NECESSARY CONDITIONS FOR THE L^p-CONVERGENCE ($0 < p < 1$) OF SINGLE AND DOUBLE TRIGONOMETRIC SERIES

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Abstract. We give necessary conditions in terms of the coefficients for the convergence of a double trigonometric series in the L^p-metric, where $0 < p < 1$. The results and their proofs have been motivated by the recent papers of A. S. Belov (2008) and F. Móricz (2010). Our basic tools in the proofs are the Hardy-Littlewood inequality for functions in H^p and the Bernstein-Zygmund inequalities for the derivatives of trigonometric polynomials and their conjugates in the L^p-metric, where $0 < p < 1$.

Keywords: trigonometric series; Hardy-Littlewood inequality for functions in H^p; Bernstein-Zygmund inequalities for the derivative of trigonometric polynomials in L^p-metric for $0 < p < 1$; necessary conditions for the convergence in L^p-metric

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1. Introduction and preliminaries

Let $f_1(x)$ be a complex-valued function, periodic with period 2π, and integrable in Lebesgue's sense, briefly $f_1 \in L^1(\mathbb{T})$. We consider its Fourier series

$$f_1(x) \sim \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx}, \quad x \in \mathbb{T} = [-\pi, \pi). \tag{1.1}$$

The symmetric partial sums of the series in (1.1) are defined by

$$s_n(f_1) = s_n(f_1; x) := \sum_{|k|\leqslant n} c_k \mathrm{e}^{\mathrm{i}kx}, \quad n \in \mathbb{N},$$

where $\mathbb{N} := \{0, 1, 2, \ldots\}$. The $L^1(\mathbb{T})$-norm of a function f_1 is defined by

$$\|f_1\| = \|f_1(x)\|_1 := \frac{1}{2\pi} \int_{\mathbb{T}} |f_1(x)|\,\mathrm{d}x.$$

The following theorem was proved by A. S. Belov in [2] (also note that some generalizations of the results of [2] were given by the first author in [5]).

Theorem 1.1. *Assume* $f_1 \in L^1(\mathbb{T}^2)$ *and*

$$\|s_n(f_1) - f_1\| \to 0 \quad \text{as } n \to \infty.$$

Then

$$\sum_{k=[n/2]}^{2n} \frac{|c_k| + |c_{-k}|}{(|k-n|+1)} \to 0 \quad \text{as } n \to \infty. \tag{1.2}$$

Now, we shall discuss some known results for double Fourier series. Let us suppose that $f_2(x,y)$ is a complex-valued function, periodic with period 2π in each variable, and integrable in Lebesgue's sense, briefly $f_2 \in L^1(\mathbb{T}^2)$. We consider its double Fourier series

$$f_2(x,y) \sim \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} c_{kl} \mathrm{e}^{\mathrm{i}(kx+ly)}, \quad (x,y) \in \mathbb{T}^2. \tag{1.3}$$

The symmetric rectangular partial sums of the double series in (1.3) are defined by

$$s_{mn}(f_2) = s_{mn}(f_2; x, y) := \sum_{|k|\leqslant m} \sum_{|l|\leqslant n} c_{kl} \mathrm{e}^{\mathrm{i}(kx+ly)}, \quad (m,n) \in \mathbb{N}^2.$$

The $L^1(\mathbb{T}^2)$-norm of a function f_2 is defined by

$$\|f_2\| = \|f_2(x,y)\|_1 := \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} |f_2(x,y)| \,\mathrm{d}x \,\mathrm{d}y.$$

The next statement is due to the third author [6]; it extends the results of A. S. Belov from single to double Fourier series.

Theorem 1.2. *Assume* $f_2 \in L^1(\mathbb{T}^2)$ *and*

$$\|s_{mn}(f_2) - f_2\| \to 0 \quad \text{as } m, n \to \infty$$

independently of one another. Then

$$\sum_{k=[m/2]}^{2m} \sum_{l=[n/2]}^{2n} \frac{|c_{kl}| + |c_{-k,l}| + |c_{k,-l}| + |c_{-k,-l}|}{(|k-m|+1)(|l-n|+1)} \to 0 \quad \text{as } m, n \to \infty. \tag{1.4}$$

For $f_1 \in L^p(\mathbb{T})$, $0 < p < 1$, the L^p-metric is defined by

$$\|f_1\|_{L^p} = \|f_1\|_p = \Big(\frac{1}{2\pi}\int_{\mathbb{T}} |f_1(x)|^p \,\mathrm{d}x\Big)^{1/p}.$$

Similarly, for $f_2 \in L^p(\mathbb{T}^2)$, $0 < p < 1$, the L^p-metric is defined by

$$\|f_2\|_{L^p} = \|f_2\|_p = \Big(\frac{1}{4\pi^2}\iint_{\mathbb{T}^2} |f_2(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}.$$

We recall that for $0 < p < 1$, $\|\cdot\|_p$ is not a norm since it does not satisfy the triangle inequality, but it is known as a quasi-norm.

