HOMOGENIZATION OF QUADRATIC COMPLEMENTARY ENERGIES: A DUALITY EXAMPLE

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Abstract. We study an example in two dimensions of a sequence of quadratic functionals whose limit energy density, in the sense of Γ-convergence, may be characterized as the dual function of the limit energy density of the sequence of their dual functionals. In this special case, Γ-convergence is indeed stable under the dual operator. If we perturb such quadratic functionals with linear terms this statement is no longer true.

Keywords: Γ-convergence, oscillatory behaviour, Young measure, conjugate functional

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1. Introduction

Consider the family of quadratic functionals I_ε defined in $H_0^1(\Omega)$ by

$$I_\varepsilon(u) = \int_\Omega \frac{a_\varepsilon(x)}{2} |\nabla u(x)|^2 \, \mathrm{d}x, \tag{1.1}$$

where Ω is an open bounded set in $\mathbb{R}^2$, and the sequence of functions a_ε stands for a first order laminate, normal to the unit vector $e_1 = (1,0)$, given by

$$a_\varepsilon(x) = \alpha \chi_{(0,t)}\Big(\frac{x_1}{\varepsilon}\Big) + \beta\Big(1 - \chi_{(0,t)}\Big(\frac{x_1}{\varepsilon}\Big)\Big) \tag{1.2}$$

for every x in Ω, with $\alpha, \beta > 0$. Here $\chi_{(0,t)}(\cdot)$ is the characteristic function of the interval $(0,t) \subset (0,1)$ extended by periodicity to $\mathbb{R}$. It is known that, when ε approaches to 0, the sequence of functionals I_ε is Γ-convergent to the quadratic functional I defined by

$$I(u) = \int_\Omega \nabla u(x) \frac{A_0}{2} \nabla u(x) \, \mathrm{d}x, \tag{1.3}$$

where A_0 is the homogenized 2×2-matrix defined by

$$A_0 = \begin{pmatrix} \dfrac{\alpha\beta}{(1-t)\alpha + t\beta} & 0 \\ 0 & t\alpha + (1-t)\beta \end{pmatrix}.$$

(See [4], [10] and the references therein.)

Γ-convergence is a variational convergence for sequences of functionals introduced by De Giorgi and Franzoni in [6]. Since that many authors have made important contributions in this field (see [2], [3], [5] and the references therein). Γ-convergence of a family of functionals I_ε is based on the analysis of the asymptotic behaviour of the minimizers of each functional I_ε, when taking ε tending to 0. Whenever a sequence $\{I_\varepsilon\}$ is Γ-convergent to a functional I, it follows that the sequence of solutions of each associated Euler-Lagrange equation converges to the solution of the Euler-Lagrange equation associated with the Γ-limit functional. In this way, from the Γ-convergence of functionals (1.1) to functional (1.3), we know that the sequence of solutions u_ε of the elliptic problem

$$\begin{cases} \operatorname{div}[a_\varepsilon(x)\nabla u_\varepsilon(x)] = 0 & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

converges weakly in $H_0^1(\Omega)$ to the solution u_0 of the homogenized problem

$$\begin{cases} \operatorname{div}[A_0\nabla u_0(x)] = 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, consider the conjugate functional of I_ε, denoted by I_ε^*, defined in the Lebesgue space $L^2(\Omega;\mathbb{R}^2)$ by putting

$$I_\varepsilon^*(U) = \int_\Omega \frac{1}{2a_\varepsilon(x)}|U(x)|^2\,\mathrm{d}x \tag{1.4}$$

under the constraint $\operatorname{div} U = 0$ in Ω. Notice that, for a.e. $x \in \Omega$, the density $g_\varepsilon(x,\varrho) = \frac{1}{2}(a_\varepsilon(x))^{-1}|\varrho|^2$ is the conjugate function of $f_\varepsilon(x,\varrho) = \frac{1}{2}a_\varepsilon(x)|\varrho|^2$ in $\mathbb{R}^2$. An interesting question that arises is: how may we explicitly characterize the Γ-limit energy density of the sequence $\{I_\varepsilon^*\}$ knowing the representation of the limit energy density of the sequence $\{I_\varepsilon\}$? The aim of this work is to give an answer to this question, and to study whether a generalization is possible.

2. Some preliminaries

Let us recall in this section some definitions and results used throughout this work.

