122 (1997) MATHEMATICA BOHEMICA No. 4, 383–399

ON THE MEASURES OF DIPERNA AND MAJDA

Martin Kružík and Tomáš Roubíček, Praha

(Received June 11, 1996)

Abstract. DiPerna and Majda [5] generalized Young measures (Young [19]) so that it is possible to describe "in the limit" oscillation as well as concentration effects of bounded sequences in L^p-spaces. Here the complete description of all such measures is stated, showing that the "energy" put at "infinity" by concentration effects can be described in the limit basically by an arbitrary positive Radon measure. Moreover, it is shown that concentration effects are intimately related to rays (in a suitable locally convex geometry) in the set of all DiPerna-Majda measures. Finally, a complete characterization of extreme points and extreme rays is established.

Keywords: bounded sequences in Lebesgue spaces, oscillations, concentrations, Young measures, DiPerna and Majda's measures, rays, extreme points, extreme rays

MSC 1991: 28C15, 40A30

0. Introduction—notation

The DiPerna-Majda measures (DiPerna and Majda [5], see also DiPerna and Majda [6,], Greengard and Thomann [9], Kružík and Roubíček [12], Roubíček [13, 14]) represent a modern mathematical tool to hold a certain "limit" information about oscillations and concentrations in nonlinear problems admitting only L^p—but not L^∞—apriori estimates, which arise quite often in variational calculus, partial differential equations, optimal control theory, game theory etc.; cf. e.g. Roubíček [13], Warga [18].

They represent a deep generalization of Young measures (Young [19]) which can record in the limit only fast oscillation effects but not the concentration ones.

While the precise characterization of all Young measures attainable by bounded sequences in Lebesgue spaces has been known (see Kružík and Roubíček [12]), a similar characterization of all DiPerna-Majda measures was missing so far. The main goal of this paper is to fill this gap. Besides, we will investigate basic geometrical properties of the convex set of all DiPerna-Majda measures; in particular, using our explicit characterization, we will be able to describe all extreme points, all rays and all extreme rays.

Let us start with a few definitions. For $p \geqslant 0$ we define the following subspace of the space $C(\mathbb{R}^m)$ of all continuous functions on $\mathbb{R}^m$:

$$C_p(\mathbb{R}^m) = \{v \in C(\mathbb{R}^m); v(s) = o(|s|^p) \text{ for } |s| \to \infty\}.$$

Let us take a complete (i.e. containing constants, separating points from closed subsets and closed with respect to the Chebyshev norm) separable (i.e. containing a dense countable subset) ring $\mathcal{R}$ of continuous bounded functions $\mathbb{R}^m \to \mathbb{R}$. It is known that there is a one-to-one correspondence $\mathcal{R} \mapsto \beta_{\mathcal{R}}\mathbb{R}^m$ between such rings and metrizable compactifications of $\mathbb{R}^m$; by a compactification we mean here a compact set, denoted by $\beta_{\mathcal{R}}\mathbb{R}^m$, into which $\mathbb{R}^m$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^m$ and its image in $\beta_{\mathcal{R}}\mathbb{R}^m$.

Besides, we will consider an open bounded domain Ω in $\mathbb{R}^n$. Then its closure, denoted by $\overline{\Omega}$, is compact and $\mathrm{rca}(\overline{\Omega}) \cong C(\overline{\Omega})^*$ will denote the space of all regular countably additive set functions on the Borel σ-algebra on $\overline{\Omega}$, i.e. the so-called Radon measures, see Dunford and Schwartz [8]. Having a positive Radon measure σ on $\overline{\Omega}$, we denote by $L^\infty_{\mathrm{w}}(\overline{\Omega}, \sigma; \mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^m))$ the space of mappings $\hat{\nu}\colon x \mapsto \hat{\nu}_x\colon \overline{\Omega} \to \mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^m)$ which are weakly σ-measurable (i.e., for any $v_0 \in \mathcal{R}$, the mapping $\overline{\Omega} \to \mathbb{R}\colon x \mapsto \int_{\beta_{\mathcal{R}}\mathbb{R}^m} v_0(s)\hat{\nu}_x(\mathrm{d}s)$ is σ-measurable in the usual sense) and σ-essentially bounded. Note that we have used informally the fact that every $v_0 \in \mathcal{R}$ has a uniquely defined continuous extension on $\beta_{\mathcal{R}}\mathbb{R}^m$. Besides, let us denote by $\mathcal{Y}(\overline{\Omega}, \sigma; \beta_{\mathcal{R}}\mathbb{R}^m)$ the set of $\hat{\nu} \in L^\infty_{\mathrm{w}}(\overline{\Omega}, \sigma; \mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^m))$ such that $\hat{\nu}_x \in \mathrm{rca}^+_1(\beta_{\mathcal{R}}\mathbb{R}^m)$ for σ-a.a. $x \in \overline{\Omega}$ where "$\mathrm{rca}^+_1(\cdot)$" stands for probability Radon measures; for such $\hat{\nu}$ the collection $\{\hat{\nu}_x\}_{x\in\overline{\Omega}}$ is called a Young measure on $(\overline{\Omega}, \sigma)$, see Young [19] or also Alibert and Bouchitté [1], Ball [4], Roubíček [13], Tartar [16], Valadier [17], Warga [18].

As usual, $L^p(\Omega; \mathbb{R}^m)$ with $1 \leqslant p < +\infty$ will denote the space of Lebesgue measurable functions $\Omega \to \mathbb{R}^m$ whose p-power is integrable.

DiPerna and Majda [5] showed that, having a bounded sequence $\{u_k\}_{k\in\mathbb{N}}$ in $L^p(\Omega; \mathbb{R}^m)$, there exist its subsequence (denoted by the same indices), a positive $\sigma \in \mathrm{rca}(\overline{\Omega})$ and a Young measure $\hat{\nu} \in \mathcal{Y}(\overline{\Omega}, \sigma; \beta_{\mathcal{R}}\mathbb{R}^m)$ such that the couple $(\sigma, \hat{\nu})$ is attainable by this subsequence $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)$ in the sense

(1)

$$\forall g \in C(\overline{\Omega})\ \forall v_0 \in \mathcal{R}\colon \lim_{k\to\infty} \int_\Omega g(x)v(u_k(x))\,\mathrm{d}x = \int_{\overline{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^m} g(x)v_0(s)\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x),$$

where $v(s) = v_0(s)(1+|s|^p)$. In particular, putting $v_0 = 1 \in \mathcal{R}$ in (1) we can see that

$$(2) \qquad \lim_{k\to\infty} (1 + |u_k|^p) = \sigma \quad \text{ weakly* in } \operatorname{rca}(\bar\Omega).$$

Let us denote by $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ the set of all pairs $(\sigma,\hat\nu) \in \operatorname{rca}(\bar\Omega) \times \mathcal{Y}(\bar\Omega,\sigma;\beta_{\mathcal{R}}\mathbb{R}^m)$ attainable by sequences from $L^p(\Omega;\mathbb{R}^m)$; note that, taking $v_0 = 1$ in (1), one can see that these sequences must be inevitably bounded in $L^p(\Omega;\mathbb{R}^m)$. The explicit description of the elements from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, called DiPerna-Majda measures, was done by the authors (Kružík and Roubíček [12, Theorem 2], see also Roubíček [13, Proposition 3.2.13]) only for the case when $\hat\nu$ is essentially not supported on the remainder $\beta_{\mathcal{R}}\mathbb{R}^m \setminus \mathbb{R}^m$ in the sense of (9) below.