The aim of this paper is to obtain necessary conditions for the convergence of the trigonometric series (1.1) and (1.3) in the L^p-metric.

Our main tools in proving the main results are the Bernstein-Zygmund inequality and the Hardy-Littlewood theorem in the spaces L^p $(0 < p < 1)$ and H^p $(0 < p < 1)$, respectively. We also need the Bernstein-Zygmund inequality for trigonometric polynomials and their conjugates. We recall that

$$\widetilde{T}_n(x) = \sum_{|k|\leqslant n} (-\mathrm{i}\,\mathrm{sign}\,k) c_k \mathrm{e}^{\mathrm{i}kx}$$

is said to be the conjugate to the polynomial

$$T_n(x) = \sum_{|k|\leqslant n} c_k \mathrm{e}^{\mathrm{i}kx}.$$

Moreover, the rth derivative of a function $f(x)$ is denoted by $f^{(r)}(x)$.

Lemma 1.3 ([1] or [3, p. 63]). *Let $T_n(x)$ be a trigonometric polynomial of order n and $0 < p < 1$. Then the inequality*

$$\|T_n^{(r)}\|_p \leqslant n^r \|T_n\|_p$$

holds true.

Lemma 1.4 (see [7]). *Let $T_n(x)$ be a trigonometric polynomial of order n and $r > 0$. Then the inequality*

$$\|\widetilde{T}_n^{(r)}\|_p \leqslant K_{p,r}\, n^r \|T_n\|_p$$

holds true if and only if $p > 1/(r+1)$.

Lemma 1.5 ([4, Theorem 16]). *If* $\varphi(z) = \sum\limits_{k=0}^{\infty} c_k z^k$, $|z| < 1$ *and* $\varphi \in H^p$, $0 < p < 1$, *then* $\sum\limits_{k=0}^{\infty} (k+1)^{p-2}|c_k|^p \leqslant K_p\|\varphi\|_p^p$.

Throughout this paper, K_p, or $K_{p,r}$, denotes a positive constant which depends only on p, or p and r, respectively, but not necessarily the same at each occurrence.

2. Main results on single trigonometric series

We begin with the following auxiliary statements.

Lemma 2.1. *For every* $n \in \mathbb{N}$ *and* $0 < p < 1$, *we have*

$$\min\left\{\left\|\sum_{k=0}^{n} c_k \mathrm{e}^{\mathrm{i}kx}\right\|_p^p, \left\|\sum_{k=0}^{n} c_k \mathrm{e}^{-\mathrm{i}kx}\right\|_p^p\right\} \geqslant K_p \max\left\{\sum_{k=0}^{n}(k+1)^{p-2}|c_k|^p, \sum_{k=0}^{n}(n-k+1)^{p-2}|c_k|^p\right\}.$$

Proof. The proof of this lemma is an immediate consequence of Lemma 1.5. Indeed, we have

$$\begin{aligned}\left\|\sum_{k=0}^{n} c_k \mathrm{e}^{\mathrm{i}kx}\right\|_p^p &= \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{n} c_k \mathrm{e}^{\mathrm{i}kx}\right|^p \mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{n} \bar{c}_k \mathrm{e}^{-\mathrm{i}kx}\right|^p \mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\mathrm{e}^{\mathrm{i}nx}\sum_{k=0}^{n} \bar{c}_k \mathrm{e}^{-\mathrm{i}kx}\right|^p \mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{n} \bar{c}_k \mathrm{e}^{\mathrm{i}(n-k)x}\right|^p \mathrm{d}x \\ &= \left\|\sum_{k=0}^{n} \bar{c}_k \mathrm{e}^{\mathrm{i}(n-k)x}\right\|_p^p \geqslant K_p\sum_{k=0}^{n}(n-k+1)^{p-2}|c_k|^p.\end{aligned}$$

An analogous inequality holds true for $\sum\limits_{k=0}^{n} c_k \mathrm{e}^{-\mathrm{i}kx}$ since

$$\left\|\sum_{k=0}^{n} c_k \mathrm{e}^{-\mathrm{i}kx}\right\|_p = \left\|\sum_{k=0}^{n} \bar{c}_k \mathrm{e}^{\mathrm{i}kx}\right\|_p.$$

□

Lemma 2.2. *For all* $-1 \leqslant n < \nu$, $r = 1, 2, \ldots$, *and* $1/(r+1) < p < 1$, *we have*

$$\max\left\{\left\|\sum_{k=n+1}^{\nu} k^r c_k \mathrm{e}^{\mathrm{i}kx}\right\|_p, \left\|\sum_{k=n+1}^{\nu} (-k)^r c_{-k} \mathrm{e}^{-\mathrm{i}kx}\right\|_p\right\} \leqslant K_{p,r}\nu^r\|s_\nu - s_n\|_p.$$