We say that the sequence $\{I_\varepsilon\}$ is Γ-convergent with respect to the weak topology of $W^{1,p}(\Omega)$, with $p > 1$, to the functional I if and only if for every function u in $W^{1,p}(\Omega)$:

1. for every sequence $\{u_\varepsilon\}$ converging weakly to u in $W^{1,p}(\Omega)$,

$$I(u) \leqslant \liminf_{\varepsilon \searrow 0} I_\varepsilon(u_\varepsilon);$$

2. there exists a sequence $\{u_\varepsilon\}$ converging weakly to u in $W^{1,p}(\Omega)$ for which

$$I(u) = \lim_{\varepsilon \searrow 0} I_\varepsilon(u_\varepsilon).$$

In the same way, we say that the sequence of functionals J_ε, defined in $L^p(\Omega;\mathbb{R}^n)$ under the divergence-free constraint, is Γ-convergent with respect to the weak topology of $L^p(\Omega;\mathbb{R}^n)$ to the functional J if and only if for every function U in $L^p(\Omega;\mathbb{R}^n)$ such that $\operatorname{div} U = 0$ in Ω:

a) for every sequence $\{U_\varepsilon\}$ converging weakly to U in $W^{1,p}(\Omega)$ and satisfying $\operatorname{div} U_\varepsilon = 0$ for every ε,

$$J(U) \leqslant \liminf_{\varepsilon \searrow 0} J_\varepsilon(U_\varepsilon);$$

b) there exists a sequence $\{U_\varepsilon\}$ converging weakly to U in $L^p(\Omega;\mathbb{R}^n)$ for which $\operatorname{div} U_\varepsilon = 0$ and

$$J(U) = \lim_{\varepsilon \searrow 0} J_\varepsilon(U_\varepsilon).$$

Since we are interested in the Γ-convergence of conjugate functionals, we define the conjugate function (Young-Fenchel transform) of $f\colon \mathbb{R}^n \to \mathbb{R}$ as the function $f^*\colon \mathbb{R}^n \to \mathbb{R}$ by putting

$$f^*(\varrho^*) = \sup_{\varrho \in \mathbb{R}^n} \left\{\varrho^* \cdot \varrho - f(\varrho)\right\},$$

so that $(f^*)^* = f$ whenever f is a lower semicontinuous convex function. Therefore, given a functional I defined in $H^1(\Omega)$ by

$$I(u) = \int_\Omega f(x, \nabla u(x))\, \mathrm{d}x,$$

where $f\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function convex in the second variable and with growth of order 2, its conjugate functional I^* is defined by

$$I^*(U) = \int_\Omega f^*(x, U(x))\,\mathrm{d}x$$

for every U in $L^2(\Omega; \mathbb{R}^n)$ such that $\operatorname{div} U = 0$ in Ω, where $f^*(x,\cdot)$ stands for the conjugate function of $f(x,\cdot)$ in $\mathbb{R}^n$ for a.e. $x \in \Omega$.

Young measures (see [1], [9], [13]) associated with relevant sequences turn out to be our main tool to study Γ-convergence of conjugate functionals. We say that a family of probability measures $\nu = \{\nu_x\}_{x\in\Omega}$ supported on $\mathbb{R}^n$ is the Young measure associated with a p-equi-integrable sequence of functions $\{u_\varepsilon\}$ if and only if

$$\lim_{\varepsilon\searrow 0} \int_E \varphi(u_\varepsilon(x))\,\mathrm{d}x = \int_E \int_{\mathbb{R}^n} \varphi(\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x$$

for every continuous function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ with growth of order p, and every open subset E of Ω. It can be proved (see [9]) that if $\nu = \{\nu_x\}_{x\in\Omega}$ is the Young measure associated with the sequence of functions $\{u_\varepsilon\}$, then

$$\liminf_{\varepsilon\searrow 0} \int_\Omega f(x, u_\varepsilon(x))\,\mathrm{d}x \geqslant \int_\Omega \int_{\mathbb{R}^n} f(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x$$

for every Carathéodory function $f\colon \Omega\times\mathbb{R}^n \to \mathbb{R}$. Moreover, the sequence of functions a_ε defined in (1.2) generates a homogeneous Young measure σ given by

$$\sigma = t\delta_\alpha + (1-t)\delta_\beta.$$

3. Pure quadratic case

In this section we present our main result on the explicit characterization of the limit energy density of the sequence of conjugate functionals (1.4).

