON-EXISTENCE OF SIGN-CHANGING
SOLUTIONS FOR A CLASS OF TWO-POINT BOUNDARY VALUE
PROBLEMS INVOLVING ONE-DIMENSIONAL p -LAPLACIAN

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Cordially dedicated to Professor Manabu Naito on his 60th birthday

Abstract. We consider the boundary value problem involving the one dimensional p -Laplacian, and establish the precise intervals of the parameter for the existence and non-existence of solutions with prescribed numbers of zeros. Our argument is based on the shooting method together with the qualitative theory for half-linear differential equations.

Keywords: boundary value problem, half-linear differential equation, Sturm comparison theorem, half-linear Prüfer transformation

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1. INTRODUCTION

In this paper we consider the existence and non-existence of sign-changing solutions for the one-dimensional p -Laplacian boundary value problem

$$(1.1) \quad (|u'|^{p-2}u')' + \lambda a(x)f(u) = 0, \quad 0 < x < 1,$$

$$(1.2) \quad u(0) = u(1) = 0,$$

where $p > 1$ and $\lambda > 0$ is a parameter. Problems of the form (1.1)–(1.2) describe some nonlinear phenomena in mathematical sciences and have been studied in recent years by many authors (see [1], [2], [6], [7], [9], [11], [13], [14] and references therein).

In (1.1) we assume that a satisfies

$$a \in C^1[0, 1], \quad a(x) > 0 \quad \text{for } 0 \leq x \leq 1,$$

and that f satisfies $f \in C(\mathbb{R})$, $sf(s) > 0$ for $s \neq 0$, f is locally Lipschitz continuous on $\mathbb{R} \setminus \{0\}$; moreover, there exist limits f_0 and f_∞ with $f_0, f_\infty \in [0, \infty]$ such that

$$f_0 = \lim_{|s| \rightarrow 0} \frac{f(s)}{|s|^{p-2}s} \quad \text{and} \quad f_\infty = \lim_{|s| \rightarrow \infty} \frac{f(s)}{|s|^{p-2}s}.$$

By a solution u of (1.1) we mean a function $u \in C^1[0, 1]$ with $|u'|^{p-2}u' \in C^1[0, 1]$ which satisfies (1.1) at all points in $(0, 1)$. For each $k \in \mathbb{N}$ we denote by S_k^+ (S_k^-) the set of all solutions u for (1.1)–(1.2) which have exactly $k - 1$ zeros in $(0, 1)$ and satisfy $u'(0) > 0$ (respectively, $u'(0) < 0$).

Let λ_k be the k -th eigenvalue of

$$(1.3) \quad \begin{cases} (|\varphi'|^{p-2}\varphi')' + \lambda a(x)|\varphi|^{p-2}\varphi = 0, & 0 < x < 1, \\ \varphi(0) = \varphi(1) = 0, \end{cases}$$

and let φ_k be an eigenfunction corresponding to λ_k . It is known that

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \lambda_{k+1} < \dots, \quad \lim_{k \rightarrow \infty} \lambda_k = \infty,$$

and that φ_k has exactly $k - 1$ zeros in $(0, 1)$. (See, e.g., [3], [4], [8].) For convenience, we put $\lambda_0 = 0$.

By [12, Theorem 1], if there exists an integer $k \in \mathbb{N}$ such that either

$$\lambda f_0 < \lambda_k < \lambda f_\infty \quad \text{or} \quad \lambda f_\infty < \lambda_k < \lambda f_0,$$

then $S_k^+ \neq \emptyset$ and $S_k^- \neq \emptyset$. As a consequence, in the case $f_0 \neq f_\infty$, if either

$$(1.4) \quad \lambda \in (\lambda_k/f_\infty, \lambda_k/f_0) \quad \text{or} \quad \lambda \in (\lambda_k/f_0, \lambda_k/f_\infty)$$

for some $k \in \mathbb{N}$, then $S_k^+ \neq \emptyset$ and $S_k^- \neq \emptyset$. Here, we agree that $1/0 = \infty$ and $1/\infty = 0$.