P r o o f. Let us denote

$$\tilde{s}_n := \sum_{|k|\leqslant n} (-\mathrm{i}\,\mathrm{sign}\, k) c_k \mathrm{e}^{\mathrm{i}kx}, \qquad s_n := \sum_{|k|\leqslant n} c_k \mathrm{e}^{\mathrm{i}kx}$$

and

$$\tilde{s}(n,\nu) := \tilde{s}_\nu - \tilde{s}_n, \qquad s(n,\nu) := s_\nu - s_n.$$

Then it is obvious that

$$s(n,\nu)^{(r)} + \mathrm{i}\tilde{s}(n,\nu)^{(r)} = 2\mathrm{i}^r \sum_{k=n+1}^{\nu} k^r c_k \mathrm{e}^{\mathrm{i}kx},$$

$$s(n,\nu)^{(r)} - \mathrm{i}\tilde{s}(n,\nu)^{(r)} = 2\mathrm{i}^r \sum_{k=n+1}^{\nu} (-k)^r c_{-k} \mathrm{e}^{-\mathrm{i}kx}.$$

In what follows, we shall use the well-known inequality

$$|a+b|^\beta \leqslant |a|^\beta + |b|^\beta \quad \text{if } 0 < \beta < 1.$$

Since

$$2^p \Big\| \sum_{k=n+1}^{\nu} k^r c_k \mathrm{e}^{\mathrm{i}kx} \Big\|_p^p = \frac{1}{2\pi} \int_{\mathbb{T}} |s(n,\nu)^{(r)} + \mathrm{i}\tilde{s}(n,\nu)^{(r)}|^p \,\mathrm{d}x$$
$$\leqslant \frac{1}{2\pi} \int_{\mathbb{T}} |s(n,\nu)^{(r)}|^p \,\mathrm{d}x + \frac{1}{2\pi} \int_{\mathbb{T}} |\mathrm{i}\tilde{s}(n,\nu)^{(r)}|^p \,\mathrm{d}x,$$

we may apply Lemmas 1.3 and 1.4 to obtain the first required inequality, i.e.,

$$\Big\| \sum_{k=n+1}^{\nu} k^r c_k \mathrm{e}^{\mathrm{i}kx} \Big\|_p^p \leqslant K_{p,r} \nu^{rp} \|s(n,\nu)\|_p^p = K_{p,r} \nu^{rp} \|s_\nu - s_n\|_p^p.$$

The other inequality can be proved in a similar way. □

Lemma 2.3. *For $0 \leqslant n < \nu$ and $0 < p < 1$, we have*

$$\Big\| \sum_{k=n+1}^{\nu} k^r c_k \mathrm{e}^{\mathrm{i}kx} \Big\|_p^p \geqslant K_p \max\Big\{ \sum_{k=n+1}^{\nu} (k-n)^{p-2} |k^r c_k|^p, \sum_{k=n+1}^{\nu} (\nu - k + 1)^{p-2} |k^r c_k|^p \Big\}$$

and

$$\Big\| \sum_{k=n+1}^{\nu} (-k)^r c_{-k} \mathrm{e}^{-\mathrm{i}kx} \Big\|_p^p$$
$$\geqslant K_p \max\Big\{ \sum_{k=n+1}^{\nu} (k-n)^{p-2} |k^r c_{-k}|^p, \sum_{k=n+1}^{\nu} (\nu - k + 1)^{p-2} |k^r c_{-k}|^p \Big\}.$$

P r o o f. In a way similar to the proof of [6, Lemma 5], we can prove that

$$\Bigl\|\sum_{k=n+1}^{\nu} k^r c_k \mathrm{e}^{\mathrm{i}kx}\Bigr\|_p = \Bigl\|\sum_{k_1=0}^{\nu-n-1} k^r c_k \mathrm{e}^{\mathrm{i}k_1x}\Bigr\|_p,$$

where $k_1 = k-n-1$. Applying Lemma 2.1 gives the first inequality. The second inequality can be proved in a similar way. □

Now, we establish our main result.

Theorem 2.4. *Assume that* $f_1 \in L^p(\mathbb{T})$, $0<p<1$, *and*

$$\|s_n(f_1)-f_1\|_p \to 0 \quad \text{as } n\to\infty.$$

Then

$$\sum_{k=[n/2]}^{2n} \frac{|c_k|^p+|c_{-k}|^p}{(|k-n|+1)^{2-p}} \to 0 \quad \text{as } n\to\infty. \tag{2.1}$$

P r o o f. Let $n \geqslant 2$ and set

$$C_k(p) := |c_k|^p + |c_{-k}|^p \quad \text{for } k\in\mathbb{N} \quad \text{and} \quad r := [(1/p)-1]+1,$$

where $[\cdot]$ denotes the integer part of a real number. Applying Lemmas 2.2 and 2.3 with $\nu := 2n$ we obtain

$$\begin{aligned}\sum_{k=n+1}^{2n} \frac{k^{rp}C_k(p)}{(k-n)^{2-p}} &\leqslant K_p\Bigl\{\Bigl\|\sum_{k=n+1}^{2n} k^r c_k \mathrm{e}^{\mathrm{i}kx}\Bigr\|_p^p + \Bigl\|\sum_{k=n+1}^{2n} (-k)^r c_{-k}\mathrm{e}^{-\mathrm{i}kx}\Bigr\|_p^p\Bigr\}\\ &\leqslant K_{p,r} n^{rp}\|s_{2n}(f_1)-s_n(f_1)\|_p^p = K_p n^{rp}\|s_{2n}(f_1)-s_n(f_1)\|_p^p.\end{aligned}$$

