0-DISTRIBUTIVE POSETS

Khalid A. Mokbel, Al Hudaydah, Vilas S. Kharat, Pune

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Abstract. Several characterizations of 0-distributive posets are obtained by using the prime ideals as well as the semiprime ideals. It is also proved that if every proper l-filter of a poset is contained in a proper semiprime filter, then it is 0-distributive. Further, the concept of a semiatom in 0-distributive posets is introduced and characterized in terms of dual atoms and also in terms of maximal annihilator. Moreover, semiatomic 0-distributive posets are defined and characterized. It is shown that a 0-distributive poset P is semiatomic if and only if the intersection of all non dense prime ideals of P equals $(0]$. Some counterexamples are also given.

Keywords: 0-distributive poset, ideal, semiprime ideal, prime ideal, semiatom, semiatomic 0-distributive poset

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1. Introduction

The concept of a 0-distributive lattice is introduced by Grillet and Varlet [3]; a lattice L with 0 is called 0-*distributive* if, for all $a, b, c \in L$, $a \wedge b = 0 = a \wedge c$ implies $a \wedge (b \vee c) = 0$. Dually, one can define 1-*distributive* lattices; also see Varlet [14]. Independently, Varlet [15] and Pawar and Thakare [12] extended the concept of 0-distributivity in lattices to semilattices by different definitions; see also Jayaram [6], Rachůnek [13] and Pawar [10].

Pawar and Dhamke [11] extended the concept of 0-distributivity in lattices to posets. Joshi and Waphare [7] have also introduced and studied the concept of a 0-distributive poset which is completely independent of the definition introduced by Pawar and Dhamke [11]. Jayaram [6] introduced the concept of a *semiatom* in semilattices with 0 as a nonzero element a of a semilattice L with 0 if, for any pair $x, y \in L$, $x \wedge y = 0$ implies either $a \wedge x = 0$ or $a \wedge y = 0$. Further, he characterized semiatoms and semiatomicity in 0-distributive semilattices. We note that the 0-distributive lattices and 0-distributive semilattices have been studied by many authors with help of prime ideals.

In this paper we generalize some results of Varlet [14], Jayaram [6] and Pawar [10] for lattices and semilattices to posets by using the prime ideals as well as the semiprimene ideals. Further, we introduce the concept of semiatoms in posets, and characterize them in 0-distributive posets. Moreover, semiatomic 0-distributive posets are defined and characterized.

We begin with the necessary concepts and terminology. For undefined notation and terminology the reader is referred to Grätzer [2].

Let $A \subseteq P$. The set $A^u = \{x \in P;\ x \geqslant a \text{ for every } a \in A\}$ is called the *upper cone* of A. Dually, we have the concept of the *lower cone* A^l of A. We shall write A^{ul} instead of $\{A^u\}^l$ and dually. The upper cone $\{a\}^u$ is simply denoted by a^u and $\{a,b\}^u$ is denoted by $(a,b)^u$. Similar notation is used for lower cones. Further, for $A, B \subseteq P$, $\{A \cup B\}^u$ is denoted by $\{A, B\}^u$ and for $x \in P$, the set $\{A \cup \{x\}\}^u$ is denoted by $\{A, x\}^u$. Similar notation is used for lower cones. We note that $A \subseteq A^{ul}$ and $A \subseteq A^{lu}$. If $A \subseteq B$, then $B^l \subseteq A^l$ and $B^u \subseteq A^u$. Moreover, $A^{lul} = A^l$, $A^{ulu} = A^u$ and $\{a^u\}^l = \{a\}^l = a^l$.

2. 0-distributive posets

In this paper, we consider the definition of a 0-distributive poset introduced by Joshi and Waphare [7] as follows.

Definition 2.1. A poset P with 0 is called 0-*distributive* if, for $x, y, z \in P$, $(x,y)^l = \{0\}$ and $(x,z)^l = \{0\}$ together imply $\{x, (y,z)^u\}^l = \{0\}$.

