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0. **Outline of the dissertation.**

The dissertation is the account of my work on the connections between Shelah's theory of proper forcing in set theory, and descriptive set theory. It turns out that in a suitably definable context the development of the theory of proper forcing is extremely smooth. Furthermore, the questions raised connect the formerly purely combinatorial forcing theory to such fields of mathematics as abstract analysis, geometric measure theory, potential theory, Borel equivalence relations, pcf theory and more. The dissertation is just a beginning of a development that has sadly been neglected for too long.

The point of departure of the dissertation is that every suitably definable proper forcing is isomorphic to the factor algebra of Borel sets modulo a suitable sigma-ideal I. Three intertwining lines of investigation lead from there:

1. Identify the sigma-ideals connected with the traditional proper forcing notions and determine their topological and descriptive properties.
2. For sigma-ideals that occur in the various fields of mathematics decide whether the associated factor algebra is a proper notion of forcing, and what forcing properties it may have.
3. See how operations on ideals and operations on forcings are connected.

The dissertation presents significant progress in all three directions. Regarding the first item, it computes and investigates ideals associated with Silver forcing, Mathias forcing, and Laver forcing, as well as the more familiar cases such as the Sacks or Miller forcing. In the second direction, it shows that the ideals associated with Hausdorff measures, porosities and certain Borel equivalences generate proper forcings. After the dissertation was published I proved that all capacities in potential theory generate proper forcings as well. Finally, in the third direction it is shown that the countable support iteration of forcings is connected with transfinite Fubini powers of ideals and that there are canonical illfounded iterations of forcings. Later I showed that the side by side products are connected with certain Ramsey theoretic properties of ideals which in turn are implied by some forcing properties of the factor forcings.

The dissertation concludes with several applications. It turns out that the theory developed in the first four chapters makes it possible to identify certain models of set theory which are canonical with respect to certain kinds of independence results. In other words, given just the syntactical form of the sentence one wishes to prove consistent with the axioms of set theory, in many cases there is a model of set theory in which the sentence must hold if it is not outright inconsistent. Moreover, these canonical models have long been known and used, except their "canonicity" seemed not to rise above the level of a heuristic. The dissertation offers a precise quantification of the phenomenon.

1. Results

1.1. Absoluteness Theorems.

The first and still the most important results obtained by the method described in the thesis are absoluteness theorems which allow the mathematician to identify the "correct" Boolean algebras for whole classes of independence results. The independence results concern inequalities between cardinal invariants of the real line. Cardinal invariants were invented to discern between various Borel structures on the real line. Given Borel structures **S**, **T** would be assigned infinite cardinal numbers **x(S)**, **x(T)**, and then inequalities of the form **x(S)<x(T)** would indicate a significant difference between the structures. However, under the assumption of the Continuum Hypothesis most of these cardinal invariants coincide and are equal to $\aleph_1$. This means that sharp inequalities between invariants can be achieved only as consistency results. Soon after the discovery of forcing many consistency results of this kind appeared, and the field started to grow chaotically. The results in the thesis bring some measure of order to the field of cardinal invariants. In fact, it turns out that the comparison of cardinal invariants frequently necessarily brings us back to the projective properties of the Borel structures concerned, without encountering any new information from, say, combinatorics on uncountable cardinals. In such a way the thesis brings closure to some parts of the field, and at the same time indicates parts for which such a swift closure is not yet possible.

Definition. A *tame cardinal invariant* is a cardinal number defined as the minimal size of a set **A** of reals with properties

$$\varphi(A) \text{ and } \forall\, x\in\Re\ \exists\, y\in A\ \ \theta(x, y)$$

where the formula φ quantifies only over natural numbers and elements of the set **A** and the formula θ quantifies only over real numbers and does not make reference to the set **A**.

Most cardinal invariants used in mathematical practice today are tame, such as **a, b, c, d, e**… [2] Others, such as **g, h** are not tame, but they form a distinct minority. Comparing tame invariants with certain other invariants is made easy by the theorems contained in the thesis.

Theorem. (ZFC+LC) Suppose that **x** is a tame invariant. If **x<c** holds in some forcing extension then it holds in the iterated Sacks extension.

This theorem is a special case of Theorem 5.1.14 in the thesis. Recall that **c** denotes the size of the continuum. The statement of the theorem needs explanation.