Now, repeating the above reasoning with $[n/2]-1$ in place of n in the lower limit of the summation and with n in place of $2n$ in the upper limit of the summation, we get

$$\sum_{k=[n/2]}^{n} \frac{k^{rp}C_k(p)}{(n-k+1)^{2-p}} \leqslant K_p n^{rp}\|s_n(f_1)-s_{[n/2]-1}(f_1)\|_p^p.$$

Using the previous two inequalities yields

$$\begin{aligned}\sum_{k=[n/2]}^{2n} \frac{C_k(p)}{(|k-n|+1)^{2-p}} &\leqslant \frac{K_p}{n^{rp}}\Bigl\{\sum_{k=n+1}^{2n}\frac{k^{rp}C_k(p)}{(k-n)^{2-p}} + \sum_{k=[n/2]}^{n}\frac{k^{rp}C_k(p)}{(n-k+1)^{2-p}}\Bigr\}\\ &\leqslant K_p \max_{[n/2]-1\leqslant\nu_1<\nu_2\leqslant 2n}\|s_{\nu_2}(f_1)-s_{\nu_1}(f_1)\|_p.\end{aligned}$$

This proves (2.1). □

We begin with the following four lemmas.

Lemma 3.1. *For all* $(m,n)\in\mathbb{N}^2$ *and* $0<p<1$, *we have*

$$\begin{aligned}\min\Bigg\{&\Big\|\sum_{k=0}^{m}\sum_{l=0}^{n}c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\Big\|_p^p,\ \Big\|\sum_{k=0}^{m}\sum_{l=0}^{n}c_{kl}\mathrm{e}^{\mathrm{i}(-kx+ly)}\Big\|_p^p,\\ &\Big\|\sum_{k=0}^{m}\sum_{l=0}^{n}c_{kl}\mathrm{e}^{\mathrm{i}(kx-ly)}\Big\|_p^p,\ \Big\|\sum_{k=0}^{m}\sum_{l=0}^{n}c_{kl}\mathrm{e}^{\mathrm{i}(-kx-ly)}\Big\|_p^p\Bigg\}\\ \geqslant K_p\max\Bigg\{&\sum_{k=0}^{m}\sum_{l=0}^{n}((k+1)(l+1))^{p-2}|c_{kl}|^p,\sum_{k=0}^{m}\sum_{l=0}^{n}((m-k+1)(l+1))^{p-2}|c_{kl}|^p,\\ &\sum_{k=0}^{m}\sum_{l=0}^{n}((k+1)(n-l+1))^{p-2}|c_{kl}|^p,\\ &\sum_{k=0}^{m}\sum_{l=0}^{n}((m-k+1)(n-l+1))^{p-2}|c_{kl}|^p\Bigg\}.\end{aligned}$$

P r o o f. Applying Lemma 2.1 twice together with Fubini's theorem, we obtain

$$\begin{aligned}4\pi^2\Big\|\sum_{k=0}^{m}\sum_{l=0}^{n}c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\Big\|_p^p&=\iint_{\mathbb{T}^2}\Big|\sum_{k=0}^{m}\sum_{l=0}^{n}c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\Big|^p\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{\mathbb{T}}\Big(\int_{\mathbb{T}}\Big|\sum_{k=0}^{m}\Big(\sum_{l=0}^{n}c_{kl}\mathrm{e}^{\mathrm{i}ly}\Big)\mathrm{e}^{\mathrm{i}kx}\Big|^p\,\mathrm{d}x\Big)\,\mathrm{d}y\\ &\geqslant K_p\int_{\mathbb{T}}\sum_{k=0}^{m}(k+1)^{p-2}\Big|\sum_{l=0}^{n}c_{kl}\mathrm{e}^{\mathrm{i}ly}\Big|^p\,\mathrm{d}y=K_p\sum_{k=0}^{m}(k+1)^{p-2}\int_{\mathbb{T}}\Big|\sum_{l=0}^{n}c_{kl}\mathrm{e}^{\mathrm{i}ly}\Big|^p\,\mathrm{d}y\\ &\geqslant K_p\sum_{k=0}^{m}(k+1)^{p-2}\sum_{l=0}^{n}(l+1)^{p-2}|c_{kl}|^p=K_p\sum_{k=0}^{m}\sum_{l=0}^{n}((k+1)(l+1))^{p-2}|c_{kl}|^p.\end{aligned}$$

The other fifteen inequalities can be proved in an analogous way. We do not enter into further details. □

Next, we prove a lemma which is an extension of Lemmas 1.3 and 1.4 from single to double trigonometric polynomials.