In this paper we will consider the non-existence of solutions with prescribed numbers of zeros, and also investigate the existence of solutions in the case $f_0 = f_\infty \in (0, \infty)$. To this end we define f_* and f^* by

$$f_* = \inf_{s \in \mathbb{R} \setminus \{0\}} \frac{f(s)}{|s|^{p-2}s} \quad \text{and} \quad f^* = \sup_{s \in \mathbb{R} \setminus \{0\}} \frac{f(s)}{|s|^{p-2}s},$$

respectively. Then it follows that $f_0, f_\infty \in [f_*, f^*]$.

Theorem 1.1. Assume that $\lambda \in (0, \lambda_k/f_*) \cup (\lambda_k/f_*, \infty)$ for some $k \in \mathbb{N}$. Then $S_k^+ = \emptyset$ and $S_k^- = \emptyset$.

Corollary 1.1. Assume that $\lambda_{k-1}/f_* < \lambda_k/f_*$ for some integer $k \in \mathbb{N}$. If $\lambda \in (\lambda_{k-1}/f_*, \lambda_k/f_*)$, then the problem (1.1)–(1.2) has no nontrivial solution.

Remark 1.1. Let us consider, for instance, the case where

$$(1.5) \quad f_* = f_0 < f_\infty = f^*.$$

In this case, by (1.4) and Theorem 1.1, we find that $S_k^+ \neq \emptyset$ and $S_k^- \neq \emptyset$ if $\lambda \in (\lambda_k/f_\infty, \lambda_k/f_0)$, and that $S_k^+ = S_k^- = \emptyset$ if $\lambda \in (0, \lambda_k/f_\infty) \cup (\lambda_k/f_0, \infty)$. Hence, λ_k/f_∞ and λ_k/f_0 are critical values for the existence of solutions in S_k^+ and S_k^- . For example, if $f(s)/|s|^{p-2}s$ is nondecreasing, then (1.5) holds.

Next, let us consider the existence of solutions in the case $f_0 = f_\infty \in (0, \infty)$. In this case we require that

$$(1.6) \quad \frac{f(s)}{|s|^{p-2}s} \neq \text{constant} \quad \text{for any interval } (-\delta, \delta) \text{ with } \delta > 0.$$

It is clear that we have $f_* < f^*$, if (1.6) holds.

Theorem 1.2. Assume that $f_0 = f_\infty = f^* \in (0, \infty)$ and (1.6) holds. Let $k \in \mathbb{N}$.

- (i) If $\lambda = \lambda_k/f_*$ then $S_k^+ = \emptyset$ and $S_k^- = \emptyset$.
- (ii) There exists $\delta_k \in (\lambda_k/f_*, \lambda_k/f_*)$ such that, if $\lambda \in (\lambda_k/f_*, \delta_k)$, then the problem (1.1)–(1.2) has at least four solutions u_k^+ , v_k^+ , u_k^- , and v_k^- such that u_k^+ , $v_k^+ \in S_k^+$ and u_k^- , $v_k^- \in S_k^-$.

Theorem 1.3. Assume that $f_0 = f_\infty = f_* \in (0, \infty)$ and (1.6) holds. Let $k \in \mathbb{N}$.

- (i) If $\lambda = \lambda_k/f_*$ then $S_k^+ = \emptyset$ and $S_k^- = \emptyset$.
- (ii) There exists $\delta_k \in (\lambda_k/f_*, \lambda_k/f_*)$ such that, if $\lambda \in (\delta_k, \lambda_k/f_*)$, then the problem (1.1)–(1.2) has at least four solutions u_k^+ , v_k^+ , u_k^- , and v_k^- such that u_k^+ , $v_k^+ \in S_k^+$ and u_k^- , $v_k^- \in S_k^-$.

Remark 1.2. In Theorems 1.2 and 1.3, if $\lambda \in (0, \lambda_k/f_*) \cup (\lambda_k/f_*, \infty)$, then $S_k^+ = \emptyset$ and $S_k^- = \emptyset$ by Theorem 1.1.

