ASYMPTOTIC BEHAVIOUR OF A TWO-DIMENSIONAL DIFFERENTIAL SYSTEM WITH A NONCONSTANT DELAY UNDER THE CONDITIONS OF INSTABILITY

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(Received October 15, 2009)

Abstract. We present several results dealing with the asymptotic behaviour of a real two-dimensional system $x'(t) = \boldsymbol{A}(t)x(t) + \sum\limits_{k=1}^{m} \boldsymbol{B}_k(t)x(\theta_k(t)) + h(t, x(t), x(\theta_1(t)), \dots, x(\theta_m(t)))$ with bounded nonconstant delays $t - \theta_k(t) \geqslant 0$ satisfying $\lim\limits_{t\to\infty} \theta_k(t) = \infty$, under the assumption of instability. Here $\boldsymbol{A}$, $\boldsymbol{B}_k$ and h are supposed to be matrix functions and a vector function, respectively. The conditions for the instable properties of solutions together with the conditions for the existence of bounded solutions are given. The methods are based on the transformation of the real system considered to one equation with complex-valued coefficients. Asymptotic properties are studied by means of a suitable Lyapunov-Krasovskii functional and the Ważewski topological principle. The results generalize some previous ones, where the asymptotic properties for two-dimensional systems with one constant or nonconstant delay were studied.

Keywords: delayed differential equations, asymptotic behaviour, boundedness of solutions, Lyapunov method, Ważewski topological principle

MSC 2010: 34K12, 34K20

1. Introduction

Consider the real two-dimensional system

$$(1.1)\qquad x'(t) = \boldsymbol{A}(t)x(t) + \sum_{k=1}^{m} \boldsymbol{B}_k(t)x(\theta_k(t)) + h(t, x(t), x(\theta_1(t)), \dots, x(\theta_m(t))),$$

where $\theta_k(t)$ are real functions, $\boldsymbol{A}(t) = (a_{jk}(t))$, $\boldsymbol{B}_l(t) = (b_{jkl}(t))$ $(j,k = 1,2;$ $l = 1,2,\dots,m)$ are real square matrices and a real vector function h is given by $h(t,x,y) = (h_1(t,x,y_1,\dots,y_m), h_2(t,x,y_1,\dots,y_m))$, whereas $x = (x_1,x_2)$,

$y_k = (y_{1k}, y_{2k})$ $(k = 1, 2, \ldots, m)$. It is supposed that the functions θ_k, a_{jk} are locally absolutely continuous on $[t_0, \infty)$, b_{jkl} are locally Lebesgue integrable on $[t_0, \infty)$ and the function h satisfies the Carathéodory conditions on $[t_0, \infty) \times \mathbb{R}^{2(m+1)}$.

There are a lot of papers dealing with the stability and asymptotic behaviour of n-dimensional real vector equations with or without delay, such as [1], [2]. Since the plane has special topological properties different from those of the n-dimensional space, where $n \geqslant 3$ or $n = 1$, it is interesting to study the asymptotic behaviour of two-dimensional systems by using tools which are typical and effective for two-dimensional systems, which gives different results likely incomparable with those for general systems. A convenient tool is the combination of the method of complexification and the method of the Lyapunov-Krasovskii functional. For the case of instability, it is useful to add to this combination a Razumikhin-type version of the Ważewski topological principle formulated in the paper [13].

Using these tools, stability and asymptotic properties of the solutions of two-dimensional ordinary differential systems (a special case of (1.1)) were studied in [9] (stable case) and [8] (unstable case). Similar results for two-dimensional differential systems with one constant delay were published in [7] (stable case) and [6], [5] (unstable case). The results were extended and new corollaries were presented for systems with a finite number of constant delays in [12], [11] (stable case only). The results concerning asymptotic properties of solutions for the stable case of (1.1) with (generally unbounded) nonconstant delay can be found in [4] and in [10]. In the present paper we shall give results for the unstable case for (1.1). These results generalize those of [3] for the case of several delays.