Dually, we have the concept of a 1-*distributive* poset.

Now, we consider the concepts of an ideal and a prime ideal introduced by Halaš [4] and Halaš and Rachůnek [5].

Definition 2.2. A subset I of a poset P is called an *ideal* if $a, b \in I$ implies $(a,b)^{ul} \subseteq I$. A proper ideal I is called *prime* if $(a,b)^l \subseteq I$ implies that either $a \in I$ or $b \in I$.

Dually, we have the concepts of a *filter* and a *prime filter*. Given $a \in P$, the subset $\{x \in P;\ x \leqslant a\}$ is an ideal of P generated by a, denoted by $(a]$; we shall call $(a]$ a *principal ideal*. Dually, a filter $[a)$ generated by a is called a principal *filter*.

A nonempty subset Q of a poset P is called an *up directed set*, if $Q \cap (x,y)^u \neq \emptyset$ for any $x, y \in Q$. Dually, we have the concept of a *down directed set*. If an ideal I (filter F) is an up (down) directed set of P, then it is called a *u-ideal* (*l-filter*).

Beran [1] defined the concept of an I-atom in lattices and has shown that this concept plays a crucial role in the study of ideals.

Definition 2.3. Let I be an ideal of a poset P. An element $i \in P$ is called an *I-atom* if the following conditions hold.

(i) $i \notin I$, and

(ii) for $x \in P$, if $x < i$, then $x \in I$.

For the sake of completeness we note that an element p of a poset P is called an *atom* if

(i) $0 \prec p$ if 0 is the least element of P, or

(ii) p is a minimal element of P if P has no least element,

where $0 \prec p$ means there is no element $x \in P$ such that $0 < x < p$ holds. Dually, we have the concept of a *coatom* of P.

Remarks 2.4. (1) Consider the ideal $I = (a]$ of the poset P depicted in Figure 1. Observe that b is an I-atom of P but not an atom. Also, a is an atom of P but not an I-atom and c is both an I-atom and an atom.

(2) Let P be a poset. From the definitions of an atom and an I-atom we observe the following.

(i) If P has the least element 0, then $i \in P$ is a $(0]$-atom if and only if i is an atom of P.

(ii) If P has no least element, then $i \in P$ is a φ-atom if and only if i is an atom of P.

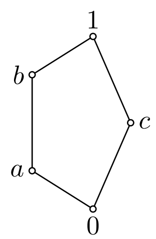

Figure 1

Throughout this section, P denotes a poset with 0. Now, we consider the concept of a semiprime ideal in posets introduced by Kharat and Mokbel [8].

Definition 2.5. An ideal I of a poset P is called *semiprime* if $(a,b)^l \subseteq I$ and $(a,c)^l \subseteq I$ together imply $\{a,(b,c)^u\}^l \subseteq I$.

Dually, we have the concept of a *semiprime filter*. The set of all semiprime ideals of a poset P forms a complete lattice with respect to set inclusion (see Kharat and Khalid [9]).

For an ideal I and a nonempty subset A of a poset P, define a subset $I : A$ of P as follows:

$$I : A = \{z \in P;\ (a, z)^l \subseteq I, \quad \forall\, a \in A\};$$

if $A = \{a\}$, then we write $I : a$ instead of $I : \{a\}$. It is clear that $I : A = \bigcap_{a \in A} I : a$ and $I \subseteq I : x\ \forall\, x \in P$.

From the definition of a semiprime ideal, it is clear that a poset P is 0-distributive if and only if $(0]$ is semiprime.