Alternatively, DiPerna and Majda [5] worked with measures from $\operatorname{rca}(\bar\Omega \times \beta_{\mathcal{R}}\mathbb{R}^m)$; let us put here

$$\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m) = \Big\{\eta \in \operatorname{rca}(\bar\Omega \times \beta_{\mathcal{R}}\mathbb{R}^m)\,;\ \exists \{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$$
$$\forall h_0 \in C(\bar\Omega \times \beta_{\mathcal{R}}\mathbb{R}^m)\colon \langle \eta, h_0\rangle = \lim_{k\to\infty} \int_\Omega h_0(x,u_k(x))(1+|u_k(x)|^p)\mathrm{d}x\Big\}.$$

Without causing any misunderstanding, the elements of $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ will be also addressed as DiPerna-Majda measures. We write $\eta \cong (\sigma,\hat\nu)$ for $\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and $(\sigma,\hat\nu) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ if

$$\langle \eta, h_0\rangle \equiv \int_{\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^m} h_0(x,s)\eta(\mathrm{d}x\,\mathrm{d}s) = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m} h_0(x,s)\hat\nu_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)$$

for any $h_0 \in C(\bar\Omega \times \beta_{\mathcal{R}}\mathbb{R}^m)$. It is known (Roubíček [13]) that $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is a convex, closed, non-compact but locally compact and locally sequentially compact subset of the locally convex space $\operatorname{rca}(\bar\Omega \times \beta_{\mathcal{R}}\mathbb{R}^m)$ considered in its weak* topology. Besides, $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ contains no straight line, see Corollary 3. As such, $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ inevitably contains a ray (i.e. half-line) and coincides with the closed convex hull of all its extreme points and extreme rays, cf. Köthe [11, Section 25.5]. Recall that $\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is an extreme point if $\eta = \frac{1}{2}\eta_1 + \frac{1}{2}\eta_2$ for some $\eta_1,\eta_2 \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ implies $\eta_1 = \eta_2$. A ray $R = \{\eta_0 + t\eta;\ t > 0\} \subset \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is extreme if every open interval in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ which intersects R is a subset of R.

Our main results (see Theorems 3 and 4 below) give the explicit characterization of the measures from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ in the general case. To this end, in Section 1 we first investigate in detail some properties of the first component of $(\sigma,\hat\nu) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, i.e. of the measure σ. Such explicit characterization has, beside its own theoretical value, an importance in implementation of (discretized) DiPerna-Majda measures on computers to solve effectively problems where concentration and oscillations can appear; however, the numerical approximation will not be treated in this paper. Eventually, Section 3 deals with geometrical properties of the set $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and, in particular, we will fully characterize its extreme points, rays, and extreme rays. Let us remark that the precise knowledge of extreme points can be exploited to establish existence of solutions to special nonconvex optimization problems via Bauer's extremal principle, as proposed by Balder [3].

1. Some properties of σ

First we shall show that, for any DiPerna-Majda measure $(\sigma,\hat\nu)$, the points x for which $\hat\nu_x$ is supported purely on the remainder $\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m$ are rare:

Lemma 1. *Let $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and let*

$$A_{\hat\nu}=\left\{x\in\bar\Omega;\ \int_{\mathbb{R}^m}\hat\nu_x(\mathrm{d}s)=0\right\}.$$

Then the Lebesgue measure of $A_{\hat\nu}$ is zero.

Proof. It follows from (2) that the Lebesgue measure is absolutely continuous with respect to σ in the sense that (cf. Halmos [10, §30])

$$E\subset\bar\Omega,\quad \sigma(E)=0\qquad\Longrightarrow\qquad \operatorname{meas}(E)=0.$$

On the other hand, the Lebesgue measure and σ are finite on $\bar\Omega$ and therefore, due to the Radon-Nikodým theorem, the Lebesgue measure has the density with respect to σ denoted by d_λ. Always, v_0 defined by $v_0(s)=1/(1+|s|^p)$ belongs to $\mathcal{R}$ because the ring $\mathcal{R}$ is complete and $\lim_{|s|\to\infty}v_0(s)$ does exist. Thus we can put $v=1$, i.e. $v_0(s)=1/(1+|s|^p)$, into (1) to get that the density d_λ has the form

$$d_\lambda(x)=\int_{\mathbb{R}^m}\frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p}.$$

This density is inevitably zero on $A_{\hat\nu}$ and we obtain

$$\operatorname{meas}(A_{\hat\nu})=\int_{A_{\hat\nu}}\mathrm{d}x=\int_{A_{\hat\nu}}\int_{\mathbb{R}^m}\frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\sigma(\mathrm{d}x)=0.$$

□

Let us recall that for any $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ there is precisely one $(\sigma^\circ,\hat{\nu}^\circ) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, called a p-nonconcentrating modification of $(\sigma,\hat{\nu})$, such that

$$\int_{\bar{\Omega}}\int_{\mathbb{R}^m} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x) = \int_{\bar{\Omega}}\int_{\mathbb{R}^m} v_0(s)\hat{\nu}^\circ_x(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x)$$

for any $v_0 \in C_0(\mathbb{R}^m)$ and any $g \in C(\bar{\Omega})$ and $(\sigma^\circ,\hat{\nu}^\circ)$ is attainable by a sequence $\{u_k\}_{k\in\mathbb{N}}$ such that the set $\{|u_k|^p;\, k \in \mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$; see Kružík and Roubíček [12], Roubíček [13, 14] for details.

Let us emphasize that any sequence $\{u_k\}_{k\in\mathbb{N}}$ which generates $(\sigma^\circ,\hat{\nu}^\circ)$ has the property that $\{|u_k|^p;\, k \in \mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$. This is quite a different situation in comparison with e.g. L^p-Young measures. Moreover, whenever $(\sigma,\hat{\nu}) \neq (\sigma^\circ,\hat{\nu}^\circ)$ then there is no sequence $\{v_k\}_{k\in\mathbb{N}}$ which generates $(\sigma,\hat{\nu})$ and $\{|v_k|^p;\, k \in \mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$; cf. Kružík and Roubíček [12, proof of Lemma].

The following assertion shows the relation between σ° and σ.

Theorem 1. *Let $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and let $(\sigma^\circ,\hat{\nu}^\circ) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ be its p-nonconcentrating modification. Then σ° is absolutely continuous with respect to σ and*

$$\sigma^\circ(\mathrm{d}x) = \left(\int_{\mathbb{R}^m} \hat{\nu}_x(\mathrm{d}s)\right)\sigma(\mathrm{d}x).$$

In other words, the Radon-Nikodým derivative $\mathrm{d}\sigma^\circ/\mathrm{d}\sigma$ is just the σ-integrable function $x \mapsto \int_{\mathbb{R}^m}\hat{\nu}_x(\mathrm{d}s)$.

Proof. First, note that σ and σ° are finite, $\sigma \geqslant \sigma^\circ$ and therefore σ° is absolutely continuous with respect to σ. It is a straightforward consequence of the Radon-Nikodým theorem that σ° has a density with respect to σ, i.e. that the Radon-Nikodým derivative $\mathrm{d}\sigma^\circ/\mathrm{d}\sigma$ exists. Let us denote (in this proof only) the Lebesgue measure by λ.

We can write (see Halmos [10, §32, Theorem A]) that

$$\frac{\mathrm{d}\sigma^\circ}{\mathrm{d}\sigma} = \frac{\mathrm{d}\sigma^\circ}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma}. \tag{3}$$

However, we know due to Kružík and Roubíček [12, Theorem 3] that

$$\frac{\mathrm{d}\sigma^\circ}{\mathrm{d}\lambda}(x) = \frac{\int_{\mathbb{R}^m}\hat{\nu}_x(\mathrm{d}s)}{\int_{\mathbb{R}^m}\frac{\hat{\nu}_x(\mathrm{d}s)}{1+|s|^p}}.$$

On the other hand, the density d_λ from Lemma 1 equals the Radon-Nikodým derivative $\mathrm{d}\lambda/\mathrm{d}\sigma$ (see Halmos [10, §32]), which means that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\sigma}(x) = \int_{\mathbb{R}^m} \frac{\hat{\nu}_x(\mathrm{d}s)}{1+|s|^p}.$$

The assertion now follows from (3). $\square$

The next theorem characterizes the support of the singular part of σ and, in particular, it can also characterize the situation where this singular part vanishes, which means just that σ is absolutely continuous with respect to the Lebesgue measure.