2. Preliminaries

Throughout the paper we use the following notation:

$\mathbb{R}$ ($\mathbb{C}$)	set of all real (complex) numbers,
$\mathbb{R}_+$ ($\mathbb{R}_+^0$, $\mathbb{R}_-$, $\mathbb{R}_-^0$)	set of all positive (non-negative, negative, non-positive) numbers,
$\mathcal{C}$	class of all continuous functions $[-r, 0] \to \mathbb{C}$,
$AC_{\mathrm{loc}}(I, M)$	class of all locally absolutely continuous functions $I \to M$,
$L_{\mathrm{loc}}(I, M)$	class of all locally Lebesgue integrable functions $I \to M$,
$K(I \times \Omega, M)$	class of all functions $I \times \Omega \to M$ satisfying the Carathéodory conditions on $I \times \Omega$,
$\operatorname{Re} z$ ($\operatorname{Im} z$)	real (imaginary) part of z,
$\overline{z}$	complex conjugate of z.

Introducing complex variables $z = x_1 + \mathrm{i}x_2$, $w_k = y_{1k} + \mathrm{i}y_{2k}$ $(k = 1, 2, \dots, m)$, we can rewrite (1.1) into an equivalent equation with complex-valued coefficients

$$
\begin{aligned}
z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{k=1}^{m}[A_k(t)z(\theta_k(t)) + B_k(t)\overline{z}(\theta_k(t))] \\
+ g\big(t, z(t), z(\theta_1(t)), \dots, z(\theta_m(t))\big),
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
a(t) &= \frac{1}{2}(a_{11}(t) + a_{22}(t)) + \frac{\mathrm{i}}{2}(a_{21}(t) - a_{12}(t)),\\
b(t) &= \frac{1}{2}(a_{11}(t) - a_{22}(t)) + \frac{\mathrm{i}}{2}(a_{21}(t) + a_{12}(t)),\\
A_k(t) &= \frac{1}{2}(b_{11k}(t) + b_{22k}(t)) + \frac{\mathrm{i}}{2}(b_{21k}(t) - b_{12k}(t)),\\
B_k(t) &= \frac{1}{2}(b_{11k}(t) - b_{22k}(t)) + \frac{\mathrm{i}}{2}(b_{21k}(t) + b_{12k}(t)),\\
g(t, z, w_1, \dots, w_m) &= h_1\Big(t, \frac{1}{2}(z + \overline{z}), \frac{1}{2\mathrm{i}}(z - \overline{z}), \frac{1}{2}(w_1 + \overline{w_1}), \frac{1}{2\mathrm{i}}(w_1 - \overline{w_1}), \dots,\\
&\frac{1}{2}(w_m + \overline{w_m}), \frac{1}{2\mathrm{i}}(w_m - \overline{w_m})\Big) + \mathrm{i}h_2\Big(t, \frac{1}{2}(z + \overline{z}), \frac{1}{2\mathrm{i}}(z - \overline{z}), \frac{1}{2}(w_1 + \overline{w_1}),\\
&\frac{1}{2\mathrm{i}}(w_1 - \overline{w_1}), \dots, \frac{1}{2}(w_m + \overline{w_m}), \frac{1}{2\mathrm{i}}(w_m - \overline{w_m})\Big).
\end{aligned}
$$

Conversely, the equation (2.1) can be written in the real form (1.1).

3. Assumptions

Consider the equation

$$
\begin{aligned}
z'(t) = a(t)z(t) + b(t)\overline{z}(t) + \sum_{k=1}^{m}[A_k(t)z(\theta_k(t)) + B_k(t)\overline{z}(\theta_k(t))] \\
+ g\big(t, z(t), z(\theta_1(t)), \dots, z(\theta_m(t))\big),
\end{aligned}
\tag{3.1}
$$

where $\theta_k \in AC_{\mathrm{loc}}(J, \mathbb{R})$, $a, b \in AC_{\mathrm{loc}}(J, \mathbb{C})$, $A_k, B_k \in L_{\mathrm{loc}}(J, \mathbb{C})$, $g \in K(J \times \mathbb{C}^{m+1}, \mathbb{C})$, $J = [t_0, \infty)$. Hereafter we shall suppose that (3.1) satisfies the uniqueness property of solutions.