In the proofs of Theorems 1.1, 1.2 and 1.3, we first consider the solution $u(x; \mu)$ of (1.1) satisfying the initial condition with a parameter $\mu \in \mathbb{R}$, and then we investigate the behavior of $u(x; \mu)$ as $\mu \rightarrow 0$ and $\mu \rightarrow \infty$. We will show the non-existence of solutions by employing variants of the Sturm comparison theorem for half-linear

differential equations, and prove the existence of solutions with prescribed numbers of zeros by making use of the half-linear Prüfer transformation which involves the generalized trigonometric functions.

This paper is organized as follows. In Section 2 we give some variants of the Sturm comparison theorem, and in Section 3 we prove Theorem 1.1. In Section 4 we give the proofs of Theorems 1.2 and 1.3.

2. COMPARISON LEMMAS

Let us consider a pair of half-linear differential equations

$$(2.1) \quad (|u'|^{p-2}u')' + c(x)|u|^{p-2}u = 0, \quad 0 \leq x \leq 1,$$

and

$$(2.2) \quad (|U'|^{p-2}U')' + C(x)|U|^{p-2}U = 0, \quad 0 \leq x \leq 1,$$

where $c, C \in C[0, 1]$ satisfy $C(x) \geq c(x)$ for $x \in [0, 1]$. The Sturm comparison theorem for the half-linear differential equation is formulated as follows: [4, Theorem 1.2.4] (See also [3], [5] and [10].)

Lemma 2.1. *Assume that a nontrivial solution u of (2.1) satisfies $u(x_1) = u(x_2) = 0$ with some $0 \leq x_1 < x_2 \leq 1$. Then every nontrivial solution U of (2.2) has a zero in (x_1, x_2) or it is a multiple of the solution u on $[x_1, x_2]$. The latter possibility is excluded if $C(x) \neq c(x)$ for $x \in [x_1, x_2]$.*

We will give some variants of Lemma 2.1.

Lemma 2.2. *Assume that a solution u of (2.1) satisfies $u(0) = u(1) = 0$ and has exactly $k - 1$ zeros in $(0, 1)$. Let U be a solution of (2.2) satisfying $U(0) = 0$ and $U'(0) \neq 0$. Then U possesses one of the following properties:*

- (i) U has at least k zeros in $(0, 1)$;
- (ii) U is a constant multiple of u on $[0, 1]$ and $c \equiv C$ on $[0, 1]$.

In both cases (i) and (ii), U has at least k zeros in $(0, 1)$.

Proof. In the case where $c \equiv C$ on $[0, 1]$, it is clear that (ii) holds. Hence it suffices to show that (i) must hold in the case $c \neq C$ on $[0, 1]$. Let $\{x_i\}_{i=0}^k$ be zeros of u satisfying $0 = x_0 < x_1 < \dots < x_{k-1} < x_k = 1$. Assume that $c \neq C$ on $[x_{i_0-1}, x_{i_0}]$ for some $i_0 \in \{1, 2, \dots, k\}$. Then Lemma 2.1 implies that U has at least one zero in (x_{i_0-1}, x_{i_0}) . By Lemma 2.1, U has at least one zero in each interval $[x_{i-1}, x_i]$ for $i = i_0 + 1, i_0 + 2, \dots, k$ and $(x_{i-1}, x_i]$ for $i = 1, 2, \dots, i_0 - 1$. Thus U has at least k zeros in $(0, 1)$. □

Lemma 2.3. Assume that a solution U of (2.2) satisfies $U(0) = U(1) = 0$ and has exactly $k - 1$ zeros in $(0, 1)$. Let u be a solution of (2.1) satisfying $u(0) = 0$ and $u'(0) \neq 0$. Then u possesses one of the following properties:

- (i) u has at most $k - 1$ zeros in $(0, 1]$;
- (ii) u is a constant multiple of U on $[0, 1]$ and $c \equiv C$ on $[0, 1]$.

In both cases (i) and (ii), u has at most $k - 1$ zeros in $(0, 1)$.