Lemma 2.6. (Kharat and Mokbel [8]). *Let I be an ideal of a poset P. Then I is semiprime if and only if $I : x$ is an ideal for all $x \in P$, in fact, a semiprime ideal. Moreover, if P is finite, then I is semiprime if and only if $I : i$ is a principal prime ideal for all I-atoms of P.*

An immediate consequence of Lemma 2.6:

Corollary 2.7. *Let P be a poset with 0. Then the following statements are equivalent:*

(i) *P is a 0-distributive poset,*
(ii) *$(0] : x$ is an ideal for all $x \in P$,*
(iii) *$(0] : A$ is an ideal for every nonempty subset A of P,*
(iv) *$(0] : x$ is a semiprime ideal for all $x \in P$,*
(v) *$(0] : A$ is a semiprime ideal for every nonempty subset A of P.*

We need the following result to obtain a characterization of 0-distributive posets.

Proposition 2.8. *Let P be a poset with 0. If every proper l-filter of a poset P is contained in a proper semiprime filter, then P is 0-distributive.*

Proof. Suppose that every proper l-filter of a poset P is contained in a proper semiprime filter and $(x, y)^l = \{0\} = (x, z)^l$. Suppose on the contrary that there exists a nonzero element $a \in P$ such that $a \in \{x, (y, z)^u\}^l$. We have $\{x, (y, z)^u\}^{lu} \subseteq [a)$ and since $[a)$ is a proper l-filter of P, there exists a proper semiprime filter F of P such that $[a) \subseteq F$. But $x \in [a) \subseteq F$ and $(y, z)^u \subseteq [a) \subseteq F$, so we have $(x, z)^u \subseteq F$ and $(y, z)^u \subseteq F$. By semiprimeness of F, we obtain $\{z, (x, y)^l\}^u \subseteq F$. Since $(x, y)^l = \{0\}$, we get $z^u = \{z, 0\}^u \subseteq F$ and so $z \in F$. Now, since $x, z \in F$ and $(x, z)^l = \{0\}$, we get $P = \{0\}^u = (x, z)^{lu} \subseteq F$. Thus $F = P$, which is a contradiction to the fact that F is proper. □

The following corollary is an immediate consequence of Proposition 2.8.

Corollary 2.9. *Let P be a poset with* 0*. If every proper l-filter of the poset P is contained in a prime filter, then P is* 0*-distributive.*

Lemma 2.10 (Kharat and Mokbel [8])**.** *Let I be a semiprime ideal and K an l-filter of a finite poset P for which $I \cap K = \emptyset$. Then there exists a semiprime filter F of P such that $K \subseteq F$ and $I \cap F = \emptyset$.*

As a consequence of Proposition 2.8 and Lemma 2.10, we have the following characterization of 0-distributivity in finite posets.

Corollary 2.11. *Let P be a finite poset with* 0*. Then P is* 0*-distributive if and only if every proper l-filter of a poset P is contained in a proper semiprime filter.*

The following result due to Halaš and Rachůnek [5], is useful to characterize 0-distributive posets.

Lemma 2.12 (Halaš and Rachůnek [5])**.** *Let I be a prime ideal of a poset P. Then $P - I$ is a filter in P. Moreover, $P - I$ is a prime filter if and only if I is an u-ideal. In this case, $P - I$ is an l-filter.*

Lemma 2.13 (Kharat and Mokbel [9])**.** *Every l-filter of a finite poset P is principal.*

Let I be a proper ideal of a poset P. Then I is said to be a maximal ideal of P if the only ideal properly containing I is P. A maximal filter, more usually known as an ultrafilter, is defined dually. Also, we have the concepts of minimal ideal and minimal filter.

Now, we establish the following characterization.

Theorem 2.14. *Let P be a finite poset with* 0*. Then the following statements are equivalent:*

(i) *P is* 0*-distributive,*

(ii) *every maximal l-filter is prime,*

(iii) *the set theoretic complement of every maximal l-filter is a minimal prime u-ideal,*

(iv) *every proper l-filter is disjoint with some prime u-ideal.*

P r o o f. (i) $\Rightarrow$ (ii) Suppose that P is 0-distributive and K is a maximal l-filter of P. Since P is finite, K is principal by Lemma 2.13, say $K = [q)$, where q is an atom in P. We are going to prove that K is a prime filter. Now, suppose that $(x, y)^u \subseteq [q)$ and $x, y \notin [q)$. We must have $(x, q)^l = \{0\} = (y, q)^l$; otherwise, if $(x, q)^l \neq \{0\}$, then there exists a nonzero element $z \in P$ such that $z \in (x, q)^l$. Since q is an atom, we get $z = q$, and this implies $x \in [q)$, a contradiction to the assumption. Now, by 0-distributivity we get $\{q, (x, y)^u\}^l = \{0\}$. But $(x, y)^u \subseteq [q)$ implies $q \in (x, y)^{ul}$ and consequently we have $q = 0$, a contradiction to the fact that q is an atom.