We shall consider the case

$$
\liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0, \qquad t - r \leqslant \theta_k(t) \leqslant t \quad \text{for} \quad t \geqslant t_0 + r,
$$

where $r > 0$ is a constant. Our assumptions imply the existence of numbers $T \geqslant t_0 + r$ and $\mu > 0$ such that

$$(3.2)\quad |a(t)| > |b(t)| + \mu \quad \text{for } t \geqslant T - r, \qquad t \geqslant \theta_k(t) \geqslant t - r \quad \text{for } t \geqslant T \ (k = 1, 2, \ldots, m).$$

Denote

$$\gamma(t) = |a(t)| + \sqrt{|a(t)|^2 - |b(t)|^2}, \quad c(t) = \frac{\bar{a}(t)b(t)}{|a(t)|}. \tag{3.3}$$

Since $\gamma(t) > |a(t)|$ and $|c(t)| = |b(t)|$, the inequality

$$\gamma(t) > |c(t)| + \mu \tag{3.4}$$

is valid for $t \geqslant T - r$. It can be easily verified that $\gamma, c \in AC_{\rm loc}([T - r, \infty), \mathbb{C})$.

Throughout the paper we denote

$$\begin{aligned} \alpha(t) &= 1 - \left|\frac{b(t)}{a(t)}\right| \operatorname{sgn} \operatorname{Re} a(t), \\ \vartheta(t) &= \frac{\operatorname{Re}(\gamma(t)\gamma'(t) - \bar{c}(t)c'(t)) - |\gamma(t)c'(t) - \gamma'(t)c(t)|}{\gamma^2(t) - |c(t)|^2}. \end{aligned} \tag{3.5}$$

The equation (3.1) will be studied subject to suitable subsets of the following assumptions:

(i) The numbers $T \geqslant t_0 + r$ and $\mu > 0$ are such that (3.2) holds.

(ii) There exist functions $\varkappa, \kappa_k, \varrho\colon [T, \infty) \to \mathbb{R}$ such that

$$\begin{aligned} |\gamma(t)g(t, z, w_1, \ldots, w_m) + c(t)\overline{g}(t, z, w_1, \ldots, w_m)| \leqslant \varkappa(t)|\gamma(t)z + c(t)\overline{z}| \\ + \sum_{k=1}^{m} \kappa_k(t)|\gamma(\theta_k(t))w_k + c(\theta_k(t))\overline{w_k}| + \varrho(t) \end{aligned}$$

for $t \geqslant T$, $z, w_k \in \mathbb{C}$ $(k = 1, 2, \ldots, m)$, where ϱ is continuous on $[T, \infty)$.

(ii$_0$) There exist numbers $R_0 \geqslant 0$ and functions $\varkappa_0, \kappa_{0k}\colon [T, \infty) \to \mathbb{R}$ such that

$$\begin{aligned} &|\gamma(t)g(t, z, w_1, \ldots, w_m) + c(t)\overline{g}(t, z, w_1, \ldots, w_m)| \\ &\qquad \leqslant \varkappa_0(t)|\gamma(t)z + c(t)\overline{z}| + \sum_{k=1}^{m} \kappa_{0k}(t)|\gamma(\theta_k(t))w_k + c(\theta_k(t))\overline{w_k}| \end{aligned}$$

for $t \geqslant \tau_0 \geqslant T$, $|z| + \sum\limits_{k=1}^{m} |w_k| > R_0$.

(iii) $\beta \in AC_{\rm loc}([T,\infty), \mathbb{R}^0_-)$ is a function satisfying

$$\theta_k'(t)\beta(t) \leqslant -\lambda_k(t) \quad \text{a.e. on } [T,\infty), \tag{3.6}$$

where λ_k is defined for $t \geqslant T$ by

$$\lambda_k(t) = \kappa_k(t) + (|A_k(t)| + |B_k(t)|)\frac{|\gamma(t)| + |c(t)|}{|\gamma(\theta_k(t))| - |c(\theta_k(t))|}. \tag{3.7}$$

(iii$_0$) $\beta_0 \in AC_{\rm loc}[T,\infty), \mathbb{R}^0_-)$ is a function satisfying

$$\theta_k'(t)\beta_0(t) \leqslant -\lambda_{0k}(t) \quad \text{a.e. on } [\tau_0,\infty), \tag{3.8}$$

where λ_{0k} is defined for $t \geqslant T$ by

$$\lambda_{0k}(t) = \kappa_{0k}(t) + (|A_k(t)| + |B_k(t)|)\frac{|\gamma(t)| + |c(t)|}{|\gamma(\theta_k(t))| - |c(\theta_k(t))|}. \tag{3.9}$$