The theorem is proved under a suitable large cardinal assumption, indicated by LC; in the given case a proper class of measurable Woodin cardinals is a sufficient assumption. The thesis also contains a variation of the theorem provable without such assumptions. It uses a somewhat more technically defined class of invariants as compared to the tame invariants, which nevertheless contains all the tame invariants used in present day practice. The users of the theorem are unlikely to be disturbed by the presence of large cardinal assumptions.

The iterated Sacks extension is a particular forcing extension, or perhaps a particular Boolean algebra, which has been studied for decades. It was understood on an intuitive level that it has properties similar to those described in the theorem, however the theorem still came as a surprise in its generality. The reason why Sacks forcing enters the scene is its close relationship to the quotient algebra of Borel subsets of the reals modulo the ideal of countable sets.

There are several important consequences of the theorem. First, it eliminates a search for a suitable Boolean algebra for a consistency result of the form $\mathbf{x}<\mathbf{c}$. Such a search is normally perhaps the most difficult part in proving a consistency result, but here that Boolean algebra is identified by the theorem. The proof of the theorem even gives a hint as to the methods of comparison of **x** and **c** in the extension. Second, it proves a weak mutual consistency result. If **x, y** are tame invariants and $\mathbf{x}<\mathbf{c}, \mathbf{y}<\mathbf{c}$ can both be forced, they can be forced in conjunction—they both hold in the iterated Sacks extension. And finally, the theorem implies that if a tame cardinal **x** can be forced less than **c**, then it can be forced that $\mathbf{x}=\aleph_1<\mathbf{c}=\aleph_2$.

Many versions of the theorem for invariants other than **c** (such as **b, d, h**…) can be found in Chapter 5 of the dissertation.

1.2. Duality Theorems.

Every σ-ideal **I** on a Polish space **X** has four cardinal invariants associated with it. **cov(I)**=the smallest number of sets in **I** necessary to cover the space **X, non(I)**=the smallest cardinality of an **I**-positive set, **add(I)**=the smallest size of a collection of **I**-small sets whose union is not **I**-small anymore, and **cof(I)**=the smallest size of a basis of the sigma ideal **I**. Among these, **cov** and **non**, and **add** and **cof** are dual in a vague sense.Workers in the area have long used so-called duality heuristic which says roughly: if **cov(I)** is provably large then **non(I)** is provably small, similarly for other dual pairs. In fact every cardinal inequality concerning these invariants can be dualized, and the heuristic then says that an inequality is provable if and only if its dual form is provable. This is not a theorem and really it is false in general, but it still serves as a valuable tool in the field. The method introduced in the dissertation makes it possible to prove some limited versions of the duality heuristic.

Theorem. If **I** is a projectively defined sigma-ideal such that ZFC+LC proves $\mathbf{cov(I)}=\mathbf{c}$ then ZFC+LC proves $\mathbf{non(I)}\leq\aleph_2$.

Similarly for **non** and **cov**, **cof** and **add**, and **add** and **cof** in place of **cov** and **non**. Again the theorem uses large cardinal assumptions denoted by LC in its statement, this time ω_1

many Woodin cardinals suffice. There is a version of the theorem for a smaller class of sigma ideals that does not use the large cardinal assumptions. There are versions of the theorem with **c** and $\aleph_2$ replaced by other cardinals, however it turns out that they are more complicated to obtain and state.

1.3. Interpolation theorems

The interpolation theorems in logic usually state something to the following effect: if φ, ψ are formulas and $\varphi\rightarrow\psi$ is provable, then there is a simple formula θ such that $\varphi\rightarrow\theta$ and $\theta\rightarrow\psi$ are both provable. It turns out that the method contained in the dissertation can be used to identify a number of interpolation theorems for set theory.

Theorem. Suppose that **x** is a tame cardinal invariant. If ZFC+LC proves that **x**$=\aleph_1$ implies the existence of a Lusin set, then ZFC+LC proves that **x**$=\aleph_1$ implies **cof(meager)**$=\aleph_1$.