Proof. We will show that u has at most $k - 1$ zeros in $(0, 1]$ when $c \neq C$ on $[0, 1]$. Let $\{x_i\}_{i=0}^k$ be zeros of U satisfying $0 = x_0 < x_1 < \dots < x_{k-1} < x_k = 1$. Assume to the contrary that u has k zeros in $(0, 1]$. Let $\{y_i\}_{i=0}^k$ be zeros of u satisfying $0 = y_0 < y_1 < \dots < y_{k-1} < y_k \leq 1$. By applying Lemma 2.2 on the interval $(0, y_k)$, we conclude that the solution U has at least k zeros in $(0, y_k) \subset (0, 1)$. This is a contradiction. Thus u has at most $k - 1$ zeros in $(0, 1]$, and (i) holds. \square

We will need the following lemma [12, Lemma 3.3] in the proof of Theorem 1.1.

Lemma 2.4. Let λ_k be the k -th eigenvalue of (1.3), and let $\{x_i\}_{i=0}^k$ be the zeros of the corresponding eigenfunction φ_k such that

$$(2.3) \quad 0 = x_0 < x_1 < x_2 < \dots < x_{k-1} < x_k = 1.$$

Assume that $\tilde{\lambda} > \lambda_k$. Then for each $i \in \{1, 2, \dots, k\}$ there is a solution w_i of the equation

$$(2.4) \quad (|w'|^{p-2}w')' + \tilde{\lambda}a(x)|w|^{p-2}w = 0$$

which has at least two zeros in (x_{i-1}, x_i) .

3. PROOF OF THEOREM 1.1

Let $\lambda > 0$. We denote by $u(x; \mu, \lambda)$ the solution of the problem (1.1) and

$$(3.1) \quad u(0) = 0 \quad \text{and} \quad u'(0) = \mu,$$

where $\mu \in \mathbb{R}$ is a parameter. By [12, Proposition 2.1] we obtain the following:

Lemma 3.1. For each $\mu \in \mathbb{R}$ and $\lambda > 0$, the solution $u(x; \mu, \lambda)$ exists on $[0, 1]$ and is unique. Furthermore, $u(x; \mu, \lambda)$ and $u'(x; \mu, \lambda)$ are continuous on $(x, \mu, \lambda) \in [0, 1] \times \mathbb{R} \times (0, \infty)$, and the number of zeros of $u(x; \mu, \lambda)$ in $[0, 1]$ is finite for each $\mu \in \mathbb{R} \setminus \{0\}$ and $\lambda > 0$.

The *generalized sine function* \sin_p is defined by the solution to the problem

$$(|S'|^{p-2}S')' + (p-1)|S|^{p-2}S = 0, \quad S(0) = 0 \text{ and } S'(0) = 1.$$

The function \sin_p is defined on \mathbb{R} and is periodic with period $2\pi_p$, where $\pi_p = (2\pi)/(p \sin(\pi/p))$. The *generalized cosine function* \cos_p is defined by $\cos_p x = (\sin_p x)'$. For simplicity, we denote by $u(x; \mu)$ the solution of the problem (1.1) and (3.1) with fixed $\lambda > 0$. We define functions $r(x; \mu)$ and $\theta(x; \mu)$ by

$$\begin{cases} u(x; \mu) = r(x; \mu) \sin_p \theta(x; \mu), \\ u'(x; \mu) = r(x; \mu) \cos_p \theta(x; \mu), \end{cases}$$

where $' = d/dx$. It can be shown that

$$\theta'(x; \mu) = |\cos_p \theta(x; \mu)|^p + \frac{\lambda a(x) f(r(x; \mu) \sin_p \theta(x; \mu)) \sin_p \theta(x; \mu)}{(p-1)[r(x; \mu)]^{p-1}} > 0$$

for $x \in [0, 1]$, which implies that $\theta(x; \mu)$ is strictly increasing in $x \in [0, 1]$ for each fixed $\mu > 0$. (See, for example, [3] or [4].) The initial condition (3.1) yields that $\theta(0; \mu) \equiv 0 \pmod{2\pi_p}$. For simplicity we take $\theta(0; \mu) = 0$. Lemma 3.1 implies that $\theta(x; \mu)$ is continuous in $(x; \mu) \in [0, 1] \times (0, \infty)$. We easily see that $u(x; \mu)$ has exactly $k-1$ zeros in $(0, 1)$ if and only if $(k-1)\pi_p < \theta(1; \mu) \leq k\pi_p$.