(ii) $\Rightarrow$ (iii) Suppose that every maximal l-filter of P is prime and K is a maximal l-filter. We have to show that $I = P - K$ is a minimal prime u-ideal. By assumption, K is a prime l-filter and by the dual of Lemma 2.12, I is a prime u-ideal. Now, if there exists a prime u-ideal J of P such that $J \subset I$, then there is an element $x \in P$ such that $x \in I = P - K$ and $x \notin J$. By Lemma 2.12, $P - J$ is an l-filter and $K \subset P - J$, as $x \in P - J$ and $x \notin K$. This is a contradiction to the maximality of K. Thus I is a minimal prime u-ideal as required.

(iii) $\Rightarrow$ (iv) Suppose that the set theoretic complement of every maximal l-filter of P is a minimal prime u-ideal and K is an arbitrary proper l-filter. Observe that for every such K, $(0] \cap K = \emptyset$. Since P is finite, there exists a maximal l-filter, say F, such that $K \subseteq F$ and $(0] \cap F = \emptyset$. In fact, F is a maximal l-filter of P. Hence $I = P - F$ is a prime u-ideal and $I \cap K = \emptyset$.

(iv) $\Rightarrow$ (i) Suppose that every proper l-filter of P is disjoint with some prime u-ideal and $(x, y)^l = \{0\} = (x, z)^l$. If there exists a nonzero element a of P such that $a \in \{x, (y, z)^u\}^l$, then we have $a \in x^l \cap (y, z)^{ul}$, and so $x \in [a)$ and $(y, z)^u \subseteq [a)$. Since $[a)$ is an l-filter, there exists a prime u-ideal I such that $I \cap [a) = \emptyset$. By Lemma 2.12, $D = P - I$ is a prime filter which also contains $[a)$. Hence $x \in D$ and $(y, z)^u \subseteq D$, and by primeness of D we must have either $x, y \in D$ or $x, z \in D$. Suppose $x, y \in D$, then we have $P = 0^u = (x, y)^{lu} \subseteq D$, a contradiction to the fact that D is a proper subset being prime. Similarly, we get a contradiction in the case when $x, z \in D$. Consequently, we must have $a = 0$, and so P is 0-distributive. □

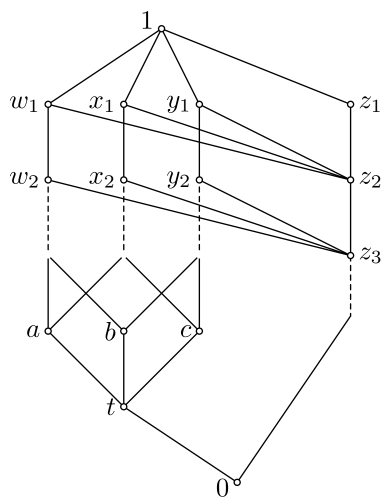

Figure 2

R e m a r k 2.15. Consider the infinite 0-distributive poset Q depicted in Figure 2. Observe that the filter $F = \bigcup\{\{w_i, x_i, y_i, z_i\};\ i = 1, 2, \ldots\} \cup \{1\}$ is a maximal l-filter of Q. However, it is not prime as $(a, b)^u \subseteq F$ and neither a nor b is in F. Therefore, the condition of finiteness on P in the statement of Theorem 2.14 is necessary.