Here a Lusin set is an uncountable set of reals that has countable intersection with every meager set. It is easy to prove that **cof(meager)**$=\aleph_1$ implies the existence of a Lusin set. The theorem shows that this is a critical provable implication of this form: every other provable implication **x**$=\aleph_1\rightarrow$ Lusin set can be factored into two provable implications **x**$=\aleph_1\rightarrow$ **cof(meager)**$=\aleph_1\rightarrow$ Lusin set. Again the large cardinal hypotheses (this time, ω_1 Woodin cardinals) can be eliminated for a class of invariants a little bit more restrictive than the tame invariants.

2. Methods

To a certain degree, the methods used in the dissertation as well as in the papers immediately preceding or following it are more important and interesting than the results themselves. It is a mixture of methods of descriptive set theory, determinacy, forcing, abstract analysis, large cardinals, pcf theory and other fields. The main idea is to link the descriptive and topological properties of σ-ideals with the forcing properties of the related quotient algebra. There are three different issues facing the theory.

2.1. Examples

The basic stepping stone is of course the identification of many σ-ideals for which the quotient algebra is a proper [39] and interesting notion of forcing, and the correspondence of traditional combinatorially obtained posets with the quotient algebra. A rich structure appears here. The posets in the left hand column are in the forcing sense equivalent to the quotient algebras of ideals in the right hand column:

Sacks forcing	Countable sets
Miller forcing	σ-compact subsets of the Baire space
Laver forcing	Non-dominating subsets of the Baire space
Cohen forcing	Meager sets
Solovay forcing	Lebesgue null sets
Mathias forcing	Sets nowhere dense in P(ω) mod fin
Silver forcing	Sets of countable Borel chromatic number in a certain graph

Moreover, it turns out that quotient algebras of σ-ideals obtained as σ-porous ideals for various notions of porosity, ideals of σ-finite Hausdorff measure sets, ideals generated by closed sets, and ideals of null sets associated with various capacities and submeasures are all proper notions of forcing. At this point, the verification of properness still proceeds more or less on a case by case basis, since there are natural ideals for which the quotient algebra is not proper and they do not seem to be easy to identify. There are very many σ-ideals for which the properness status of the quotient algebra has not been cleared.

2.2. Topology-forcing connection

It turns out that many traditional properties of forcing notions have a natural topological restatement in the context of the quotient algebras, making the terminology of the field much more compact, exact and readily understandable. A simple example:

Theorem. Suppose that **I** is a σ-ideal on a Polish space such that the quotient forcing is proper. The following are equivalent:

- The poset $\mathbf{P_I}$ is bounding
- Compact sets are dense (every Borel **I**-positive set has a compact positive subset) and the continuous reading of names (every Borel function on a positive Borel domain has a continuous restriction with a positive Borel domain).

The bounding property of partial orders has long been used in various constructions. These constructions invariably used posets consisting of finitely branching trees, which is now fully understandable in view of the theorem. The continuous reading of names used to be just a forcing slang for a certain trick in proving properness, which in the context of quotient algebras has the above exact and natural reformulation.

2.3. Operations on ideals vs. operations on forcings

The connection between operations on ideals and forcings turns out to be very natural and it is the most important object of study in the dissertation.

The most natural operation on proper forcings is the countable support iteration. It is closely connected to the transfinite Fubini product of ideals. This notion of product naturally extends the Fubini product and the dissertation is apparently the first publication in which it appears. Let me avoid the simple formal definition and just mention that given an ideal **I** on a Polish space **X** and a countable ordinal β, the β-th Fubini power of **I** is an ideal I_β on the space X^β. For finite values of β this is the same as the usual iterated Fubini power.

Theorem. (ZFC+LC) Suppose that **I** is a projectively definable σ-ideal on a Polish space such that it is almost full and the forcing P_I is proper. For every countable ordinal β the countable support iteration of the poset P_I of length β is in forcing sense equivalent to P_J, where $J=I_\beta$.

The large cardinal hypothesis LC (in this case ω_1 Woodin cardinals) is necessary already for $\beta=2$ and very simple ideal **I**. This is the main reason why large cardinal assumptions are an important tool in the dissertation. The large cardinal hypotheses can be eliminated for Π_1^1 on Σ_1^1 ideals. Note that for ideals without a suitably absolute definition the whole idea of iteration does not make much sense. The theorem can be extended to give a similar information for iterations of several different forcings.