Theorem 3.1. *The sequence of conjugate functionals*

$$I_\varepsilon^*(U) = \int_\Omega \frac{1}{2a_\varepsilon(x)} |U(x)|^2\,\mathrm{d}x,$$

defined in $L^2(\Omega;\mathbb{R}^2)$ under the divergence-free constraint, is Γ-convergent with respect to the weak topology in $L^2(\Omega;\mathbb{R}^2)$ to the conjugate functional I^* given by

$$I^*(U) = \int_\Omega U(x) \frac{A_0^{-1}}{2} U(x)\,\mathrm{d}x,$$

where A_0^{-1} is the inverse matrix of A_0 defined by

$$A_0^{-1} = \begin{pmatrix} \dfrac{t}{\alpha} + \dfrac{1-t}{\beta} & 0 \\ 0 & \dfrac{1}{t\alpha + (1-t)\beta} \end{pmatrix}.$$

P r o o f. According to the definition of Γ-convergence, we should prove first a lower inequality, and then the existence of a recovering sequence.

So, consider a sequence $\{U_\varepsilon\}$ converging weakly to U in $L^2(\Omega;\mathbb{R}^2)$ such that $\operatorname{div} U_\varepsilon = 0 = \operatorname{div} U$ in Ω for every ε. There exists a joint Young measure $\nu = \{\nu_x\}_{x\in\Omega}$ associated with the sequence of pairs $\{(a_\varepsilon, U_\varepsilon)\}$, which may be decomposed as

$$\nu_x = t\mu_{\alpha,x}\otimes\delta_\alpha + (1-t)\mu_{\beta,x}\otimes\delta_\beta \qquad \text{for a.e. } x\in\Omega,$$

for some probability measures $\mu_{\alpha,x}$ and $\mu_{\beta,x}$ supported on $\mathbb{R}^2$, since the Young measure associated with $\{a_\varepsilon\}$ is the homogeneous measure $\sigma = t\delta_\alpha + (1-t)\delta_\beta$. Therefore we have

$$\begin{aligned} &\liminf_{\varepsilon\searrow 0}\int_\Omega \frac{1}{2a_\varepsilon(x)}|U_\varepsilon(x)|^2\,\mathrm{d}x \\ &\quad\geqslant \int_\Omega\left[t\int_{\mathbb{R}^2}\frac{1}{2\alpha}|\varrho|^2\,\mathrm{d}\mu_{\alpha,x}(\varrho) + (1-t)\int_{\mathbb{R}^2}\frac{1}{2\beta}|\varrho|^2\,\mathrm{d}\mu_{\beta,x}(\varrho)\right]\mathrm{d}x \\ &\quad\geqslant \int_\Omega\left[t\frac{1}{2\alpha}|\varphi(\alpha,x)|^2 + (1-t)\frac{1}{2\beta}|\varphi(\beta,x)|^2\right]\mathrm{d}x \end{aligned}$$

applying the Jensen inequality in the last inequality, and taking $\varphi\colon\ \Omega\times\{\alpha,\beta\}\to\mathbb{R}^2$ defined by

$$\varphi(x,\lambda) = \int_{\mathbb{R}^2}\varrho\,\mathrm{d}\mu_{\lambda,x}(\varrho).$$

Notice that the sequence $\{U_\varepsilon\}$ converges weakly to U in $L^2(\Omega;\mathbb{R}^2)$ so that

$$U(x) = t\varphi(x,\alpha) + (1-t)\varphi(x,\beta) \qquad \text{for a.e. } x\in\Omega.$$

If we define the function $g\colon\ \mathbb{R}^2\to\mathbb{R}$ by putting

$$g(\varrho) = \min_{A,B\in\mathbb{R}^2}\left\{\frac{t}{2\alpha}|A|^2 + \frac{1-t}{2\beta}|B|^2\colon\ \varrho = tA + (1-t)B,\ (A-B)\cdot e_1 = 0\right\},$$

we realize that

$$\liminf_{\varepsilon\searrow 0}\int_\Omega\frac{1}{2a_\varepsilon(x)}|U_\varepsilon(x)|^2\,\mathrm{d}x \geqslant \int_\Omega g(U(x))\,\mathrm{d}x. \tag{3.1}$$

Solving the previous minimization problem we conclude that, for every $\varrho \in \mathbb{R}^2$,

$$g(\varrho) = \varrho \frac{A_0^{-1}}{2} \varrho.$$

We point out that $g(\cdot)$ is the conjugate function of $f(\varrho) = \varrho \frac{A_0}{2} \varrho$ in $\mathbb{R}^2$. In this way we prove the lower inequality.