Theorem 2.16. *Let P be a finite poset with 0. Then the following statements are equivalent:*

(i) *P is 0-distributive,*

(ii) *if $(0] : x \cap F = \emptyset$ for every l-filter F and for every $x \in P$, then there exists a prime filter D in P containing F and disjoint with $(0] : x$.*

P r o o f. (i) $\Rightarrow$ (ii) Suppose that P is 0-distributive and for $x \in P$, denote $I = (0] : x$. Suppose F is an l-filter such that $I \cap F = \emptyset$. By Lemma 2.13, F is principal, say $F = [d)$. Now $d \notin I$, therefore there exists an I-atom i of P such that $i \leqslant d$ and $i \notin I$. Observe that $d \notin I : i$, as if $d \in I : i$, then $i \in (d, i)^l \subseteq I$, a contradiction to the fact that i is an I-atom. In view of Lemma 2.6, $I : i$ is a principal prime ideal. We claim that $D = P - I : i$ is the required filter. By Lemma 2.12, D is prime. Since $d \notin I : i$, we have $d \in D$ and hence $F = [d) \subseteq D$. Finally, since $I \subseteq I : i$, we get $I \cap D = \emptyset$.

(ii) $\Rightarrow$ (i) Suppose $(x, y)^l = \{0\} = (x, z)^l$ and there exists a nonzero element a of P such that $a \in \{x, (y, z)^u\}^l$. Since $a \leqslant x$, we have $(0] : x \cap [a) = \emptyset$, as if $b \in (0] : x \cap [a)$, then $(x, b)^l = \{0\}$ and $a \leqslant b$, and hence $(x, a)^l = \{0\}$ which implies $a = 0$, a contradiction. Observe that $[a)$ is an l-filter, and by (ii) there exists a prime filter D such that $[a) \subseteq D$ and $(0] : x \cap D = \emptyset$. Since D is prime and $(y, z)^u \subseteq D$, we have $y \in D$ or $z \in D$. Suppose $y \in D$. Since $x \in D$, we have $P = \{0\}^u = (x, y)^{lu} \subseteq D$ and thus $D = P$, a contradiction to the fact that D is a proper subset being prime. Similarly, we get a contradiction in the case when $z \in D$. Consequently, we must have $a = 0$, and therefore P is 0-distributive. □

R e m a r k 2.17. We note that for the proof of (ii) $\Rightarrow$ (i), the condition of finiteness on P is not necessary, but it is necessary for the proof of (i) $\Rightarrow$ (ii). Indeed, consider the infinite 0-distributive poset Q depicted in Figure 2 and an l-filter $F = \{1\} \cup \{w_1, w_2, ...\}$. Observe that $(0] : z_1 \cap F = \emptyset$, where $(0] : z_1 = \{0, t, a, b, c\}$. But there does not exist a prime filter D of Q for which $F \subseteq D$ and $(0] : z \cap D = \emptyset$ hold.

3. Semiatomic 0-distributive posets

Definition 3.1. A nonzero element a of a poset P with 0 is called a *semiatom* if for any pair $x, y \in P$, $(x,y)^l = \{0\}$ implies either $(a,x)^l = \{0\}$ or $(a,y)^l = \{0\}$.

Clearly, every atom is a semiatom but the converse is not true in general. Consider the poset P depicted in Figure 1 and observe that b is a semiatom of P but not an atom. For a poset P, introduce the set $A(P) = \{(0] : x;\ x \in P\}$. Observe that $(A(P), \subseteq)$ is a poset with P as the greatest element and for $x \leqslant y$ in P, $(0] : y \subseteq (0] : x$. An ideal I of P is called *dense* if $(0] : I = \{0\}$, where $(0] : I = \{z \in P;\ (z,x)^l \subseteq (0]\ \forall\, x \in I\}$, otherwise it is called *non dense*. An element x of P is *dense* if $(0] : x = \{0\}$. Also, the set $(0] : I$ is called a *maximal annihilator* if $(0] : I \neq P$ and $(0] : I \subseteq (0] : B \neq P$ together imply $(0] : I = (0] : B$ for any nonempty subset B of P.