Another natural operation on partial orders is the side-by-side product. Several ideals resulting from a product of partial orders were computed in the dissertation and a general theory was exhibited in a later paper. Given a ideals **I, J** on Polish spaces **X, Y**, let $I\times J$ be the sigma ideal on $X\times Y$ generated by Borel sets without a subset of the form $B\times C$ where **B** and **C** are Borel **I** or **J**-positive sets respectively. A similar definition can give a side by side product of countably infinitely many various ideals. The main problem here is that $I\times J$ is in many cases trivial.

Theorem. (ZFC+LC) Suppose that **I, J** are projectively definable ideals such that the quotient posets are proper and preserve **cof(meager)**. Then $I\times J$ is a nontrivial ideal, its quotient forcing is proper, preserves **cof(meager)** and is in the forcing sense equivalent to $P_I\times P_J$.

A similar theorem holds for infinite side-by-side products. The large cardinal hypothesis here is again that of ω_1 Woodin cardinals. Preservation of **cof(meager)** means that every meager set in the extension is a subset of a meager set coded in the ground model. Note that the theorem has a Ramsey-theoretic content: in the given case, if the plane is partitioned into countably many Borel pieces then one of them must contain a rectangle with Borel positive sides. Since it contains a determinacy argument, the proof of the theorem works for definable ideals only.

Several other operations on ideals and posets are identified in the dissertation. There is the illfounded iteration of definable partial orders preserving **non(meager)**, and towers of ideals. It seems that other natural operations will be found soon.

3. Literature used.

1. U. Abraham, S. Shelah: Forcing closed unbounded sets. J. Symbolic Logic 48 (1983), 643-657
2. T. Bartoszynski, H. Judah: Set theory. A K Peters, Wellesley MA, 1995
3. H. Becker: The restriction of a Borel equivalence relation to a sparse set. Preprint
4. A. Blass, S. Shelah: Near coherence of filters III. A simplified consistency proof. Notre Dame J. Formal Logic 30 (1989), 530-538
5. J. Brendle: Mad families and iteration theory. Preprint
6. J. Brendle: Mob families and mad families. Archive Math. Logic 37 (1998), 183-197
7. J. Cichon, M. Morayne, J. Pawlikowski, S. Solecki: Decomposing Baire functions. J. Symbolic Logic 56 (1991), 1273-1283
8. K. Ciesielski, J. Pawlikowski: The covering axiom CPA. Preprint
9. J. Feldman, C. Moore: Ergodic equivalence relations, cohomology and von Neumann algebras, Trans. Amer. Math. Soc. 234 (1977), 289-324
10. Q. Feng, M. Magidor, H. Woodin: Universally Baire sets of reals. In Set theory of the continuum, Springer, New York 1992
11. D. Fremlin: Measure algebras. In Handbook of Boolean algebras, North Holland, New York, 1989
12. S. Geschke, M. Goldstern, M. Kojman: Continuous Ramsey theory and covering the plane by functions. Preprint
13. S. Geschke, M. Kojman, W. Kubis, R. Schipperus: Convex decompositions of the plane and continuous pair colorings of the irrationals. Israel J. Math 131 (2002), 285-317
14. M. Gitik: Nonsplitting subsets of Pkk+. J. Symbolic Logic 50 (1985), 881-894
15. L. Harrington, A. Kechris, A. Louveau: A Glimm-Effros dichotomy for Borel equivalence relations. J. Amer. Math. Soc. 3 (1990), 903-928
16. T. Jech: Set Theory. Academic Press, San Diego, 1978
17. T. Jech, S. Shelah: A complete Boolean algebra that has no proper atomless subalgebra. J. Algebra 182 (1996)
18. A. Kanamori: The higher infinite. Springer, New York 1994
19. V. Kanovei: On non-well-founded iterations of the perfect set forcing. J. Symbolic Logic 64 (1999), 191-207
20. A. Kechris: On a notion of smallness for subsets of the Baire space. Trans. Amer. Math. Soc. 229 (1977), 191-207
21. A. Kechris, A. Louveau: The structure of hypersmooth Borel equivalence relations. J. Amer. Math. Soc. 10 (1997), 215-242
22. A. Kechris: Classical descriptive set theory. Springer, New York 1994
23. R. Laver: On the consistency of Borel's conjecture. Acta Math. 137 (1987), 151-169
24. D. A. Martin: A purely inductive proof of Borel determinacy. Amer. Math. Soc., Providence RI, 1985
25. I. Neeman: A book in preparation.