Lemma 3.2. (i) Assume that $\lambda f(s)/(|s|^{p-2}s) > \lambda_k$ for $s \in \mathbb{R} \setminus \{0\}$ with some $k \in \mathbb{N}$. Then for each $\mu \neq 0$ the solution $u(x; \mu)$ has at least k zeros in $(0, 1)$.

(ii) Assume that $\lambda f(s)/(|s|^{p-2}s) < \lambda_k$ for $s \in \mathbb{R} \setminus \{0\}$ with some $k \in \mathbb{N}$. Then for each $\mu \neq 0$ the solution $u(x; \mu)$ has at most $k-1$ zeros in $(0, 1]$.

Proof. (i) We observe that $u = u(x; \mu)$ satisfies the equation

$$(3.2) \quad (|u'|^{p-2}u')' + b(x; \lambda)|u|^{p-2}u = 0,$$

where

$$(3.3) \quad b(x; \lambda) = \lambda a(x) \frac{f(u(x; \mu))}{|u(x; \mu)|^{p-2}u(x; \mu)}.$$

Note that $f(s)/(|s|^{p-2}s)$ is continuous at $s = 0$ if $f_0 < \infty$. Then the function $b(x; \lambda)$ is continuous for $x \in [0, 1]$ if $f_0 < \infty$.

First, assume that $f_0 < \infty$. Then $b(x; \lambda)$ given by (3.3) is continuous for $x \in [0, 1]$, and satisfies

$$b(x; \lambda) \geq \lambda_k a(x), \quad b(x; \lambda) \not\equiv \lambda_k a(x) \quad \text{for } 0 \leq x \leq 1.$$

By Lemma 2.2, the solution $u(x; \mu)$ has at least k zeros in $(0, 1)$.

Next, assume that $f_0 = \infty$. Let φ_k be an eigenfunction corresponding to λ_k , and let $\{x_j\}_{j=0}^k$ be zeros of φ_k satisfying (2.3). We will show that $u(x; \mu)$ has at least one zero in each interval (x_{i-1}, x_i) for $i = 1, 2, \dots, k$, which implies that $u(x; \mu)$ has at least k zeros in $(0, 1)$. Assume to the contrary that $u(x; \mu)$ has no zero in (x_{i_0-1}, x_{i_0}) for some $i_0 \in \{0, 1, 2, \dots, k\}$. Then $b(x; \lambda)$ given by (3.3) is continuous for $x \in (x_{i_0-1}, x_{i_0})$ and satisfies $b(x; \lambda) > \lambda_k a(x)$ for $x_{i_0-1} < x < x_{i_0}$. We observe that, due to $f_0 = \infty$, there exists $\tilde{\lambda} > \lambda_k$ such that

$$b(x; \lambda) > \tilde{\lambda} a(x) \quad \text{for } x_{i_0-1} < x < x_{i_0},$$

even if $u(x_{i_0-1}; \mu) = 0$ or $u(x_{i_0}; \mu) = 0$. By Lemma 2.4, Eq. (2.4) has a nontrivial solution w such that $w(t_1) = w(t_2) = 0$ with $t_1, t_2 \in (x_{i_0-1}, x_{i_0})$. Lemma 2.1 implies that $u(x; \mu)$ has at least one zero in $(t_1, t_2) \subset (x_{i_0-1}, x_{i_0})$. This is a contradiction. Thus $u(x; \mu)$ has at least one zero in each interval (x_{i-1}, x_i) for $i = 1, 2, \dots, k$, and hence $u(x; \mu)$ has at least k zeros in $(0, 1)$.

(ii) By the assumption, $f_0 < \infty$. Then the function $b(x; \lambda)$ given by (3.3) is continuous for $x \in [0, 1]$ and satisfies

$$b(x; \lambda) \leq \lambda_k a(x), \quad b(x; \lambda) \not\equiv \lambda_k a(x) \quad \text{for } 0 \leq x \leq 1.$$

By Lemma 2.3, the solution $u(x; \mu)$ has at most $k - 1$ zeros in $(0, 1]$. □

Proof of Theorem 1.1. Assume that $\lambda \in (0, \lambda_k/f^*)$. In this case, we have

$$\lambda f(s)/(|s|^{p-2}s) < \lambda_k \quad \text{for } s \in \mathbb{R} \setminus \{0\}.$$

Then, by Lemma 3.2 (ii), the solution $u(x; \mu)$ has at most $k - 1$ zeros in $(0, 1]$ for every $\mu \neq 0$. This implies that $S_k^+ = S_k^- = \emptyset$. In the case $\lambda \in (\lambda_k/f_*, \infty)$ we obtain $S_k^+ = S_k^- = \emptyset$ by a similar argument with a slight modification.