It remains to prove the existence of a recovering sequence $\{V_\varepsilon\}$ in $L^2(\Omega; \mathbb{R}^2)$ for which the inequality (3.1) is indeed an equality, and $\operatorname{div} V_\varepsilon = 0$ in Ω for every ε. If we take U in $L^2(\Omega; \mathbb{R}^2)$ such that $\operatorname{div} U = 0$ in Ω, then from the definition of the function $g(\cdot)$, for a.e. $x \in \Omega$ we may consider the minimizer pair $(A(U(x)), B(U(x))) \in \mathbb{R}^2 \times \mathbb{R}^2$ for which

$$\begin{aligned} g(U(x)) &= \frac{t}{2\alpha} |A(U(x))|^2 + \frac{1-t}{2\beta} |B(U(x))|^2 \\ U(x) &= tA(U(x)) + (1-t)B(U(x)) \\ 0 &= [A(U(x)) - B(U(x))] \cdot e_1. \end{aligned}$$

Thus, let us consider the family of probability measures $\tau = \{\tau_x\}_{x \in \Omega}$ supported on $\mathbb{R}^2 \times \{\alpha, \beta\}$ given by

$$\tau_x(\lambda, \varrho) = t\delta_\alpha(\lambda) \otimes \delta_{A(U(x))}(\varrho) + (1-t)\delta_\beta(\lambda) \otimes \delta_{B(U(x))}(\varrho).$$

We know there exists a sequence $\{V_\varepsilon\}$ converging weakly in $L^2(\Omega; \mathbb{R}^2)$, under the divergent-free constraint, such that the sequence of pairs $\{(a_\varepsilon, V_\varepsilon)\}$ generates the Young measure τ. Thus we conclude that

$$\begin{aligned} \int_\Omega g(U(x))\, \mathrm{d}x &= \int_\Omega \left[\frac{t}{2\alpha} |A(U(x))|^2 + \frac{(1-t)}{2\beta} |B(U(x))|^2 \right] \mathrm{d}x \\ &= \int_\Omega \int_{\mathbb{R}^2 \times \mathbb{R}} \frac{1}{2\lambda} |\varrho|^2 \, \mathrm{d}\tau_x(\lambda, \varrho)\, \mathrm{d}x \\ &= \lim_{\varepsilon \searrow 0} \int_\Omega \frac{1}{2a_\varepsilon(x)} |V_\varepsilon(x)|^2 \, \mathrm{d}x. \end{aligned}$$

□

However, if we add some perturbation to the quadratic functional (1.1), then the limit energy density of the sequence of its conjugate functionals is no longer the conjugate function. Precisely, consider the sequence of quadratic functionals

$$F_\varepsilon(u) = \int_\Omega \left[\frac{a_\varepsilon(x)}{2}|\nabla u(x)|^2 + b_\varepsilon(x)\cdot\nabla u(x)\right]\,\mathrm{d}x,$$

where the sequence of functions $b_\varepsilon\colon\ \Omega \to \mathbb{R}^2$ is given by

$$b_\varepsilon(x) = \gamma\chi_{(0,t)}\Big(\frac{x_1}{\varepsilon}\Big) + \theta\Big(1 - \chi_{(0,t)}\Big(\frac{x_1}{\varepsilon}\Big)\Big).$$

Notice that the sequences $\{a_\varepsilon\}$ and $\{b_\varepsilon\}$, i.e. the quadratic and linear coefficients, oscillate at the same length scale ε. It is known (see [11]) that the sequence F_ε is Γ-convergent to the functional F defined by

$$F(u) = \int_\Omega \Big[\nabla u(x)\frac{A_0}{2}\nabla u(x) + B_0\cdot\nabla u(x) + c\Big]\,\mathrm{d}x, \tag{4.1}$$

where

$$B_0 = t\gamma + (1-t)\theta - \Big[\frac{t(1-t)(\alpha-\beta)^2}{(1-t)\alpha + t\beta}(\gamma-\theta)\cdot e_1\Big]e_1,$$
$$c = -\frac{t(1-t)(\alpha-\beta)^2}{2((1-t)\alpha+t\beta)}\left[(\gamma-\theta)\cdot e_1\right]^2,$$

and the homogenized linear term B_0 is a constant vector depending on the values of the sequence $\{a_\varepsilon\}$. On the other hand, it follows that the sequence of minimizers u_ε of F_ε, which are solutions of the elliptic equation

$$\begin{cases} \operatorname{div}\left[a_\varepsilon(x)\nabla u_\varepsilon(x) + b_\varepsilon(x)\right] = 0 & \text{in } \Omega,\\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

converges weakly in $H^1_0(\Omega)$ to the solution of the homogenized equation

$$\begin{cases} \operatorname{div}\left[A_0\nabla u_0(x)\right] = 0 & \text{in } \Omega,\\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Now let us present the result on the Γ-convergence of the sequence of the dual functionals.