26. I. Neeman, J. Zapletal: Proper forcing and absoluteness in L(R). Commentationes Math. Univ. Carolinae 39 (1998), 281-301
27. J. Pelant, M. Zeleny: The structure of the sigma ideal of sigma porous sets. T. Amer. Math. Soc., submitted
28. C. A. Rogers: Hausdorff measures. Cambridge University Press, New York, 1970
29. A. Roslanowski, S. Shelah: Norms on possibilities I. Memoirs Amer. Math. Soc. 141, 1999
30. S. Shelah, J. Zapletal: Duality and pcf theory. Math. Res. Lett. 9 (2002), 585-595
31. S. Shelah, J. Zapletal: Games with creatures. Commentationes Math. Univ. Carolinae.
32. S. Shelah, J. Zapletal: Canonical models for aleph one combinatorics. Ann. Pure Appl. Logic 98 (1999), 217-259
33. S. Shelah: Analytical guide and updates for Cardinal arithmetic, Shelah [Sh:E12]
34. S. Shelah: Non-elementary proper forcing notions. J. Appl. Analysis.
35. S. Shelah: Non-structure theory. Oxford University Press, accepted
36. S. Shelah: Vive la difference I. In Set theory of the continuum, Springer, New York, 1992
37. S. Shelah: Are a and d your cup of tea? Acta Math., submitted
38. S. Shelah: Cardinal arithmetic. Clarendon Press, Oxford, 1994
39. S. Shelah: Proper and improper forcing. Springer, New York, 1998
40. S. Shelah, O. Spinas: The distributivity numbers of P(w) mod fin and its square.Trans. Amer. Math. Soc., accepted
41. R. Sikorski: Boolean algebras. Springer, New York, 1964
42. M. Sion, D. Sjerve: Approximation properties of measures generated by continuous set functions. Mathematika 9 (1969), 145-156
43. S. Solecki: Covering analytic sets by families of closed sets. J. Symbolic Logic 59 (1994), 1022-1031
44. S. Solecki: Decomposing Borel sets and functions and the structure of Baire class one functions. J. Amer. Math. Soc. 11 (1998), 521-550
45. J. Steprans: A very discontinuous Borel function. J. Symbolic Logic 58 (1993), 1268-1283
46. W. H. Woodin: Unpublished works
47. W. H. Woodin: Supercompact cardinals, sets of reals and weakly homogeneous trees. Proc. Nat. Acad. Sci. USA 85 (1988), 6587-6591
48. W. H. Woodin: The axiom of determinacy, forcing axiom, and the nonstationary ideal. Walter de Gruyter, New York, 1999
49. J. Zapletal: Terminal notions in set theory. Ann. Pure Appl. Logic 109 (2001), 89-116
50. J. Zapletal: Forcing with ideals of closed sets. Commentationes Math. Univ. Carolinae 43 (2002), 181-188
51. J. Zapletal: Isolating cardinal invariants. J. Math. Logic 3 (2003), 143-162

3. Selected publications of the author.

Between Maharam's and von Neumann's Problem, with Ilijas Farah, Mathematical Research Letters 11 (2004), 673-684

Descriptive Set Theory and Definable Forcing, Memoirs of American Mathematical Society 793, Providence 2004

Forcing with Ideals of Closed Sets, Commentationes Mathematicae Universitatis Carolinae 43 (2002), 181-188

Isolating Cardinal Invariants, Journal of Mathematical Logic 3 (2003), 143-162

The Nonstationary Ideal and the Other σ-ideals on ω_1, Transactions of American Mathematical Society 352 (2000), 3981-3993
Terminal Notions, Bulletin of Symbolic Logic 5 (1999), 470-484

Canonical Models for $\aleph_1$ Combinatorics, with Saharon Shelah, Annals of Pure and Applied Logic 98 (1999), 217-259

Proper Forcing and Absoluteness in L(R), with Itay Neeman, Commentationes Mathematicae Universitatis Carolinae 39 (1998), 281-301