Theorem 2. *Let $A_{\hat{\nu}}$ be defined as in Lemma 1. Let further σ_s be the singular part of σ from the Lebesgue decomposition of σ. Then the support of σ_s, denoted by $\operatorname{supp}\sigma_s$, is equal to*

$$A^\sigma = A_{\hat{\nu}} \setminus \bigcup_{B\subset A_{\hat{\nu}},\sigma(B)=0} B$$

in the sense that $\sigma_s(A) = \sigma_s(\bar{\Omega})$ for any Borel set A such that $A^\sigma \subset A \subset \bar{\Omega}$.

Let us start with the following auxiliary lemma.

Lemma 2. *Let $\mu_1, \mu_2 \in \mathrm{rca}(\bar{\Omega})$ be positive finite measures and let μ_2 be absolutely continuous with respect to the Lebesgue measure. Moreover, let μ_{1s} be the singular part of μ_1 from the Lebesgue decomposition of μ_1 and let the following implication hold for a Borel set $E \subset \bar{\Omega}$:*

$$\forall \text{ Borel set } F \subset E\colon \mu_2|_E(F) = 0 \Longrightarrow \mu_1|_E(F) = 0.$$

Then $\mu_{1s}(E) = 0$.

Proof. The above implication says that $\mu_1|_E$ is absolutely continuous with respect to $\mu_2|_E$ and because $\mu_2|_E$ is absolutely continuous with respect to the Lebesgue measure it follows that $\mu_1|_E$ is absolutely continuous with respect to the Lebesgue measure, too. It means that $\mu_1|_E$ has a density with respect to the Lebesgue measure and therefore $\operatorname{supp}\mu_{1s} \cap E = \emptyset$ and $\mu_{1s}(E) = 0$. $\square$

Proof of Theorem 2. First, note that A^σ is correctly defined, i.e. that it is independent of an element of the class of σ-equivalent measure-valued functions $\hat{\nu}$ which we choose. Moreover, it is closed because it cannot have interior elements due to the fact that the Lebesgue measure of A^σ is zero (see Lemma 1). Therefore, all elements of it make its boundary.

We are to prove that $E \subset \overline{\Omega}$ a Borel set and $E \cap A^\sigma = \emptyset$ imply that $\sigma_s(E) = 0$. Due to Lemma 2 with $\mu_1 = \sigma$ and $\mu_2 = \sigma^\circ$ it suffices to prove that $E \cap A^\sigma = \emptyset$ yields that

$$\forall \text{ Borel set } F \subset E \quad \sigma^\circ|_E(F) = 0 \Longrightarrow \sigma|_E(F) = 0. \tag{4}$$

Let us prove (4). Supposing that $\sigma^\circ(E \cap F) = 0$, due to Theorem 1 we have

$$0 = \int_{E\cap F} \sigma^\circ(\mathrm{d}x) = \int_{E\cap F} \int_{\mathbb{R}^m} \hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x).$$

But $\int_{\mathbb{R}^m} \hat{\nu}_x(\mathrm{d}s) > 0$ for σ-a.a. $x \in E \cap F$. Clearly, for the set

$$C = \left\{x \in E \cap F;\ \int_{\mathbb{R}^m} \hat{\nu}_x(\mathrm{d}s) = 0\right\}$$

such that $\sigma(C) > 0$ we have $A^\sigma \cap C \neq \emptyset$ contradicting the fact that $E \cap A^\sigma = \emptyset$. Therefore $\sigma|_E(F) = \sigma(E \cap F) = 0$. Altogether we have proved that (4) holds and $\sigma_s(E) = 0$. ☐

Remark 1. Theorem 2 implies that $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^m)$ with σ absolutely continuous with respect to the Lebesgue measure if and only if $A^\sigma = \emptyset$.

Remark 2. It follows immediately from Theorems 1, 2 and Kružík and Roubíček [12, Theorem 2] that the density d_{σ_r} with respect to the Lebesgue measure of σ_r (i.e. of the absolutely continuous part of σ), is given by

$$d_{\sigma_r}(x) = \left(\int_{\mathbb{R}^m} \frac{\hat{\nu}_x(\mathrm{d}s)}{1 + |s|^p}\right)^{-1}.$$

2. A characterization of the DiPerna-Majda measures

The next theorem establishes a sufficient condition for a pair $(\sigma, \hat{\nu})$ to form a DiPerna-Majda measure.

Theorem 3. *Let $\mathcal{R}$ be a separable complete subring of the ring of all continuous bounded functions on $\mathbb{R}^m$, let $\Omega \subset \mathbb{R}^n$ be a bounded open domain, $1 \leqslant p < \infty$ and let $(\sigma, \hat{\nu}) \in \mathrm{rca}(\overline{\Omega}) \times \mathcal{Y}(\overline{\Omega}, \sigma; \beta_{\mathcal{R}}\mathbb{R}^m)$ with $\int_{\mathbb{R}^m} \hat{\nu}_x(\mathrm{d}s) > 0$ for a.a. $x \in \Omega$ be such that the measure $\sigma_{\hat{\nu}}(\mathrm{d}x) = (\int_{\mathbb{R}^m} \hat{\nu}_x(\mathrm{d}s))\sigma(\mathrm{d}x) \in \mathrm{rca}(\overline{\Omega})$ has the density $d_{\sigma_{\hat{\nu}}}$ with respect to the Lebesgue measure, for a.a. $x \in \Omega$, given by the function*

$$d_{\sigma_{\hat{\nu}}}(x) = \left(\int_{\mathbb{R}^m} \frac{\hat{\nu}_x(\mathrm{d}s)}{1 + |s|^p}\right)^{-1} \int_{\mathbb{R}^m} \hat{\nu}_x(\mathrm{d}s). \tag{5}$$

Then $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^m)$.

P r o o f. We are to construct explicitly a sequence in $L^p(\Omega;\mathbb{R}^m)$ which attains $(\sigma,\hat\nu)$ in the sense of (1).

As $\bar\Omega$ and $\beta_{\mathcal R}\mathbb{R}^m\setminus\mathbb{R}^m$ are metrizable compact sets, for every $l\in\mathbb{N}$ there exist finite partitions $\mathcal{P}_l=\{\Omega_l^j\}_{j=1}^{J(l)}$ of $\bar\Omega$ and $\tilde{\mathcal{P}}_l=\{\Gamma_l^j\}_{j=1}^{\tilde J(l)}$ of $\beta_{\mathcal R}\mathbb{R}^m\setminus\mathbb{R}^m$ such that $\Omega_l^{j_1}\cap\Omega_l^{j_2}=\emptyset$, $1\leqslant j_1<j_2\leqslant J(l)$ and $\Gamma_l^{j_1}\cap\Gamma_l^{j_2}=\emptyset$, $1\leqslant j_1<j_2\leqslant\tilde J(l)$ and moreover all Ω_l^j and Γ_l^j are measurable with $\operatorname{diam}(\Omega_l^j)<1/l$ and $\operatorname{diam}(\Gamma_l^j)<1/l$ for all j. Besides, we may suppose that, for any $l\in\mathbb{N}$, the partitions $\mathcal{P}_{l+1}$ and $\tilde{\mathcal{P}}_{l+1}$ are respectively refinements of the partitions $\mathcal{P}_l$ and $\tilde{\mathcal{P}}_l$ and that $\operatorname{int}(\Omega_l^j)\neq\emptyset$ for all j. We shall denote by $\bar v_0$ the continuous extension of $v_0\in\mathcal{R}$ on $\beta_{\mathcal R}\mathbb{R}^m$, i.e. $\bar v_0\in C(\beta_{\mathcal R}\mathbb{R}^m)$.