(iv$_0$) Λ_0 is a real locally Lebesgue integrable function satisfying the inequalities $\beta_0'(t) \geqslant \Lambda_0(t)\beta_0(t)$, $\Theta_0(t) \geqslant \Lambda_0(t)$ for almost all $t \in [\tau_0,\infty)$, where Θ_0 is defined by

$$\Theta_0(t) = \alpha(t)\operatorname{Re} a(t) + \vartheta(t) - \varkappa_0(t) + m\beta_0(t). \tag{3.10}$$

Furthermore, denote

$$\Theta(t) = \alpha(t)\operatorname{Re} a(t) + \vartheta(t) - \varkappa(t). \tag{3.11}$$

Obviously, if A_k, B_k, κ_k, θ_k' are locally absolutely continuous on $[T,\infty)$ and $\lambda_k(t) \geqslant 0$, $\theta_k'(t) > 0$ with $\theta_k'^{-1}$ bounded on any compact subinterval of $[T,\infty)$, the choice $\beta(t) = -\max\limits_{k=1,\dots,m}[\lambda_k(t)(\theta_k'(t))^{-1}]$ is admissible in (iii). Similarly, if A_k, B_k, κ_{0k}, θ_k' are locally absolutely continuous on $[T,\infty)$ and $\lambda_{0k}(t) \geqslant 0$, $\theta_k'(t) > 0$ with $\theta_k'^{-1}$ bounded on any compact subinterval of $[T,\infty)$, the choice $\beta_0(t) = -\max\limits_{k=1,\dots,m}[\lambda_{0k}(t)(\theta_k'(t))^{-1}]$ is admissible in (iii$_0$). Moreover, it can be easily verified that the function ϑ is locally Lebesgue integrable on $[T,\infty)$, assuming that (i) holds true. If the relations $\beta_0 \in AC_{\rm loc}([T,\infty), \mathbb{R}_-)$, $\varkappa_0 \in L_{\rm loc}([T,\infty), \mathbb{R})$ and $\beta_0'(t)/\beta_0(t) \leqslant \Theta_0(t)$ for almost all $t \geqslant \tau_0$ together with the conditions (i), (ii$_0$) are fulfilled, then we can choose $\Lambda_0(t) = \Theta_0(t)$ for $t \in [T,\infty)$ in (iv$_0$).

4. Results

The asymptotic properties for the solutions of the equation (3.1) with one bounded nonconstant delay under the conditions of instability were studied in [3]. Here we give the generalization of some of these results to the case of a finite number of delays.

Theorem 4.1. *Let the assumptions* (i), (ii$_0$), (iii$_0$), (iv$_0$) *be fulfilled for some* $\tau_0 \geqslant T$. *Suppose there exist* $t_1 \geqslant \tau_0$ *and* $\nu \in (-\infty, \infty)$ *such that*

$$\inf_{t \geqslant t_1} \left[\int_{t_1}^{t} \Lambda_0(s)\,\mathrm{d}s - \ln(\gamma(t) + |c(t)|) \right] \geqslant \nu. \tag{4.1}$$

If $z(t)$ *is any solution of* (3.1) *satisfying*

$$\min_{\theta(t_1) \leqslant s \leqslant t_1} |z(s)| > R_0, \qquad \Delta(t_1) > R_0 \mathrm{e}^{-\nu}, \tag{4.2}$$

where $\theta(t) = \min_{k=1,2,\dots,m} \theta_k(t)$, $\Delta(t) = (\gamma(t) - |c(t)|)|z(t)| + \beta_0(t) \max_{\theta(t_1) \leqslant s \leqslant t} |z(s)| \times \sum_{k=1}^{m} \int_{\theta_k(t_1)}^{t_1} (\gamma(s) + |c(s)|)\,\mathrm{d}s$, *then*

$$|z(t)| \geqslant \frac{\Delta(t_1)}{\gamma(t) + |c(t)|} \exp\left[\int_{t_1}^{t} \Lambda_0(s)\,\mathrm{d}s \right] \tag{4.3}$$