Lemma 3.2 (Kharat and Mokbel [8]). *Let I be a semiprime ideal of a poset P. Then the following statements hold for $x, a, b \in P$:*

(i) $(a,b)^l \subseteq I : x$ *if and only if* $(x,a,b)^l \subseteq I$,

(ii) $\{x, (a,b)^u\}^l \subseteq I$ *if and only if* $(a,b)^{ul} \subseteq I : x$,

(iii) $I : x = P$ *if and only if* $x \in I$.

Note: The statement (i) *does not require semiprimeness.*

The following theorem presents several characterizations of the semiatoms of 0-distributive posets that are equivalent.

Theorem 3.3. *Let a be a nonzero element of a* 0*-distributive poset P. Then the following statements are equivalent.*

(i) *a is a semiatom of P,*

(ii) $(0] : a = (0] : b$ *for all* $0 \neq b \leqslant a$,

(iii) $(0] : a$ *is a prime ideal of P,*

(iv) $(0] : a$ *is a dual atom of the poset* $(A(P), \subseteq)$,

(v) $(0] : a$ *is a maximal annihilator of P.*

Proof. (i) $\Rightarrow$ (ii) Suppose that a is a semiatom of P and b is a nonzero element of P such that $b \leqslant a$. It is enough to show that $(0] : b \subseteq (0] : a$, as the converse inclusion is trivial. Suppose $z \in (0] : b$, then we have $(b,z)^l = \{0\}$. Since a is a semiatom and $(a,b)^l \neq \{0\}$, we must have $(a,z)^l = \{0\}$. Hence $z \in (0] : a$ as required.

(ii) $\Rightarrow$ (iii) Suppose that $(0] : a = (0] : b$ for all $0 \neq b \leqslant a$. Since $(0]$ is a semiprime ideal, by Lemma 2.6, $(0] : a$ is an ideal. To show that $(0] : a$ is prime let $(x,y)^l \subseteq (0] : a$ and $x \notin (0] : a$. We have $(a,x)^l \neq \{0\}$, therefore there exists $z \in P$ such that $z \in (a,x)^l$ and $z \neq 0$. In other words, $0 \neq z \leqslant a$. By assumption we must have $(0] : a = (0] : z$. Now, since $z \leqslant x$ and $(x,y)^l \subseteq (0] : a = (0] : z$, we get $(z,y)^l \subseteq (0] : z$. By Lemma 3.2 (i), we have $(z,z,y)^l \subseteq (0]$, thus $y \in (0] : z = (0] : a$, as required.

(iii) $\Rightarrow$ (iv) Suppose that $(0] : a$ is a prime ideal of P. We shall prove that it is a dual atom of $A(P)$. Now, suppose $(0] : a \subset (0] : x \subseteq P$. Then there exists an element $z \in (0] : x$ and $z \notin (0] : a$, hence $(x,z)^l = \{0\} \subseteq (0] : a$ and $z \notin (0] : a$. By primeness of $(0] : a$, we must have $x \in (0] : a$. Thus $x \in (0] : x$, which yields $x = 0$, and therefore $(0] : x = P$. Consequently, $(0] : a$ is a dual atom in $A(P)$.

(iv) $\Rightarrow$ (v) Suppose that $(0] : a$ is a dual atom of the poset $(A(P), \subseteq)$ and $(0] : a \subseteq (0] : B \neq P$ for a nonempty subset B of P. Observe that $B \nsubseteq (0] : a$. Indeed, if $B \subseteq (0] : a$ holds, then $B \subseteq (0] : B = \bigcap_{b \in B} (0] : b$. Thus $b \in (0] : b$ for all $b \in B$ and hence $B = \{0\}$, which implies $(0] : B = P$, a contradiction. Therefore there exists $x \in B$ such that $x \notin (0] : a$.