Theorem 4.1. *The sequence of conjugate functionals*

$$F_\varepsilon^*(U) = \int_\Omega \Big[\frac{1}{2a_\varepsilon(x)}|U(x)|^2 - \frac{1}{a_\varepsilon(x)} b_\varepsilon(x)\cdot U(x) + \frac{|b_\varepsilon(x)|^2}{2a_\varepsilon(x)}\Big]\,\mathrm{d}x,$$

defined in $L^2(\Omega;\mathbb{R}^2)$ under the divergence-free constraint, is Γ-convergent with respect to the weak topology in $L^2(\Omega;\mathbb{R}^2)$ to the conjugate functional J given by

$$J(U) = \int_\Omega \Big[U(x)\frac{A_0^{-1}}{2}U(x) - D_0\cdot U(x) + \tilde{c}\Big]\,\mathrm{d}x,$$

where

$$\begin{aligned} D_0 &= \frac{t}{\alpha}\gamma + \frac{1-t}{\beta}\theta,\\ \tilde{c} &= \frac{t}{2\alpha}|\gamma|^2 + \frac{1-t}{2\beta}|\theta|^2 - \frac{t(1-t)}{2((1-t)\alpha + t\beta)}\Big[\Big(\frac{1}{\alpha}\gamma - \frac{1}{\beta}\theta\Big)\cdot e_2\Big]^2, \end{aligned}$$

with $e_2 = (0,1)$.

It turns out that the Γ-limit functional J is not the conjugate functional of F, i.e. the density of J is no longer the conjugate function of the integrand of F.

Proof. The proof follows closely from the main ideas stated in the proof of Theorem 3.1. Namely, first we prove that for every sequence $\{U_\varepsilon\}$ converging weakly to U in $L^2(\Omega;\mathbb{R}^2)$ so that $\operatorname{div} U_\varepsilon = \operatorname{div} U = 0$ in Ω, we have

$$\liminf_{\varepsilon\searrow 0} \int_\Omega \Big[\frac{1}{2a_\varepsilon(x)}|U_\varepsilon(x)|^2 - \frac{1}{a_\varepsilon(x)} b_\varepsilon(x)\cdot U_\varepsilon(x) + \frac{|b_\varepsilon(x)|^2}{2a_\varepsilon(x)}\Big]\,\mathrm{d}x \geqslant \int_\Omega g(U(x))\,\mathrm{d}x,$$

where $g\colon \mathbb{R}^2 \to \mathbb{R}$ is the function defined by

$$\begin{aligned} g(\varrho) = \min_{A,B\in\mathbb{R}^2} \Big\{ & t\Big(\frac{1}{2\alpha}|A|^2 - \frac{1}{\alpha}\gamma\cdot A + \frac{|\gamma|^2}{2\alpha}\Big) + (1-t)\Big(\frac{1}{2\beta}|B|^2 - \frac{1}{\beta}\theta\cdot B + \frac{|\theta|^2}{2\beta}\Big)\colon \\ & \varrho = tA + (1-t)B,\ (A-B)\cdot e_1 = 0\Big\}. \end{aligned}$$

Solving the previous minimizing problem we obtain the expression for $g(\cdot)$:

$$g(\varrho) = \varrho\frac{A_0^{-1}}{2}\varrho - D_0\cdot\varrho + \tilde{c}, \qquad \text{for every } \varrho\in\mathbb{R}^2,$$

so that $g(\cdot)$ is not the conjugate function of the density

$$f(\varrho) = \varrho\frac{A_0}{2}\varrho + B_0\cdot\varrho + c$$

of the Γ-limit F in (4.1).

Secondly, we prove that there exists a sequence $\{V_\varepsilon\}$ converging weakly to U in $L^2(\Omega;\mathbb{R}^2)$, so that $\operatorname{div} V_\varepsilon = \operatorname{div} U = 0$ in Ω, and

$$\lim_{\varepsilon\searrow 0}\int_\Omega \Big[\frac{1}{2a_\varepsilon(x)}|V_\varepsilon(x)|^2 - \frac{1}{a_\varepsilon(x)}b_\varepsilon(x)\cdot V_\varepsilon(x) + \frac{|b_\varepsilon(x)|^2}{2a_\varepsilon(x)}\Big]\,\mathrm{d}x = \int_\Omega g(U(x))\,\mathrm{d}x.$$

□

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