for all $t \geqslant t_1$ *for which* $z(t)$ *is defined.*

Proof. Let $z(t)$ be any solution of (3.1) satisfying (4.2). Denote $w_k(t) = z(\theta_k(t))$ and use a Lyapunov function

$$V(t) = U(t) + \beta_0(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} U(s)\,\mathrm{d}s, \tag{4.4}$$

where $U(t) = |\gamma(t) z(t) + c(t) \overline{z}(t)|$. Proceeding similarly to [3, Theorem 1], we obtain

$$\begin{aligned} U' &\geqslant U(\alpha \operatorname{Re} a + \vartheta - \varkappa_0) - (\gamma + |c|) \sum_{k=1}^{m} (|A_k| + |B_k|)|w_k| \\ &\quad - \sum_{k=1}^{m} \kappa_{0k} |\gamma(\theta_k(t)) w_k + c(\theta_k(t)) \overline{w_k}| \\ &\geqslant U(\alpha \operatorname{Re} a + \vartheta - \varkappa_0) - \sum_{k=1}^{m} \lambda_{0k} |\gamma(\theta_k(t)) w_k + c(\theta_k(t)) \overline{w_k}| \end{aligned}$$

for almost all $t \in \mathcal{K} := \left\{ t \geqslant t_1 \colon\ z(t) \text{ exists}, \min_{s \in \langle t_1, t \rangle} |z(s)| > R_0 \right\}$. Consequently, using (4.4) and (3.8), we obtain

$$V'(t) \geqslant U(t) \Theta_0(t) + \beta_0'(t) \sum_{k=1}^{m} \int_{\theta_k(t)}^{t} |\gamma(s) z(s) + c(s) \overline{z}(s)|\,\mathrm{d}s.$$

Hence, in view of (iv_0), we get $V'(t) - \Lambda_0(t)V(t) \geqslant 0$ for almost all $t \in \mathcal{K}$. With respect to (4.4), we have

$$(\gamma(t) + |c(t)|)|z(t)| \geqslant V(t) \geqslant V(t_1)\exp\left[\int_{t_1}^{t} \Lambda_0(s)\,\mathrm{d}s\right] \geqslant \Delta(t_1)\exp\left[\int_{t_1}^{t} \Lambda_0(s)\,\mathrm{d}s\right]$$

on any interval $[t_1, \omega)$ where the solution $z(t)$ exists and satisfies the inequality $|z(t)| > R_0$. If (4.2) is fulfilled, there is an $R > R_0$ such that $\Delta(t_1) > R\mathrm{e}^{-\nu}$. By virtue of (4.1) and (4.2) we can easily see that

$$|z(t)| \geqslant \frac{\Delta(t_1)}{\gamma(t) + |c(t)|}\exp\left[\int_{t_1}^{t} \Lambda_0(s)\,\mathrm{d}s\right] \geqslant R\mathrm{e}^{-\nu}\mathrm{e}^{\nu} = R$$

for all $t \geqslant t_1$ for which $z(t)$ is defined. $\square$

Corollary 4.1. *Let the assumptions of Theorem* 4.1 *be fulfilled with* $R_0 > 0$. *If*

$$\lim_{t\to\infty}\left[\int_{t_1}^{t} \Lambda_0(s)\,\mathrm{d}s - \ln(\gamma(t) + |c(t)|)\right] = \infty,$$

then for any $\varepsilon > 0$ *there exists a* $t_2 \geqslant t_1$ *such that* $|z(t)| > \varepsilon$ *holds for all* $t \geqslant t_2$ *for which* $z(t)$ *is defined.*

The proof of the next theorem is based on the results of K. P. Rybakowski [13] on a Ważewski topological principle for retarded functional differential equations of Carathéodory type.