We know due to Kružík and Roubíček [12, Theorem 2] that there exists a DiPerna-Majda measure $(\sigma^\circ,\hat\nu^\circ)$ where the density of σ° is defined by the right hand side of (5) and $\hat\nu_x^\circ$ by

$$\hat\nu_x^\circ(\mathrm{d}s)=\frac{[\hat\nu_x|_{\mathbb{R}^m}](\mathrm{d}s)}{\int_{\mathbb{R}^m}\hat\nu_x(\mathrm{d}s)}.$$

Having now a pairwise disjoint decomposition of $\bar\Omega$ and $\beta_{\mathcal R}\mathbb{R}^m\setminus\mathbb{R}^m$ we can define

$$a_{ij}^l=\int_{\Omega_l^i}\int_{\Gamma_l^j}\hat\nu_x(\mathrm{d}s)\,\sigma(\mathrm{d}x),\ 1\leqslant i\leqslant J(l),\ 1\leqslant j\leqslant\tilde J(l). \tag{6}$$

Let us choose $x_{ij}\in\ \operatorname{int}(\Omega_l^i)$, $1\leqslant i\leqslant J(l)$, $1\leqslant j\leqslant\tilde J(l)$, $x_{ij}\neq x_{im}$, $j\neq m$, $s_j\in\Gamma_j^l$ and define a measure $\eta^l\in\operatorname{rca}(\bar\Omega\times\beta_{\mathcal R}\mathbb{R}^m)$ by the formula

$$\int_{\bar\Omega\times\beta_{\mathcal R}\mathbb{R}^m}g(x)\bar v_0(s)\,\eta^l(\mathrm{d}x\,\mathrm{d}s)=\int_{\bar\Omega}\int_{\mathbb{R}^m}v_0(s)\hat\nu(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)+\sum_{i=1}^{J(l)}\sum_{j=1}^{\tilde J(l)}\bar v_0(s_j)g(x_{ij})a_{ij}^l \tag{7}$$

for any $g\in C(\bar\Omega)$ and any $v_0\in\mathcal{R}$; here we have used also the facts that $\mathbb{R}^m$ is a Borel subset in $\beta_{\mathcal R}\mathbb{R}^m$ and that the linear hull $\{g\otimes\bar v_0\,;\,g\in C(\bar\Omega),\ v_0\in\mathcal{R}\}$ is dense in $C(\bar\Omega\times\beta_{\mathcal R}\mathbb{R}^m)$.

Now let us take $l\in\mathbb{N}$ fixed. As $\mathbb{R}^m$ (if embedded into $\beta_{\mathcal R}\mathbb{R}^m$) is dense in $\beta_{\mathcal R}\mathbb{R}^m$, for any j there is a sequence $\{s_j^k\}_{k\in\mathbb{N}}\subset\mathbb{R}^m$ such that $\lim\limits_{k\to\infty}s_j^k=s_j$ in $\beta_{\mathcal R}\mathbb{R}^m$, which means precisely that $\lim\limits_{k\to\infty}v_0(s_j^k)=\bar v_0(s_j)$ for any $v_0\in\mathcal{R}$. Inevitably, $\lim\limits_{k\to\infty}|s_j^k|=+\infty$. We can define neighborhoods N_{ij}^k of points x_{ij} for $k\in\mathbb{N}$, $1\leqslant i\leqslant J(l)$ and $1\leqslant j\leqslant\tilde J(l)$ by $N_{ij}^k=\left\{x;|x-x_{ij}|<\left(a_{ij}^l/|s_j^k|^pB(1)\right)^{1/n}\right\}$ where $B(1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Note that, since $|s_j^k|\to+\infty$ for $k\to\infty$, N_{ij}^k are pairwise disjoint and $N_{ij}^k\subset\Omega_l^i$, $1\leqslant i\leqslant J(l)$ and $1\leqslant j\leqslant\tilde J(l)$ whenever k is large enough.

Now we are to prove that η^l is generated by a bounded sequence $\{w_k^l\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ in the sense that

$$\lim_{k\to\infty}\int_\Omega v(w_k^l(x))g(x)\,\mathrm{d}x = \int_{\overline{\Omega}\times\beta_{\mathcal{R}}\mathbb{R}^m} g(x)\bar{v}_0(s)\,\eta^l(\mathrm{d}x\,\mathrm{d}s)$$

for any $g \in C(\overline{\Omega})$ and $v(s) = v_0(s)(1+|s|^p)$ with $v_0 \in \mathcal{R}$. Let us seek a generating sequence $\{w_k^l\}_{k\in\mathbb{N}}$ for η^l in the form

$$w_k^l(x) = \begin{cases} u_k(x) & \text{if } x \in \Omega \setminus \bigcup\limits_{i=1}^{J(l)}\bigcup\limits_{j=1}^{\tilde{J}(l)} N_{ij}^k \\ s_j^k & \text{if } x \in N_{ij}^k \end{cases}$$

where $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ is a generating sequence of (σ°,ν°). We have

$$\begin{aligned}
\lim_{k\to\infty}\int_\Omega v(w_k^l(x))g(x)\,\mathrm{d}x &= \lim_{k\to\infty}\Big(\int_{\Omega\setminus\bigcup\limits_{i=1}^{J(l)}\bigcup\limits_{j=1}^{\tilde{J}(l)} N_{ij}^k} g(x)v_0(u_k(x))(1+|u_k(x)|^p)\,\mathrm{d}x \\
&\quad + \sum_{i=1}^{J(l)}\sum_{j=1}^{\tilde{J}(l)}\int_{N_{ij}^k} g(x)v_0(s_j^k)(1+|s_j^k|^p)\,\mathrm{d}x\Big) \\
&= \int_{\bar{\Omega}}\int_{\mathbb{R}^m} v_0(s)\hat{\nu}_x^\circ(\mathrm{d}s)g(x)\,\sigma^\circ(\mathrm{d}x) + \sum_{i=1}^{J(l)}\sum_{j=1}^{\tilde{J}(l)}\bar{v}_0(s_j)g(x_{ij})a_{ij}^l \\
&= \int_{\bar{\Omega}}\int_{\mathbb{R}^m} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x) + \sum_{i=1}^{J(l)}\sum_{j=1}^{\tilde{J}(l)}\bar{v}_0(s_j)g(x_{ij})a_{ij}^l
\end{aligned}$$

because $\mathrm{meas}(N_{ij}^k) = a_{ij}^l/|s_j^k|^p$, which follows from the fact that the volume of the ball of the radius r in the space $\mathbb{R}^n$ is given by the formula $B(1)r^n$ and therefore

$$\lim_{k\to\infty}\ \mathrm{meas}\ (N_{i,j}^k)(1+|s_j^k|^p) = a_{ij}^l,$$

and because

$$\lim_{k\to\infty}\int_{N_{ij}^k} g(x)v_0(u_k(x))(1+|u_k(x)|^p)\,\mathrm{d}x = 0. \tag{8}$$

Note that Theorem 1 is also used. Let us prove (8). We may suppose that v_0 and g are not identically zero functions since otherwise (8) is obvious. As $\{|u_k|^p,\ k\in\mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$ (see Kružík and Roubíček [12, Lemma]) we obtain due to the Dunford-Pettis compactness criterion that it is equicontinuous.