Lemma 3.2. *For all* $(m,n)\in\mathbb{N}^2$, $r_1,r_2=1,2,\ldots$, *and* $\max\{1/r_1,1/r_2\}<p<1$, *we have*

$$\max\Bigg\{\bigg\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\sum_{|k|\leqslant m}\sum_{|l|\leqslant n}c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\bigg\|_p,$$
$$\bigg\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\sum_{|k|\leqslant m}\sum_{|l|\leqslant n}(-\mathrm{i}\,\mathrm{sign}\,k)c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\bigg\|_p,$$
$$\bigg\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\sum_{|k|\leqslant m}\sum_{|l|\leqslant n}(-\mathrm{i}\,\mathrm{sign}\,l)c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\bigg\|_p,$$
$$\bigg\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\sum_{|k|\leqslant m}\sum_{|l|\leqslant n}(-\mathrm{i}\,\mathrm{sign}\,k)(-\mathrm{i}\,\mathrm{sign}\,l)c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\bigg\|_p\Bigg\}$$
$$\leqslant K_{p,r_1,r_2}m^{r_1}n^{r_2}\bigg\|\sum_{|k|\leqslant m}\sum_{|l|\leqslant n}c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\bigg\|_p.$$

Proof. By Lemmas 1.3 and 1.4, for example, we have

$$\bigg\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\sum_{|k|\leqslant m}\sum_{|l|\leqslant n}(-\mathrm{i}\,\mathrm{sign}\,k)c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\bigg\|_p$$
$$=\bigg\|\frac{\partial^{r_1}}{\partial x^{r_1}}\sum_{|k|\leqslant m}(-\mathrm{i}\,\mathrm{sign}\,k)\bigg(\frac{\partial^{r_2}}{\partial x^{r_2}}\sum_{|l|\leqslant n}c_{kl}\mathrm{e}^{\mathrm{i}ly}\bigg)\mathrm{e}^{\mathrm{i}kx}\bigg\|_p$$
$$\leqslant K_{p,r_1}m^{r_1}\bigg\|\frac{\partial^{r_2}}{\partial x^{r_2}}\sum_{|l|\leqslant n}\bigg(\sum_{|k|\leqslant m}c_{kl}\mathrm{e}^{\mathrm{i}kx}\bigg)\mathrm{e}^{\mathrm{i}ly}\bigg\|_p$$
$$\leqslant K_{p,r_1}m^{r_1}n^{r_2}\bigg\|\sum_{|k|\leqslant m}\sum_{|l|\leqslant n}c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\bigg\|_p.$$

This proves the second inequality. The other three inequalities can be proved in a similar way. □

Lemma 3.3. *For all* $-1\leqslant m<\mu$, $-1\leqslant n<\nu$, $r_1,r_2=1,2,\ldots$, *and* $\max\{1/r_1,1/r_2\}<p<1$, *we have*

$$\max\Bigg\{\bigg\|\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu}k^{r_1}l^{r_2}c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\bigg\|_p,\ \bigg\|\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu}(-k)^{r_1}l^{r_2}c_{-kl}\mathrm{e}^{\mathrm{i}(-kx+ly)}\bigg\|_p,$$
$$\bigg\|\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu}k^{r_1}(-l)^{r_2}c_{k,-l}\mathrm{e}^{\mathrm{i}(kx-ly)}\bigg\|_p,$$
$$\bigg\|\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu}(-k)^{r_1}(-l)^{r_2}c_{-k,-l}\mathrm{e}^{\mathrm{i}(-kx-ly)}\bigg\|_p\Bigg\}$$
$$\leqslant K_{p,r_1,r_2}\mu^{r_1}\nu^{r_2}\|s_{\mu\nu}-s_{m\nu}-s_{\mu n}+s_{mn}\|_p,$$

where

$$s_{-1,n} = s_{m,-1} = s_{-1,-1} = 0 \quad \text{and} \quad s_{mn} := \sum_{|k|\leqslant m} \sum_{|l|\leqslant n} c_{kl} e^{i(kx+ly)}, \quad (m,n) \in \mathbb{N}^2.$$

P r o o f. We introduce the notation

$$\tilde{s}^{(1,0)}_{mn} := \sum_{|k|\leqslant m} \sum_{|l|\leqslant n} (-i\,\mathrm{sign}\,k) c_{kl} e^{i(kx+ly)},$$
$$\tilde{s}^{(0,1)}_{mn} := \sum_{|k|\leqslant m} \sum_{|l|\leqslant n} (-i\,\mathrm{sign}\,l) c_{kl} e^{i(kx+ly)},$$
$$\tilde{s}^{(1,1)}_{mn} := \sum_{|k|\leqslant m} \sum_{|l|\leqslant n} (-i\,\mathrm{sign}\,k)(-i\,\mathrm{sign}\,l) c_{kl} e^{i(kx+ly)},$$