Theorem 4.2. *Let the conditions* (i), (ii), (iii) *be fulfilled and let* Λ, θ_k' ($k = 1, 2, \ldots, m$) *be continuous functions such that the inequality* $\Lambda(t) \leqslant \Theta(t)$ *holds a.e. on* $[T, \infty)$, *where* Θ *is defined by* (3.11). *Suppose that* $\xi\colon [T - r, \infty) \to \mathbb{R}$ *is a continuous function such that*

$$\Lambda(t) + \beta(t)\sum_{k=1}^{m}\theta_k'(t)\exp\left[-\int_{\theta_k(t)}^{t}\xi(s)\,\mathrm{d}s\right] - \xi(t) > \varrho(t)C^{-1}\exp\left[-\int_{T}^{t}\xi(s)\,\mathrm{d}s\right] \tag{4.5}$$

for $t \in [T, \infty]$ *and some constant* $C > 0$. *Then there exist a* $t_2 > T$ *and a solution* $z_0(t)$ *of* (3.1) *satisfying*

$$|z_0(t)| \leqslant \frac{C}{\gamma(t) - |c(t)|}\exp\left[\int_{T}^{t}\xi(s)\,\mathrm{d}s\right] \tag{4.6}$$

for $t \geqslant t_2$.

P r o o f. Let $\tau > T$. Write the equation (3.1) in the form

$$z' = F(t, z_t), \tag{4.7}$$

where $F\colon\ J \times \mathcal{C} \to \mathbb{C}$ is defined by

$$\begin{aligned} F(t,\psi) = a(t)\psi(0) + b(t)\overline{\psi}(0) + \sum_{k=1}^{m}[A_k(t)\psi(\theta_k(t)-t) + B_k(t)\overline{\psi}(\theta_k(t)-t)] \\ + g(t,\psi(0),\psi(\theta_1(t)-t),\dots,\psi(\theta_m(t)-t)) \end{aligned}$$

and z_t is the element of $\mathcal{C}$ defined by the relation $z_t(\tilde\theta) = z(t+\tilde\theta)$, $\tilde\theta \in [-r,0]$. Put

$$\begin{aligned} \widetilde{U}(t,z,\overline{z}) &= |\gamma(t)z + c(t)\overline{z}| - \varphi(t), \quad \varphi(t) = C\exp\left[\int_T^t \xi(s)\,\mathrm{d}s\right], \\ \Omega^0 &= \{(t,z)\in(\tau,\infty)\times\mathbb{C}\colon\ \widetilde{U}(t,z,\overline{z}) < 0\}, \\ \Omega_{\widetilde{U}} &= \{(t,z)\in(\tau,\infty)\times\mathbb{C}\colon\ \widetilde{U}(t,z,\overline{z}) = 0\}. \end{aligned}$$

It can be easily verified that Ω^0 is a polyfacial set generated by functions $\widehat{U}(t) = \tau - t$, $\widetilde{U}(t,z,\overline{z})$ (see Rybakowski [13, p. 134]). It holds that $\Omega_{\widetilde{U}} \subset \partial\Omega^0$. As $(\gamma(t) + |c(t)|)|z(t)| \geqslant |\gamma(t)z + c(t)\overline{z}|$, we have

$$|z| \geqslant \frac{\varphi(t)}{\gamma(t)+|c(t)|} = \frac{C}{\gamma(t)+|c(t)|}\exp\left[\int_T^t \xi(s)\,\mathrm{d}s\right] > 0$$

for $(t,z)\in\Omega_{\widetilde{U}}$. We have $D^+\widehat{U}(t) = \frac{\partial}{\partial t}(\tau - t) = -1 < 0$. Let $(t^*,\zeta)\in\Omega_{\widetilde{U}}$ and $\phi\in\mathcal{C}$ be such that $\phi(0)=\zeta$ and $(t^*+\tilde\theta,\phi(\tilde\theta))\in\Omega^0$ for all $\tilde\theta\in[-r,0)$. If $(t,\psi)\in(\tau,\infty)\times\mathcal{C}$, then

$$\begin{aligned} &D^+\widetilde{U}(t,\psi(0),\overline{\psi}(0)) \\ &\quad := \limsup_{h\to 0+}(1/h)[\widetilde{U}(t+h,\psi(0)+hF(t,\psi),\overline{\psi}(0)+h\overline{F}(t,\psi)) - \widetilde{U}(t,\psi(0),\overline{\psi}(0))] \\ &\quad = \frac{\partial\widetilde{U}(t,\psi(0),\overline{\psi}(0))}{\partial t} + \frac{\partial\widetilde{U}(t,\psi(0),\overline{\psi}(0))}{\partial z}F(t,\psi) + \frac{\partial\widetilde{U}(t,\psi(0),\overline{\psi}(0))}{\partial\overline{z}}\overline{F}(t,\psi). \end{aligned}$$