Moreover, it is only a simple observation that $\lim_{k\to\infty} \operatorname{meas}(N_{ij}^k) = \lim_{k\to\infty} a_{ij}^l/|s_j^k|^p = 0$. Let us take $\varepsilon > 0$. Due to the equicontinuity we can find $k_0 \in \mathbb{N}$ such that for $k \geqslant k_0$

$$\int_{N_{ij}^k} (1+|u_k(x)|^p)\,\mathrm{d}x < \frac{\varepsilon}{\max_{x\in\bar\Omega}|g(x)| \sup_{s\in\mathbb{R}^m}|v_0(s)|}$$

and we have the estimate

$$\Big|\int_{N_{ij}^k} g(x)v_0(u_k(x))(1+|u_k(x)|^p)\,\mathrm{d}x\Big| \leqslant \max_{x\in\bar\Omega}|g(x)| \sup_{s\in\mathbb{R}^m}|v_0(s)| \int_{N_{ij}^k}(1+|u_k(x)|^p)\,\mathrm{d}x < \varepsilon,$$

which proves (8). Altogether we have shown that η^l is attainable by $\{w_k^l\}_{k\in\mathbb{N}}$, i.e. $\eta^l \in \mathrm{DM}_{\mathcal{R}}^p(\Omega;\mathbb{R}^m)$.

Now we want to show that η^l approaches $\eta \cong (\sigma,\hat\nu)$ for $l\to\infty$ in the sense that, for any $g\in C(\bar\Omega)$ and any $v_0 \in \mathcal{R}$, we have

$$\lim_{l\to\infty}\int_{\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^m} g(x)\bar v_0(s)\,\eta^l(\mathrm{d}x\,\mathrm{d}s) = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}\bar v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x).$$

Indeed, in view of (6)–(7), this convergence follows from the estimate

$$\begin{aligned}
&\Big|\int_{\bar\Omega\times\beta_{\mathcal{R}}\mathbb{R}^m} g(x)\bar v_0(s)\,\eta^l(\mathrm{d}s\mathrm{d}x) - \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}\bar v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\Big|\\
&\qquad = \Big|\sum_{i=1}^{J(l)}\sum_{j=1}^{\tilde J(l)} \bar v_0(s_j)g(x_{ij})a_{ij}^l - \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\bar v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x)\Big|\\
&\qquad \leqslant \sum_{i=1}^{J(l)}\sum_{j=1}^{\tilde J(l)}\int_{\Omega_l^i}\int_{\Gamma_l^j}|\bar v_0(s_j)g(x_{ij}) - \bar v_0(s)g(x)|\hat\nu_x(\mathrm{d}\lambda)\,\sigma(\mathrm{d}x)\\
&\qquad \leqslant \sum_{i=1}^{J(l)}\sum_{j=1}^{\tilde J(l)}\int_{\Omega_l^i}\int_{\Gamma_l^j}\Big(M_{\bar v_0}\Big(\frac{1}{l}\Big)\|g\|_{C(\Omega_l^i)} + M_g\Big(\frac{1}{l}\Big)\|\bar v_0\|_{C(\Gamma_l^j)}\Big)\hat\nu_x(\mathrm{d}\lambda)\,\sigma(\mathrm{d}x)\\
&\qquad \leqslant \Big(M_{\bar v_0}\Big(\frac{1}{l}\Big)\|g\|_{C(\bar\Omega)} + M_g\Big(\frac{1}{l}\Big)\|\bar v_0\|_{C(\beta_{\mathcal{R}}\mathbb{R}^m)}\Big)\sigma(\bar\Omega)
\end{aligned}$$

where $M_{\bar v_0}$, $M_g\colon \mathbb{R}^+\to\mathbb{R}^+$ are respectively moduli of continuity of $\bar v_0$ and g, i.e. $|\bar v_0(s_1)-\bar v_0(s_2)| \leqslant M_{\bar v_0}(\varrho(s_1,s_2))$ and $|g(x_1)-g(x_2)| \leqslant M_g(|x_1-x_2|)$, where $\varrho(\cdot,\cdot)$ denotes a metric inducing the (metrizable) compact topology of $\beta_{\mathcal{R}}\mathbb{R}^m$. Of course, $\lim_{\varepsilon\to0^+} M_{\bar v_0}(\varepsilon) = \lim_{\varepsilon\to0^+} M_g(\varepsilon) = 0$ because $\bar v_0$ and g are uniformly continuous due to the Cantor theorem. Note also that the assumption $\sigma \geqslant 0$ has been used here.

Now we are in the situation that

$$\lim_{l\to\infty}\lim_{k\to\infty}\int_\Omega v(w_k^l(x))g(x)\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}\bar v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)$$

for any $g\in C(\bar\Omega)$ and $v(s)=v_0(s)(1+|s|^p)$ with $v_0\in\mathcal{R}$. By a suitable diagonalization procedure, one can choose a net $\{w_{k_\xi}^{l_\xi}\}_{\xi\in\Xi}\subset L^p(\Omega;\mathbb{R}^m)$ such that

$$\lim_{\xi\in\Xi}\int_\Omega v(w_{k_\xi}^{l_\xi}(x))g(x)\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}\bar v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x).$$

On the other hand, for arbitrary $l,k\in\mathbb{N}$ we have

$$\begin{aligned}\|w_k^l\|^p_{L^p(\Omega;\mathbb{R}^m)}&\leqslant\left(\|u_k\|^p_{L^p(\Omega;\mathbb{R}^m)}+\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)\right)\\&\leqslant\left(C^p+\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)\right)<+\infty,\end{aligned}$$

where C bounds $\{u_k\}_{k\in\mathbb{N}}$ in $L^p(\Omega;\mathbb{R}^m)$. This means that the whole net $\{w_{k_\xi}^{l_\xi}\}_{\xi\in\Xi}$ is bounded in $L^p(\Omega;\mathbb{R}^m)$. As $C(\bar\Omega)$ and $\mathcal{R}$ are separable we can even suppose $\Xi=\mathbb{N}$.

□

Remark 3. One can find in DiPerna and Majda [5] or Roubíček [13, Theorem 3.2.10] that any measure η^l, $l\in\mathbb{N}$ generated by an L^p-bounded sequence and defined above admits a representation $(\sigma^l,\hat\nu^l)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$.

Remark 4. Note that $(\sigma^\circ,\hat\nu^\circ)$ is the common p-nonconcentrating modification of all $\eta^l\cong(\sigma^l,\hat\nu^l)$ independently of $l\in\mathbb{N}$.

Remark 5. Note that the situation when

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m}\hat\nu_x(\mathrm{d}s)\,\sigma(\mathrm{d}x)=0\tag{9}$$

has been already completely solved in Kružík and Roubíček [12, Theorem 2].

Linking our Theorems 1 and 3 with Kružík and Roubíček [12, Theorems 2 and 3] we get a generalization of Kružík and Roubíček [12, Theorem 2].

Theorem 4. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open domain, let $\mathcal{R}$ be a separable complete subring of the ring of all continuous bounded functions on $\mathbb{R}^m$ and $(\sigma,\hat\nu)\in\mathrm{rca}(\bar\Omega)\times\mathcal{Y}(\bar\Omega,\sigma;\beta_{\mathcal{R}}\mathbb{R}^m)$. Then the following two statements are equivalent to each other:*

(i) *the pair $(\sigma,\hat\nu)$ is a DiPerna-Majda measure, i.e. $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$,*

(ii) *the following conditions are satisfied simultaneously:*

1. σ *is positive,*
2. $\sigma_{\hat\nu} \in \mathrm{rca}(\bar\Omega)$ *defined by* $\sigma_{\hat\nu}(\mathrm{d}x) = (\int_{\mathbb{R}^m} \hat\nu_x(\mathrm{d}s))\sigma(\mathrm{d}x)$ *is absolutely continuous with respect to the Lebesgue measure* ($d_{\sigma_{\hat\nu}}$ *will denote its density),*
3. *for a.a.* $x \in \Omega$,

$$\int_{\mathbb{R}^m} \hat\nu_x(\mathrm{d}s) > 0 \quad \textit{and} \quad d_{\sigma_{\hat\nu}}(x) = \left(\int_{\mathbb{R}^m} \frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\right)^{-1} \int_{\mathbb{R}^m} \hat\nu_x(\mathrm{d}s).$$

3. Extreme points and rays in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$

In this section we want to study some geometric properties of the set of the DiPerna-Majda measures $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, especially we give a characterization of the extreme points and the extreme rays in this set. We shall see the usefulness of the notion of the p-nonconcentrating modification in studying them.