and

$$s(m,n;\mu,\nu) := s_{\mu\nu} - s_{m\nu} - s_{\mu n} + s_{mn} = \sum_{m<|k|\leqslant\mu} \sum_{n<|l|\leqslant\nu} c_{kl} e^{i(kx+ly)},$$
$$\tilde{s}^{(1,0)}(m,n;\mu,\nu) := s^{(1,0)}_{\mu\nu} - s^{(1,0)}_{m\nu} - s^{(1,0)}_{\mu n} + s^{(1,0)}_{mn},$$
$$\tilde{s}^{(0,1)}(m,n;\mu,\nu) := s^{(0,1)}_{\mu\nu} - s^{(0,1)}_{m\nu} - s^{(0,1)}_{\mu n} + s^{(0,1)}_{mn},$$
$$\tilde{s}^{(1,1)}(m,n;\mu,\nu) := s^{(1,1)}_{\mu\nu} - s^{(1,1)}_{m\nu} - s^{(1,1)}_{\mu n} + s^{(1,1)}_{mn}.$$

Using the equalities (see in [6])

$$\frac{\partial^2}{\partial x \partial y}\{s(m,n;\mu,\nu) + i\tilde{s}^{(1,0)}(m,n;\mu,\nu) + i\tilde{s}^{(0,1)}(m,n;\mu,\nu) + i^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\}$$
$$= -4 \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} kl c_{kl} e^{i(kx+ly)},$$
$$\frac{\partial^2}{\partial x \partial y}\{s(m,n;\mu,\nu) - i\tilde{s}^{(1,0)}(m,n;\mu,\nu) + i\tilde{s}^{(0,1)}(m,n;\mu,\nu) - i^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\}$$
$$= -4 \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} (-k) l c_{-kl} e^{i(-kx+ly)},$$
$$\frac{\partial^2}{\partial x \partial y}\{s(m,n;\mu,\nu) + i\tilde{s}^{(1,0)}(m,n;\mu,\nu) - i\tilde{s}^{(0,1)}(m,n;\mu,\nu) - i^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\}$$
$$= -4 \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} k(-l) c_{k,-l} e^{i(kx-ly)},$$
$$\frac{\partial^2}{\partial x \partial y}\{s(m,n;\mu,\nu) - i\tilde{s}^{(1,0)}(m,n;\mu,\nu) - i\tilde{s}^{(0,1)}(m,n;\mu,\nu) + i^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\}$$
$$= -4 \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} (-k)(-l) c_{-k,-l} e^{i(-kx-ly)},$$

we obtain the following representations:

$$\begin{aligned}
&\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\{s(m,n;\mu,\nu)+\mathrm{i}\tilde{s}^{(1,0)}(m,n;\mu,\nu)+\mathrm{i}\tilde{s}^{(0,1)}(m,n;\mu,\nu)+\mathrm{i}^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\}\\
&\qquad=4\mathrm{i}^{r_1+r_2}\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu}k^{r_1}l^{r_2}c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)},\\
&\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\{s(m,n;\mu,\nu)-\mathrm{i}\tilde{s}^{(1,0)}(m,n;\mu,\nu)+\mathrm{i}\tilde{s}^{(0,1)}(m,n;\mu,\nu)-\mathrm{i}^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\}\\
&\qquad=4\mathrm{i}^{r_1+r_2}\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu}(-k)^{r_1}l^{r_2}c_{-kl}\mathrm{e}^{\mathrm{i}(-kx+ly)},\\
&\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\{s(m,n;\mu,\nu)+\mathrm{i}\tilde{s}^{(1,0)}(m,n;\mu,\nu)-\mathrm{i}\tilde{s}^{(0,1)}(m,n;\mu,\nu)-\mathrm{i}^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\}\\
&\qquad=4\mathrm{i}^{r_1+r_2}\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu}k^{r_1}(-l)^{r_2}c_{k,-l}\mathrm{e}^{\mathrm{i}(kx-ly)},\\
&\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\{s(m,n;\mu,\nu)-\mathrm{i}\tilde{s}^{(1,0)}(m,n;\mu,\nu)-\mathrm{i}\tilde{s}^{(0,1)}(m,n;\mu,\nu)+\mathrm{i}^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\}\\
&\qquad=4\mathrm{i}^{r_1+r_2}\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu}(-k)^{r_1}(-l)^{r_2}c_{-k,-l}\mathrm{e}^{\mathrm{i}(-kx-ly)}.
\end{aligned}$$