Proof of Corollary 1.1. Note that $\lambda_k < \lambda_{k+1}$ for $k = 1, 2, \dots$. Then Theorem 1.1 implies that, if $\lambda \in (\lambda_{k-1}/f_*, \infty)$, then $S_j^+ = S_j^- = \emptyset$ for each $j =$

$1, 2, \dots, k-1$, and that, if $\lambda \in (0, \lambda_k/f^*)$, then $S_j^+ = S_j^- = \emptyset$ for each $j = k, k+1, \dots$. By Lemma 3.1, the number of zeros of nontrivial solutions of (1.1)–(1.2) is finite. Hence (1.1)–(1.2) has no nontrivial solution.

4. PROOF OF THEOREMS 1.2 AND 1.3

We denote by $u(x; \mu, \lambda)$ the solution of the problem (1.1) and (3.1). As in Section 3, we define functions $r(x; \mu, \lambda)$ and $\theta(x; \mu, \lambda)$ by

$$\begin{cases} u(x; \mu, \lambda) = r(x; \mu, \lambda) \sin_p \theta(x; \mu, \lambda), \\ u'(x; \mu, \lambda) = r(x; \mu, \lambda) \cos_p \theta(x; \mu, \lambda) \end{cases}$$

with $\theta(0; \mu, \lambda) = 0$, where $' = d/dx$. We see that $\theta(x; \mu, \lambda)$ is continuous in $(x, \mu, \lambda) \in [0, 1] \times \mathbb{R} \times (0, \infty)$ by Lemma 3.1, and that $\theta(x; \mu, \lambda)$ is strictly increasing in $x \in [0, 1]$ for each fixed $\mu > 0$ and $\lambda > 0$. From $\theta(0; \mu, \lambda) = 0$ it follows that $u(x; \mu, \lambda)$ has exactly $k-1$ zeros in $(0, 1)$ if and only if $(k-1)\pi_p < \theta(1; \mu, \lambda) \leq k\pi_p$.

By Lemmas 4.1–4.4 in [12] we obtain the following.

Lemma 4.1. *Let $k \in \mathbb{N}$.*

- (i) *Assume that $\lambda f_0 < \lambda_k$. Then there exists $\mu_* > 0$ such that, for each $\mu \in (0, \mu_*]$, the solution $u(x; \mu, \lambda)$ has at most $k-1$ zeros in $(0, 1)$.*
- (ii) *Assume that $\lambda f_0 > \lambda_k$. Then there exists $\mu_* > 0$ such that, for each $\mu \in (0, \mu_*]$, the solution $u(x; \mu, \lambda)$ has at least k zeros in $(0, 1)$.*
- (iii) *Assume that $\lambda f_\infty > \lambda_k$. Then there exists $\mu^* > 0$ such that, for each $\mu \geq \mu^*$, the solution $u(x; \mu, \lambda)$ has at least k zeros in $(0, 1)$.*
- (iv) *Assume that $\lambda f_\infty < \lambda_k$. Then there exists $\mu^* > 0$ such that, for each $\mu \geq \mu^*$, the solution $u(x; \mu, \lambda)$ has at most $k-1$ zeros in $(0, 1)$.*

We will prove Theorem 1.2 only, since Theorem 1.3 can be shown by an argument similar to the proof of Theorem 1.2 with a slight modification.

Proof of Theorem 1.2. (i) We observe that $u = u(x; \mu, \lambda)$ satisfies (3.2) with

$$(4.1) \quad b(x; \lambda) = \lambda a(x) \frac{f(u(x; \mu, \lambda))}{|u(x; \mu, \lambda)|^{p-2} u(x; \mu, \lambda)} \quad \text{for } 0 \leq x \leq 1.$$

If $f_0 < \infty$, then the function $b(x; \lambda)$ is continuous for $x \in [0, 1]$.