We shall denote the p-nonconcentrating modification of η by $\mathring{\eta}$, i.e. $\mathring{\eta} \cong (\sigma^\circ, \hat\nu^\circ)$ whenever $\eta \cong (\sigma,\hat\nu)$. Let us say that $\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is *p-nonconcentrating* if $\eta = \mathring{\eta}$. This is equivalent to the fact that the appropriate $(\sigma,\hat\nu) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, such that $\eta \cong (\sigma,\hat\nu)$, is p-nonconcentrating, i.e $(\sigma,\hat\nu) = (\sigma^\circ,\hat\nu^\circ)$.

Let us denote by $\mathrm{rem}(\bar\Omega \times \beta_{\mathcal{R}}\mathbb{R}^m)$ the subset of $\mathrm{rca}(\bar\Omega \times \beta_{\mathcal{R}}\mathbb{R}^m)$ that contains positive measures supported on the set $\bar\Omega \times (\beta_{\mathcal{R}}\mathbb{R}^m \setminus \mathbb{R}^m)$. Note that it follows from the proof of Theorem 3 that $\{\eta + t\tilde\eta;\, t > 0\}$ is a ray in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ whenever $\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and $\tilde\eta \in \mathrm{rem}(\bar\Omega \times \beta_{\mathcal{R}}\mathbb{R}^m)$.

Following Ball [4] and Schonbek [15], we denote by

$$\mathcal{Y}^p(\Omega;\mathbb{R}^m) = \Big\{\nu \in L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^m));\, \nu(x) \equiv \nu_x \in \mathrm{rca}^+_1(\mathbb{R}^m) \text{ for a.a. } x \in \Omega \\ \text{and } (x \mapsto \int_{\mathbb{R}^m} |s|^p \nu_x(\mathrm{d}s)) \in L^1(\Omega)\Big\}$$

the set of the so-called L^p-Young measures; see Kružík and Roubíček [12], Roubíček [13] for details.

Lemma 3. *A Young measure* ν *is an extreme point in* $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ *if and only if it is a.e. a Dirac mass, i.e.* $\nu_x = \delta_{u(x)}$ *for a.a.* $x \in \Omega$ *with some* $u \in L^p(\Omega;\mathbb{R}^m)$.

Proof. If ν is not an extreme point in $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$, then there are $\nu^1,\nu^2 \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ such that $\nu = \frac{1}{2}\nu^1 + \frac{1}{2}\nu^2$ but $\nu^1_x \neq \nu^2_x$ for all x from a positive-measure set $\Omega_\nu \subset \Omega$. It means that ν_x is not an extreme point in the set $\mathrm{rca}^+_1(\mathbb{R}^m)$ of the probability measures on $\mathbb{R}^m$ for $x \in \Omega_\nu$, hence ν_x is not a Dirac mass for such x; cf. Köthe [11, §25.2] and realize that extremality of $\nu_x \in \mathrm{rca}_1^+(\mathbb{R}^m)$ is equivalent to the extremality in $\mathrm{rca}_1^+(\beta_\mathcal{R}\mathbb{R}^m)$ because ν_x vanishes on the remainder $\beta_\mathcal{R}\mathbb{R}^m \setminus \mathbb{R}^m$.

Conversely, suppose that ν_x is not a Dirac mass for $x \in \Omega_0$ with $\mathrm{meas}(\Omega_0) > 0$. Hence it is not an extreme point in the set of the probability measures $\mathrm{rca}_1^+(\mathbb{R}^m)$ so that there are $\nu_x^1, \nu_x^2 \in \mathrm{rca}_1^+(\mathbb{R}^m)$ such that $\nu_x = \frac{1}{2}\nu_x^1 + \frac{1}{2}\nu_x^2$ but $\nu_x^1 \neq \nu_x^2$.

Let us put

$$M(\mu) = \left\{ (\mu^1, \mu^2) \in \mathrm{rca}_1^+(\beta_\mathcal{R}\mathbb{R}^m) \times \mathrm{rca}_1^+(\beta_\mathcal{R}\mathbb{R}^m);\ \tfrac{1}{2}\mu^1 + \tfrac{1}{2}\mu^2 = \mu \right\}.$$

Note that $M(\mu)$ contains only the pairs of measures that vanish on the remainder $\beta_\mathcal{R}\mathbb{R}^m \setminus \mathbb{R}^m$ provided μ vanishes on it. Alternatively, we can also write $M(\mu) = \bigcap_{v \in \mathcal{R}_0} M_v(\mu)$ where $\mathcal{R}_0$ is a countable dense subset of $\mathcal{R}$ and

$$M_v(\mu) = \left\{ (\mu^1, \mu^2) \in \mathrm{rca}_1^+(\beta_\mathcal{R}\mathbb{R}^m) \times \mathrm{rca}_1^+(\beta_\mathcal{R}\mathbb{R}^m);\ \tfrac{1}{2}\langle \mu^1, v\rangle + \tfrac{1}{2}\langle \mu^2, v\rangle = \langle \mu, v\rangle \right\}.$$

As $x \mapsto \nu_x$ is weakly measurable, $x \mapsto \langle \nu_x, v\rangle$ is measurable, hence the multivalued mapping $x \mapsto M_v(\nu_x)$ is measurable, too, provided $\mathrm{rca}_1^+(\beta_\mathcal{R}\mathbb{R}^m) \times \mathrm{rca}_1^+(\beta_\mathcal{R}\mathbb{R}^m)$ is considered in its weak* topology, which makes it a separable complete metric space; cf. Aubin and Frankowska [2, Theorem 8.2.9]. Since $\mathcal{R}_0$ is countable, also the multivalued mapping $x \mapsto M(\nu_x) = \bigcap_{v \in \mathcal{R}_0} M_v(\nu_x)$ is measurable; cf. Aubin and Frankowska [2, Theorem 8.2.4]. Besides, this multivalued mapping is obviously convex and closed-valued, hence weakly* closed-valued as well. By Aubin and Frankowska [2, Theorem 8.1.4], this multivalued mapping is a.e. a closed union of a countable number of its measurable selections, i.e. there are $(\nu^{1,k}, \nu^{2,k})$ such that $x \mapsto (\nu_x^{1,k}, \nu_x^{2,k}) \subset M(\nu_x)$ is measurable and $M(\nu_x) = \text{w*-cl} \bigcup_{k \in \mathbb{N}} (\nu_x^{1,k}, \nu_x^{2,k})$. The convex closed set $M(\nu_x)$ is not a singleton at least if $x \in \Omega_0$ because certainly $M(\nu_x) \supset \{(\nu_x, \nu_x),\ (\nu_x^1, \nu_x^2)\}$ and therefore there is at least one measurable selection $(\nu_x^{1,k}, \nu_x^{2,k})$ which is not equal to (ν_x, ν_x) for a.a. $x \in \Omega$.

Eventually, by the obvious estimate

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^m} |s|^p \nu_x^{1,k}(\mathrm{d}s)\mathrm{d}x &= \int_\Omega \int_{\mathbb{R}^m} |s|^p (2\nu_x - \nu_x^{2,k})(\mathrm{d}s)\mathrm{d}x \\ &\leqslant 2 \int_\Omega \int_{\mathbb{R}^m} |s|^p \nu_x(\mathrm{d}s)\mathrm{d}x < +\infty\end{aligned}$$

we can see that $\nu^{1,k} \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)$. Analogously, $\nu^{2,k} \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)$ as well.