Since

$$\begin{aligned}
&4^p\Bigg\|\sum_{k=m+1}^{\mu}\sum_{l=n+1}^{\nu}k^{r_1}l^{r_2}c_{kl}\mathrm{e}^{\mathrm{i}(kx+ly)}\Bigg\|_p^p\\
&\qquad=\frac{1}{4\pi^2}\iint_{\mathbb{T}^2}\Big|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\{s(m,n;\mu,\nu)+\mathrm{i}\tilde{s}^{(1,0)}(m,n;\mu,\nu)\\
&\qquad\quad+\mathrm{i}\tilde{s}^{(0,1)}(m,n;\mu,\nu)+\mathrm{i}^2\tilde{s}^{(1,1)}(m,n;\mu,\nu)\}\Big|^p\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\leqslant\frac{1}{4\pi^2}\iint_{\mathbb{T}^2}\Big|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}s(m,n;\mu,\nu)\Big|^p\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\quad+\frac{1}{4\pi^2}\iint_{\mathbb{T}^2}\Big|\mathrm{i}\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\tilde{s}^{(1,0)}(m,n;\mu,\nu)\Big|^p\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\quad+\frac{1}{4\pi^2}\iint_{\mathbb{T}^2}\Big|\mathrm{i}\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\tilde{s}^{(0,1)}(m,n;\mu,\nu)\Big|^p\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\quad+\frac{1}{4\pi^2}\iint_{\mathbb{T}^2}\Big|\mathrm{i}^2\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}\tilde{s}^{(1,1)}(m,n;\mu,\nu)|^p\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

applying Lemma 3.2 four times gives

$$\Big\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} k^{r_1} l^{r_2} c_{kl} \mathrm{e}^{\mathrm{i}(kx+ly)} \Big\|_p^p \leqslant K_{p,r_1,r_2} \mu^{r_1 p} \nu^{r_2 p} \| s(m,n;\mu,\nu) \|_p^p$$
$$= K_{p,r_1,r_2} \mu^{r_1 p} \nu^{r_2 p} \| s_{\mu\nu} - s_{m\nu} - s_{\mu n} + s_{mn} \|_p^p.$$

This means the first inequality is proved. The other three inequalities can be proved in an analogous way. □

Lemma 3.4. *For* $0 \leqslant m < \mu$, $0 \leqslant n < \nu$, $r_1, r_2 = 1, 2, \ldots$, *and* $0 < p < 1$, *we have*

$$\Big\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} k^{r_1} l^{r_2} c_{kl} \mathrm{e}^{\mathrm{i}(kx+ly)} \Big\|_p^p$$
$$\geqslant K_{p,r_1,r_2} \max \Big\{ \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} ((k-m)(l-n))^{p-2} |k^{r_1} l^{r_2} c_{kl}|^p,$$
$$\sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} ((\mu-k+1)(l-n))^{p-2} |k^{r_1} l^{r_2} c_{kl}|^p,$$
$$\sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} ((k-m)(\nu-l+1))^{p-2} |k^{r_1} l^{r_2} c_{kl}|^p,$$
$$\sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} ((\mu-k+1)(\nu-l+1))^{p-2} |k^{r_1} l^{r_2} c_{kl}|^p \Big\},$$

and three other analogous inequalities hold involving $c_{-k,l}$, $c_{k,-l}$, *and* $c_{-k,-l}$, *respectively, in place of* c_{kl}.

P r o o f. Using the idea of the third author (see [6, in the proof of Lemma 5]), we may write

$$\Big\| \sum_{k=m+1}^{\mu} \sum_{l=n+1}^{\nu} k^{r_1} l^{r_2} c_{kl} \mathrm{e}^{\mathrm{i}(kx+ly)} \Big\|_p^p = \Big\| \sum_{k_1=0}^{\mu-m-1} \sum_{l_1=0}^{\nu-n-1} k^{r_1} l^{r_2} c_{kl} \mathrm{e}^{\mathrm{i}(k_1 x+l_1 y)} \Big\|_p^p,$$

where $k_1 = k - m - 1$ and $l_1 = l - n - 1$. Applying Lemma 3.1 in the case of the above equality, we immediately obtain the inequality of the lemma which involves c_{kl}. The other three inequalities which involve $c_{-k,l}$, $c_{k,-l}$ and $c_{-k,-l}$, respectively, can be proved in a similar way. □

Now, we pass to the main result. Indeed, the following statement holds true.

Theorem 3.5. *Assume $f_2 \in L^p(\mathbb{T}^2)$, $0 < p < 1$, and*

$$\|s_{mn}(f_2) - f_2\|_p \to 0 \quad \text{as } m, n \to \infty$$

independently of one another. Then

$$\sum_{k=[m/2]}^{2m} \sum_{l=[n/2]}^{2n} \frac{|c_{kl}|^p + |c_{-k,l}|^p + |c_{k,-l}|^p + |c_{-k,-l}|^p}{((|k-m|+1)(|l-n|+1))^{2-p}} \to 0 \quad \text{as } m, n \to \infty. \tag{3.1}$$

P r o o f. Denote

$$C_{kl}(p) := |c_{kl}|^p + |c_{-k,l}|^p + |c_{k,-l}|^p + |c_{-k,-l}|^p, \quad (k,l) \in \mathbb{N}^2.$$