Similarly to [3] we obtain

$$\begin{aligned} D^+\widetilde{U}(t,\psi(0),\overline{\psi}(0)) &\geqslant \Lambda(t)|\gamma(t)\psi(0)+c(t)\overline{\psi}(0)| \\ &+ \beta(t)\sum_{k=1}^{m}\theta_k'(t)|\gamma(\theta_k(t))\psi(\theta_k(t)-t) + c(\theta_k(t))\overline{\psi}(\theta_k(t)-t)| - \varrho(t) - \varphi'(t) \end{aligned}$$

for almost all $t\in(\tau,\infty)$ and for $\psi\in\mathcal{C}$ sufficiently close to ϕ.

Replacing t and ψ by t^* and ϕ, respectively, in the last right-hand side, we get

$$\begin{aligned}
&\Lambda(t^*)|\gamma(t^*)\phi(0)+c(t^*)\bar{\phi}(0)| \\
&\quad+\beta(t^*)\sum_{k=1}^{m}\theta_k'(t^*)|\gamma(\theta_k(t^*))\phi(\theta_k(t^*)-t^*)+c(\theta_k(t^*))\bar{\phi}(\theta_k(t^*)-t^*)|-\varrho(t^*)-\varphi'(t^*) \\
&\geqslant \Lambda(t^*)|\gamma(t^*)\zeta+c(t^*)\bar{\zeta}|+\beta(t^*)\sum_{k=1}^{m}\theta_k'(t^*)\varphi(\theta_k(t^*))-\varrho(t^*)-\varphi'(t^*) \\
&\geqslant \Lambda(t^*)\varphi(t^*)+\beta(t^*)\sum_{k=1}^{m}\theta_k'(t^*)\varphi(\theta_k(t^*))-\varrho(t^*)-\varphi'(t^*) \\
&=\Big\{\Lambda(t^*)+\beta(t^*)\sum_{k=1}^{m}\theta_k'(t^*)\exp\Big[-\int_{\theta_k(t^*)}^{t^*}\xi(s)\,\mathrm{d}s\Big]-\xi(t^*)\Big\}C\exp\Big[\int_T^{t^*}\xi(s)\,\mathrm{d}s\Big]-\varrho(t^*).
\end{aligned}$$

Clearly, the expression on the last right-hand side is positive. Therefore, in view of continuity, $D^+\widetilde{U}(t,\psi(0),\overline{\psi}(0))>0$ holds for ψ sufficiently close to ϕ and almost all t sufficiently close to t^*. Hence Ω^0 is a regular polyfacial set with respect to (4.7).

The rest of the proof is the same as that of [3, Theorem 4]. Using a topological principle for retarded functional differential equations (see Rybakowski [13, Theorem 2.1]), we infer that there is a solution $z_0(t)$ of (3.1) such that $|z_0(t)|\leqslant \varphi(t)/(\gamma(t)-|c(t)|)$ for $t\geqslant t_2$. □

As a corollary of Theorem 4.2 we obtain sufficient conditions for the existence of a bounded solution of (3.1) or the existence of a solution $z_0(t)$ of (3.1) satisfying $\lim\limits_{t\to\infty} z_0(t)=0$.

Corollary 4.2. *Let the assumptions of Theorem* 4.2 *be satisfied. If*

$$\limsup_{t\to\infty}\Big[\frac{1}{\gamma(t)-|c(t)|}\exp\Big(\int_T^t\xi(s)\,\mathrm{d}s\Big)\Big]<\infty,$$

then there is a bounded solution $z_0(t)$ *of* (3.1). *If*

$$\lim_{t\to\infty}\Big[\frac{1}{\gamma(t)-|c(t)|}\exp\Big(\int_T^t\xi(s)\,\mathrm{d}s\Big)\Big]=0,$$

then there is a solution $z_0(t)$ *of* (3.1) *such that*

$$\lim_{t\to\infty} z_0(t)=0.$$

A c k n o w l e d g m e n t. The research was supported by grant 201/08/0469 of the Czech Science Foundation. This support is gratefully acknowledged.

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