Let $\mu > 0$. Due to $f_0 = f_\infty = f^* \in (0, \infty)$, the function $b(x; \lambda)$ given by (4.1) satisfies

$$b(x; \lambda_k/f^*) \leq \lambda_k a(x) \quad \text{for } x \in [0, 1].$$

By Lemma 2.3, the solution $u(x; \mu, \lambda)$ has at most $k - 1$ zeros on $(0, 1)$, that is, $\theta(1; \mu, \lambda_k/f^*) \leq k\pi_p$. Assume that $\theta(1; \mu, \lambda_k/f^*) = k\pi_p$ with some $\mu > 0$. Then, by Lemma 2.3, we obtain

$$b(x; \lambda_k/f^*) \equiv \lambda_k a(x) \quad \text{for } x \in [0, 1],$$

which implies that

$$\frac{f(s)}{|s|^{p-2}s} \equiv f^* \quad \text{for } 0 < s \leq \max_{x \in [0,1]} u(x; \mu, \lambda).$$

This contradicts (1.6). Thus we obtain $\theta(1; \mu, \lambda_k/f^*) < k\pi_p$ for any $\mu > 0$. This implies that $S_k^+ = \emptyset$ if $\lambda = \lambda_k/f^*$. By a similar argument, we obtain $\theta(1; \mu, \lambda_k/f^*) < k\pi_p$ for any $\mu < 0$, and hence $S_k^- = \emptyset$ if $\lambda = \lambda_k/f^*$.

(ii) Put $\mu_0 > 0$. By (i) we have $\theta(1; \mu_0, \lambda_k/f^*) < k\pi_p$. By the continuity of $\theta(1; \mu_0, \lambda)$ with respect to $\lambda > 0$ there exists $\delta_k^+ > \lambda_k/f^*$ such that $\theta(1; \mu_0, \lambda) < k\pi_p$ for $\lambda \in (\lambda_k/f^*, \delta_k^+)$. Let $\lambda \in (\lambda_k/f^*, \delta_k^+)$. Then we have $\lambda f_0 = \lambda f_\infty > \lambda_k$. By Lemmas 4.1 (ii), (iii) there are μ_* , $\mu^* > 0$ such that, if either $\mu \in (0, \mu_*]$ or $\mu \in [\mu^*, \infty)$, the solution $u(x; \mu, \lambda)$ has at least k zeros in $(0, 1)$. This implies that

$$\theta(1; \mu, \lambda) > k\pi_p \quad \text{for } \mu \in (0, \mu_*] \cup [\mu^*, \infty),$$

and that $\mu_0 \in (\mu_*, \mu^*)$. Since $\theta(1; \mu, \lambda)$ is continuous in $\mu \in (0, \infty)$, there exist μ_1 and μ_2 such that

$$0 < \mu_1 < \mu_0 < \mu_2 \quad \text{and} \quad \theta(1; \mu_1, \lambda) = \theta(1; \mu_2, \lambda) = k\pi_p,$$

which means $u(x; \mu_1, \lambda), u(x; \mu_2, \lambda) \in S_k^+$.

By an argument similar to the above, there exists a sequence $\delta_k^- > \lambda_k/f^*$ such that, if $\lambda \in (\lambda_k/f^*, \delta_k^-)$, then (1.1)–(1.2) has two solutions v_1 and v_2 which have exactly $k - 1$ zeros in $(0, 1)$ and satisfy $v_1'(0) < 0$ and $v_2'(0) < 0$. This implies that $v_1, v_2 \in S_k^-$.

Finally, put $\delta_k = \min\{\delta_k^+, \delta_k^-\}$. If $\lambda \in (\lambda_k/f^*, \delta_k)$, then (1.1)–(1.2) has at least four solutions $u_k^+, v_k^+, u_k^-, v_k^-$ which satisfy $u_k^+, v_k^+ \in S_k^+$ and $u_k^-, v_k^- \in S_k^-$.

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