Let $m, n \geqslant 2$ and set $r_1 := r_2 := [1/p - 1] + 1$. Then applying Lemmas 3.3 and 3.4 with $\mu := 2m$ and $\nu := 2n$, we find that

$$\begin{aligned}
&\sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} \frac{(k^{r_1} l^{r_2})^p C_{kl}(p)}{((k-m)(l-n))^{2-p}} \\
&\quad \leqslant K_p \Bigg\{ \Bigg\| \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} k^{r_1} l^{r_2} c_{kl} \mathrm{e}^{\mathrm{i}(kx+ly)} \Bigg\|_p^p \\
&\qquad + \Bigg\| \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} (-k)^{r_1} l^{r_2} c_{-kl} \mathrm{e}^{\mathrm{i}(-kx+ly)} \Bigg\|_p^p \\
&\qquad + \Bigg\| \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} k^{r_1} (-l)^{r_2} c_{k,-l} \mathrm{e}^{\mathrm{i}(kx-ly)} \Bigg\|_p^p \\
&\qquad + \Bigg\| \sum_{k=m+1}^{2m} \sum_{l=n+1}^{2n} (-k)^{r_1} (-l)^{r_2} c_{-k,-l} \mathrm{e}^{\mathrm{i}(-kx-ly)} \Bigg\|_p^p \Bigg\} \\
&\quad \leqslant K_p m^{r_1 p} n^{r_2 p} \|s_{2m,2n}(f_2) - s_{m,2n}(f_2) - s_{2m,n}(f_2) + s_{mn}(f_2)\|_p^p.
\end{aligned}$$

Repeating the above reasoning with $[m/2] - 1$ in place of m in the lower limit of the summation and with m in place of $2m$ in the upper limit of the summation with respect to k gives

$$\begin{aligned}
&\sum_{k=[m/2]}^{m} \sum_{l=n+1}^{2n} \frac{(k^{r_1} l^{r_2})^p C_{kl}(p)}{((m-k+1)(l-n))^{2-p}} \\
&\quad \leqslant K_p m^{r_1 p} n^{r_2 p} \|s_{m,2n}(f_2) - s_{[m/2]-1,2n}(f_2) - s_{m,n}(f_2) + s_{[m/2]-1,n}(f_2)\|_p^p.
\end{aligned}$$

The symmetric counterpart of this inequality is

$$\sum_{k=m+1}^{2m}\sum_{l=[n/2]}^{n}\frac{(k^{r_1}l^{r_2})^pC_{kl}(p)}{((k-m)(n-l+1))^{2-p}} \leqslant K_pm^{r_1p}n^{r_2p}\|s_{2m,n}(f_2)-s_{m,n}(f_2)-s_{2m,[n/2]-1}(f_2)+s_{m,[n/2]-1}(f_2)\|_p^p.$$

In a similar way we obtain that

$$\sum_{k=[m/2]}^{m}\sum_{l=[n/2]}^{n}\frac{(k^{r_1}l^{r_2})^pC_{kl}(p)}{((m-k+1)(n-l+1))^{2-p}} \leqslant K_pm^{r_1p}n^{r_2p}\|s_{m,n}(f_2)-s_{[m/2]-1,n}(f_2)-s_{m,[n/2]-1}(f_2)+s_{[m/2]-1,[n/2]-1}(f_2)\|_p^p.$$

Now, it follows from the last four inequalities that

$$\begin{aligned}\sum_{k=[n/2]}^{2n}\sum_{l=[n/2]}^{2n}&\frac{C_{kl}(p)}{((|k-m|+1)(|l-n|+1))^{2-p}}\\ &\leqslant \frac{K_p}{m^{r_1p}n^{r_2p}}\Bigg\{\sum_{k=m+1}^{2m}\sum_{l=n+1}^{2n}\frac{(k^{r_1}l^{r_2})^pC_{kl}(p)}{((k-m)(l-n))^{2-p}}\\ &\quad+\sum_{k=[m/2]}^{m}\sum_{l=n+1}^{2n}\frac{(k^{r_1}l^{r_2})^pC_{kl}(p)}{((m-k+1)(l-n))^{2-p}}\\ &\quad+\sum_{k=m+1}^{2m}\sum_{l=[n/2]}^{n}\frac{(k^{r_1}l^{r_2})^pC_{kl}(p)}{((k-m)(n-l+1))^{2-p}}\\ &\quad+\sum_{k=[m/2]}^{m}\sum_{l=[n/2]}^{n}\frac{(k^{r_1}l^{r_2})^pC_{kl}(p)}{((m-k+1)(n-l+1))^{2-p}}\Bigg\}\\ &\leqslant K_p\max_{[m/2]-1\leqslant\mu_1<\mu_2\leqslant 2m}\ \max_{[n/2]-1\leqslant\nu_1<\nu_2\leqslant 2n}\|s_{\mu_2,\nu_2}(f_2)\\ &\quad-s_{\mu_1,\nu_2}(f_2)-s_{\mu_2,\nu_1}(f_2)+s_{\mu_1,\nu_1}(f_2)\|_p^p,\end{aligned}$$

and hence (3.1) is proved. □

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