In other words, we have found $\nu_x^{1,k}, \nu_x^{2,k} \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)$ such that $\nu = \frac{1}{2}\nu^{1,k} + \frac{1}{2}\nu^{2,k}$ and $\nu^{1,k} \neq \nu^{2,k}$, which shows that $\nu = \{\nu_x\}_{x \in \Omega}$ is not an extreme point in $\mathcal{Y}^p(\Omega; \mathbb{R}^m)$.
□

Lemma 4. *Let $\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and let $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ be the L^p-Young measure generated by the same sequence as η. Then η is an extreme point in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ if and only if it is p-nonconcentrating and ν is a.e. a Dirac mass, i.e. $\nu_x = \delta_{u(x)}$ for a.a. $x \in \Omega$ with some $u \in L^p(\Omega;\mathbb{R}^m)$.*

Proof. First, let us realize that an extreme point can be found only among p-nonconcentrating measures. Indeed, for any $\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ not p-nonconcentrating we can write $\eta = \mathring{\eta} + \bar{\eta}$ with $\bar{\eta} \in \mathrm{rem}(\overline{\Omega} \times \beta_{\mathcal{R}}\mathbb{R}^m)$ nonvanishing. It follows that such η lies on the ray $\{\mathring{\eta} + t\bar{\eta};\, t > 0\}$ and therefore it cannot be an extreme point.

Clearly, η being p-nonconcentrating is not an extreme point in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ implies that it is not an extreme point in the p-nonconcentrating measures, i.e. there exist η_1, $\eta_2 \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, both p-nonconcentrating, $\eta_1 \neq \eta_2$ such that $\eta = (\eta_1 + \eta_2)/2$. The converse implication is trivial.

Since there is a one-to-one affine mapping

$$\nu \leftrightarrow \eta\colon \mathcal{Y}^p(\Omega;\mathbb{R}^m) \leftrightarrow \{\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m);\, \eta \ p\text{-nonconcentrating}\}$$

(cf. (10) below), the extreme points in $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ are thus mapped uniquely onto extreme points in p-nonconcentrating DiPerna-Majda measures, hence onto extreme points in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$. However, the extreme points in $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ have been already described by Lemma 3. □

Theorem 5. *Let $\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and let $\eta \cong (\sigma,\hat{\nu})$ for $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$. Then η is an extreme point in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ if and only if $\hat{\nu}_x = \delta_{u(x)}$ and σ has the density with respect to the Lebesgue measure d_σ given by $d_\sigma(x) = 1 + |u(x)|^p$ for a.a. $x \in \Omega$ for some $u \in L^p(\Omega;\mathbb{R}^m)$.*

Proof. It is proved in (Köthe [11, §25.2]) that the set of extreme points of the set of probability measures on the compact space is equal to the set of Dirac's measures on this space. Let us take $\beta_{\mathcal{R}}\mathbb{R}^m$ as this space. On the other hand, we know that, for $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$, ν_x with $x \in \Omega$ can be supported only on $\mathbb{R}^m \subset \beta_{\mathcal{R}}\mathbb{R}^m$. As the space of probability measures supported on $\mathbb{R}^m$ is the subspace of $\mathrm{rca}^+_1(\beta_{\mathcal{R}}\mathbb{R}^m)$, it follows that ν_x is an extreme point in the set of all probability measures supported on $\mathbb{R}^m$ if it equals the Dirac measure supported at some point of $\mathbb{R}^m$, say $u(x)$, where $u\colon \Omega \to \mathbb{R}^m$. This ν defines $(\sigma;\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ by (see Kružík and Roubíček [12] and Roubíček [13])

$$\hat{\nu}_x(\mathrm{d}s) = \frac{(1+|s|^p)\nu_x(\mathrm{d}s)}{d_\sigma(x)}, \quad d_\sigma(x) = 1 + \int_{\mathbb{R}^m} |s|^p \nu_x(\mathrm{d}s) \quad \text{for a.a. } x \in \Omega, \tag{10}$$

which gives $\hat{\nu}_x = \delta_{u(x)}$ and $d_\sigma(x) = 1 + |u(x)|^p$. Lemma 4 implies that the corresponding $\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is an extreme point.

Let us prove the converse implication. Let us suppose that η is an extreme point in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$. We know due to Lemma 4 that the L^p-Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ generated by the same sequence as η is such that $\nu_x = \delta_{u(x)}$ for a.a. $x \in \Omega$ and some $u \in L^p(\Omega;\mathbb{R}^m)$. Our assertion now follows from (10). $\square$

Theorem 6. *There is no ray in the set of L^p-Young measures.*

Proof. Let $\nu^1, \nu^2 \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ such that $\nu^1 \neq \nu^2$. Moreover, let us suppose $\{t\nu^2 + (1-t)\nu^1;\, t > 0\}$ is a ray in $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$. It follows that $\{t\nu^2_x + (1-t)\nu^1_x;\, t > 0\}$ is a ray (or a point provided that $\nu^1_x = \nu^2_x$) in $\mathrm{rca}^+_1(\mathbb{R}^m)$ for a.a. $x \in \Omega$. Let us take $B \subset \mathbb{R}^m$ a compact set. Then ν^1_x and ν^2_x restricted on B are subprobability measures on B. But subprobability measures on the compact set form a weakly* compact, hence bounded set which apparently cannot contain any ray. This implies that $\nu^1_x|_B = \nu^2_x|_B$. As B is an arbitrary compact set and $\mathbb{R}^m$ is σ-compact we obtain that $\nu^1_x = \nu^2_x$ for a.a. $x \in \Omega$, contradicting the fact that $\nu^1 \neq \nu^2$. $\square$

Corollary 1. *There is no ray in the set of p-nonconcentrating DiPerna-Majda's measures, i.e. in $\{\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m);\, \eta$ p-nonconcentrating$\}$.*

Proof. It follows directly by Theorem 6 because there is a one-to-one affine mapping given by (10) between $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ and the set of p-nonconcentrating DiPerna-Majda's measures, namely

$$\nu \leftrightarrow \eta\colon \mathcal{Y}^p(\Omega;\mathbb{R}^m) \leftrightarrow \{\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m);\, \eta\ p\text{-nonconcentrating}\}$$

is affine. $\square$

Corollary 2. *Any ray in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ has the form $\{\eta + t\tilde{\eta};\, t > 0\}$ with some $\eta \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and some $\tilde{\eta} \in \mathrm{rem}(\bar{\Omega} \times \beta_{\mathcal{R}}\mathbb{R}^m)$.*

Proof. Let us take $\eta_1, \eta_2 \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and define $\eta(t) = t\eta_2 + (1-t)\eta_1$, $t > 0$. We are to find conditions which must be fulfilled by η_1 and η_2 so that $\eta(t) \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ for any t positive. Let us recall that " $\circ$ " indicates the p-nonconcentrating modification. We can write $\eta_i = \mathring{\eta}_i + \bar{\eta}_i$ and $\eta(t) = \mathring{\eta}(t) + \bar{\eta}(t)$ provided $\eta(t) \in \mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, where $\bar{\eta}_i, \bar{\eta}(t) \in \mathrm{rem}(\bar{\Omega} \times \beta_{\mathcal{R}}\mathbb{R}^m)$ $(i = 1, 2)$. This implies that

$$\mathring{\eta}(t) = t\mathring{\eta}_2 + (1-t)\mathring{\eta}_1 \quad \text{and} \quad \bar{\eta}(t) = t\bar{\eta}_2 + (1-t)\bar{\eta}_1,\ t > 0.$$

It follows by Corollary 1 that $\mathring{\eta}_1 = \mathring{\eta}_2$. So we obtain that $\eta(t) = \mathring{\eta}_1 + t\bar{\eta}_2 + (1-t)\bar{\eta}_1 = \eta_1 + t(\bar{\eta}_2 - \bar{\eta}_1)$. Putting $\tilde{\eta} = \bar{\eta}_2 - \bar{\eta}_1$ and $\eta = \eta_1$, we get the desired result. $\square$

Corollary 3. *There is no straight line in* $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$.