Now, let $y \in (0] : B$. We have to show that $y \in (0] : a$. Since $y \in (0] : B$ and $x \in B$, then we have $(x,y)^l = \{0\}$. Observe that $(a,y)^l \subset a^l$. Indeed, if $(a,y)^l = a^l$ holds, then $a \leqslant y$. Since $(x,y)^l = \{0\}$, we get $(x,a)^l = \{0\}$, and this implies $x \in (0] : a$, a contradiction to the fact that $x \notin (0] : a$. Thus there exists $z \in (a,y)^l$ and $z < a$. Now $z < a$ implies $(0] : a \subseteq (0] : z$.

We claim that $(0] : a \subset (0] : z$. Indeed, suppose $(0] : a = (0] : z$. Now from $(x,y)^l = \{0\}$ and $z \leqslant y$ we get $(x,z)^l = \{0\}$. Hence $x \in (0] : z = (0] : a$, a contradiction to the fact that $x \notin (0] : a$. Therefore $(0] : a \subset (0] : z \subseteq P$. By assumption, $(0] : z = P$ which yields $z = 0$. Therefore $(a,y)^l = \{0\}$, and so $y \in (0] : a$. Thus we obtain $(0] : B \subseteq (0] : a$, as required.

(v) $\Rightarrow$ (i) Suppose that $(0] : a$ is a maximal annihilator of P and $(x,y)^l = \{0\}$ so that $x \notin (0] : a$. To prove that a is a semiatom, it is enough to show that $y \in (0] : a$. Since $(a,x)^l \neq \{0\}$, there exists a nonzero element $z \in P$ such that $z \in (a,x)^l$. We have two cases:

(1) If $z = a$, then $a \leqslant x$ and therefore $y \in (0] : x \subseteq (0] : a$.

(2) If $z < a$, then $(0] : a \subseteq (0] : z \neq P$, as $z \neq 0$. By assumption, $(0] : a = (0] : z$. Since $(z,y)^l = \{0\}$, we have $y \in (0] : z = (0] : a$, and therefore a is a semiatom. □

Lemma 3.4. *Every non dense prime ideal of a* 0*-distributive poset* P *is of the form* $(0] : a$ *for some semiatom* a *of* P*. In fact, every nonzero element of* $(0] : I$ *is a semiatom.*

Proof. Suppose I is a non dense prime ideal of P. We claim that $I = (0] : a$ for every nonzero $a \in (0] : I$. Suppose $z \in (0] : a$, then by primeness of I we have $z \in I$, as $(a,z)^l = \{0\} \subseteq I$ and $a \notin I$. Thus $(0] : a \subseteq I$. Now, if $z \in I$ holds, then

$(0] : I \subseteq (0] : z$, and this implies $a \in (0] : z$, i.e., $z \in (0] : a$. Thus $I = (0] : a$, which is prime by assumption. Now, by Theorem 3.3, a is a semiatom of P. $\square$

We introduce the notion of a semiatomic poset as follows.

Definition 3.5. A poset P with 0 is called *semiatomic* if for each nonzero element x of P, there is a semiatom $a \in P$ such that $a \leqslant x$.

The following theorem is a characterization of semiatomic 0-distributive posets.

Theorem 3.6. *Let P be a 0-distributive poset. Then the following statements are equivalent:*

(i) *P is semiatomic,*

(ii) *each $(0] : x \in A(P)$ such that $(0] : x \neq P$ is the intersection of dual atoms in $A(P)$,*