Proof. Let $L=\{\eta+t\tilde{\eta};\, t\in\mathbb{R}\}$ be a line in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$. Then both

$$\{\eta+t\tilde{\eta};\, t>0\} \text{ and } \{\eta+t\tilde{\eta};\, t<0\}$$

would be rays in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, which implies by Corollary 2 that both $\tilde{\eta}$ and $-\tilde{\eta}$ lie in $\mathrm{rem}(\overline{\Omega}\times\beta_{\mathcal{R}}\mathbb{R}^m)$. This is possible only if $\tilde{\eta}$ vanishes, so that L is actually a singleton. □

Remark 6. Note that any ray in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is composed of DiPerna-Majda's measures that have the same p-nonconcentrating modification.

Our last theorem together with Theorem 5 characterizes completely the extreme rays in the set of DiPerna-Majda's measures.

Theorem 7. *A ray* $\{\eta+t\tilde{\eta};\, t>0\}$ *with* $\eta\in\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ *and* $\tilde{\eta}\in\mathrm{rem}(\overline{\Omega}\times\beta_{\mathcal{R}}\mathbb{R}^m)$ *is an extreme ray if and only if* η *is an extreme point in* $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ *and* $\tilde{\eta}$ *is the Dirac measure supported at some point of* $\overline{\Omega}\times(\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m)$.

Proof. The end point η of the ray in question belongs to $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and therefore the extreme ray must arise from an extreme point of $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, see Köthe [11, §25].

We shall see that the problem to find all extreme rays in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is equivalent to the problem to find all extreme rays in $\mathrm{rem}(\overline{\Omega}\times\beta_{\mathcal{R}}\mathbb{R}^m)$.

Due to the definition, $\{\eta+t\tilde{\eta};\, t>0\}$ in $\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is an extreme ray if and only if the following implication holds for $\eta_1,\ \eta_2\in\mathrm{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$:

$$\begin{aligned}\exists t_0>0\ \exists 0<r_0<1\ \text{ such that }\ \eta+t_0\tilde{\eta}=r_0\eta_1+(1-r_0)\eta_2 \quad\Longrightarrow\\ \forall 0<r<1\ \exists t>0\ \text{ such that }\ \eta+t\tilde{\eta}=r\eta_1+(1-r)\eta_2.\end{aligned}$$

We can write $\eta_1=\mathring{\eta}_1+\bar{\eta}_1$ and $\eta_2=\mathring{\eta}_2+\bar{\eta}_2$ with $\bar{\eta}_1,\bar{\eta}_2\in\mathrm{rem}(\overline{\Omega}\times\beta_{\mathcal{R}}\mathbb{R}^m)$. Using this representation, we obtain from the above implication used for $\mathring{\eta}_1=\mathring{\eta}_2=\eta$ (note that $\eta=\mathring{\eta}$) that the following implication holds for $\bar{\eta}_1,\bar{\eta}_2\in\mathrm{rem}(\overline{\Omega}\times\beta_{\mathcal{R}}\mathbb{R}^m)$:

$$\begin{aligned}\exists t_0>0\ \exists 0<r_0<1\ \text{ such that }\ t_0\tilde{\eta}=r_0\bar{\eta}_1+(1-r_0)\bar{\eta}_2 \quad\Longrightarrow\\ \forall 0<r<1\ \exists t>0\ \text{ such that }\ t\tilde{\eta}=r\bar{\eta}_1+(1-r)\bar{\eta}_2.\end{aligned}$$

The last implication says precisely that $\{t\tilde{\eta};\, t>0\}$ is an extreme ray in $\mathrm{rem}(\overline{\Omega}\times\beta_{\mathcal{R}}\mathbb{R}^m)$. It can be found in (Köthe [11, §25]) that $\{t\tilde{\eta};\, t>0\}$ is an extreme ray in $\mathrm{rem}(\overline{\Omega}\times\beta_{\mathcal{R}}\mathbb{R}^m)$ if and only if $\tilde{\eta}$ is the Dirac measure. □

A c k n o w l e d g e m e n t. We thank Professor Robert V. Kohn for several fruitful comments and suggestions.

References

[1] *J. J. Alibert, G. Bouchitté*: Non uniform integrability and generalized Young measures. J. Convex. Anal. *4* (1997), 1–19.
[2] *J. P. Aubin, H. Frankowska*: Set-valued Analysis. Birkhäuser, 1990.
[3] *E. J. Balder*: New existence results for optimal controls in the absence of convexity: the importance of extremality. SIAM J. Control Anal. *32* (1994), 890–916.
[4] *J. M. Ball*:PDEs and Continuum Models of Phase Transition (M. Rascle, D. Serre, M. Slemrod., eds.). Lecture Notes in Physics 344, Springer, Berlin, 1989, pp. 207–215.
[5] *R. J. DiPerna, A. J. Majda*: Oscillations and concentrations in weak solutions of the incompressible fluid equations. Comm. Math. Physics *108* (1987), 667–689.
[6] *R. J. DiPerna, A. J. Majda*: Concentrations in regularizations for 2-D incompressible flow. Comm. Pure Appl. Math. *40* (1987), 301–345.
[7] *R. J. DiPerna, A. J. Majda*: Reduced Hausdorff dimension and concentration cancellation for 2-D incompressible flow. J. Amer. Math. Soc. *1* (1988), 59–95.
[8] *N. Dunford, J. T. Schwartz*: Linear Operators. Part I, Interscience, New York, 1967.
[9] *C. Greengard, E. Thomann*: On DiPerna-Majda concentration sets for two-dimensional incompresssible flow. Comm. Pure Appl. Math. *41* (1988), 295–303.
[10] *P. R. Halmos*: Measure Theory. D. van Nostrand, 1950.
[11] *G. Köthe*: Topological Vector Spaces I. 2nd ed. Springer, Berlin, 1983.
[12] *M. Kružík, T. Roubíček*: Explicit characterization of L^p-Young measures. J. Math. Anal. Appl. *198* (1996), 830–843.
[13] *T. Roubíček*: Relaxation in Optimization Theory and Variational Calculus. W. de Gruyter, Berlin, 1997.
[14] *T. Roubíček*: Nonconcentrating generalized Young functionals. Comment. Math. Univ. Carolin. *38* (1997), 91–99.
[15] *M. E. Schonbek*: Convergence of solutions to nonlinear dispersive equations. Comm. Partial Differential Equations *7* (1982), 959–1000.
[16] *L. Tartar*: Compensated compactness and applications to partial differential equations. Nonlinear Analysis and Mechanics (R. J. Knops, ed.). Heriott-Watt Symposium IV, Pitman Res. Notes in Math. 39, San Francisco, 1979.
[17] *M. Valadier*: Young measures. Methods of Nonconvex Analysis (A. Cellina, ed.). Lecture Notes in Math. 1446, Springer, Berlin, 1990, pp. 152–188.
[18] *J. Warga*: Optimal Control of Differential and Functional Equations. Academic Press, New York, 1972.
[19] *L. C. Young*: Generalized curves and the existence of an attained absolute minimum in the calculus of variations. Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, Classe III *30* (1937), 212–234.

Authors' addresses: *Martin Kružík*, Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic, e-mail: `kruzik@utia.cas.cz`; *Tomáš Roubíček*, Mathematical Institute, Charles University, Sokolovská 83, CZ-186 00 Praha 8, Czech Republic and Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic, e-mail: `roubicek@karlin.mff.cuni.cz`.