(iii) *$(0] = \bigcap\{I;\ I$ is a non dense prime ideal of $P\}$,*

(iv) *$(0] : I = (0]$, where $I = \bigcup\{(0] : I_1;\ I_1 = (0] : a$ and a is a semiatom in $P\}$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that P is semiatomic and $(0] : x \in A(P)$ is such that $(0] : x \neq P$. We know from Theorem 3.3 that for every semiatom a of P, $(0] : a$ is a dual atom of $A(P)$. Consider the set $B = \bigcap\{(0] : a;\ a \leqslant x$ and a is a semiatom in $P\}$; we show that $(0] : x = B$. Suppose $z \in (0] : x$. Then $(x, z)^l = \{0\}$ which yields $(a, z)^l = \{0\}$ for any semiatom of P with $a \leqslant x$. Hence $z \in (0] : a$, in other words, $(0] : x \subseteq B$. Now, let $b \in B$. If $(x, b)^l \neq \{0\}$, then there exists a nonzero element d such that $d \in (x, b)^l$. Since P is semiatomic, there exists a semiatom c such that $c \leqslant d \leqslant b$. Now c is a semiatom, $b \in B$, so we have $b \in (0] : c$, which implies $c^l = (c, b)^l = \{0\}$, a contradiction to the fact that c is a semiatom. Therefore we must have $(x, b)^l = \{0\}$ and so $b \in (0] : x$. Consequently $(0] : x = B$.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds and $x \neq 0$. We have to show that $x \notin \bigcap\{I;\ I$ is a non dense prime ideal of $P\}$. Clearly $(0] : x \neq P$ and by (ii), there exists a dual atom $(0] : a = I_1$ (where a is a semiatom of P) of $A(P)$ such that $(0] : x \subseteq (0] : a \neq P$. Observe that $x \notin (0] : a$, otherwise $x \in (0] : a$ would imply $a \in (0] : x \subseteq (0] : a$, which yields $a = 0$, a contradiction to the fact that $a \neq 0$. Now, since $(0] : a$ is a dual atom of $A(P)$, by Theorem 3.3, I_1 is a prime ideal of P. In fact, I_1 is a non dense prime ideal, as $(0] : I_1 \neq \{0\}$ since $a \in (0] : I_1$. Thus $x \notin \bigcap\{I;\ I$ is a non dense prime ideal of $P\}$, which proves (iii).

(iii) $\Rightarrow$ (iv) Suppose that (iii) holds and $I = \bigcup\{(0] : I_1;\ I_1 = (0] : a$ and a is a semiatom in $P\}$. Suppose $(0] : I \neq (0]$, i.e., there exists a nonzero element $x \in (0] : I$. Therefore by assumption, $x \notin J$ for some non dense prime ideal J of P. By Lemma 3.4, $J = (0] : b$ for some semiatom $b \in P$ and since $x \notin J$, we have $(b, x)^l \neq \{0\}$. Since $b \in (0] : J$, we have $b \in I$. But we have $x \in (0] : I$ and $b \in I$, thus $(b, x)^l = \{0\}$, which is a contradiction.

(iv) $\Rightarrow$ (i) Suppose (iv) holds and x is a nonzero element of P. By (iv), we have $x \notin (0] : I$, where $I = \bigcup\{(0] : I_1;\ I_1 = (0] : a$ and a is a semiatom in $P\}$. Therefore $(b, x)^l \neq \{0\}$ for some $b \in I$. Consider an element $a \in (b, x)^l$ such that $a \neq 0$. We show that a is a semiatom. First, observe that in view of (iv) $b \in I$ implies $b \in (0] : I_1$, where $I_1 = (0] : c$ for some semiatom c of P. Now suppose $(y, z)^l = \{0\}$. Then either $(c, y)^l = \{0\}$ or $(c, z)^l = \{0\}$, as c is a semiatom in P, and so $y \in I_1$ or $z \in I_1$. But $a \leqslant b$ and $b \in (0] : I_1$, therefore $a \in (0] : I_1$ and y or z is in I_1. Hence $y \in (0] : a$ or $z \in (0] : a$. Thus a is a semiatom of P that satisfies $a \leqslant x$. □

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Authors' addresses: *Khalid Mokbel*, Mathematics Department, Education Faculty, Hodaidah University, Hodaidah, Yemen, e-mail: khalidalaghbari@yahoo.com; *Vilas Kharat*, Department of Mathematics, University of Pune, Pune 411 007, India, e-mail: vsk@math.unipune.ernet.in.