Measure–valued solutions for problems in fluid mechanics

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Hierarchy of solutions

Strong solution

Strong *a priori* bounds also on derivatives

Weak (distributional) solutions

L^α, $\alpha > 1$ *a priori* bounds, compactness in L^1

Measure–valued solutions

L^α, $\alpha > 1$ *a priori* bounds

Measure–valued solutions with concentration measure

L^1 *a priori* bounds

Dissipative measure–valued (DMV) solutions

A form of energy balance, concentration defect dominated by dissipation defect. (DMV)–strong uniqueness principle

Barotropic Euler/ Navier Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = \left\{ \begin{array}{l} 0 \\ \\ \mathrm{div}_x \mathbb{S} \end{array} \right.$$

$$\mathbf{m} = \varrho \mathbf{u},\ \mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Initial conditions, periodic boundary conditions

$$\varrho(0, \cdot) = \varrho_0,\ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

$$x \in \mathcal{T}^N,\ N = (1), 2, 3$$

Pressure, pressure potential

$$p = p(\varrho),\ p'(\varrho) \geq 0,\ P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

Dissipative (weak) solutions

Energy inequality

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \boxed{\leq} E(0)$$

$$E = \int_\Omega \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \ \mathrm{d}x, \ \ P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z$$

Relative energy/entropy

Lyapunov function

$$E = \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - 0|^2 + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right] \, \mathrm{d}x$$

Coercivity of the pressure potential

$$\varrho \mapsto p(\varrho) \text{ non-decreasing } \Rightarrow \ \varrho \mapsto P(\varrho) \text{ convex}$$

Relative energy (relative entropy Dafermos [1979])

$$\mathcal{E} \left(\varrho, \mathbf{u} \ \Big| \ r, \mathbf{U} \right)$$

$$= \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right] \, \mathrm{d}x$$

Dissipative solutions

Relative energy inequality

$$\left[\mathcal{E} \left(\varrho, \mathbf{u} \ \Big| \ r, \mathbf{U} \right) \right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \int_\Omega \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R} \left(\varrho, \mathbf{u} \ \Big| \ r, \mathbf{U} \right) \ \mathrm{d}t$$

Test functions

$$r > 0, \ \ r, \mathbf{U} \text{ periodic (or other relevant b.c.)}$$

Remainder

$$\int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U}\right) \ \mathrm{d}t$$

$$\int_\Omega \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\right) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x$$

$$+ \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \ \mathrm{d}x + \int_\Omega (p(r) - p(\varrho)) \, \mathrm{div}_x \mathbf{U} \ \mathrm{d}x$$

$$+ \int_\Omega \left[(r - \varrho) \partial_t P'(r) + \nabla_x P'(r) \cdot (r \mathbf{U} - \varrho \mathbf{u}) \right] \ \mathrm{d}x$$

Weak (distributional) solutions

Navier–Stokes system

- P.L.Lions [1998] - global in time existence for $p \approx a\varrho^{\gamma}$, $\gamma \geq \frac{9}{5}(N = 3)$
- EF, A.Novotný, H.Petzeltová [2000] - extension to $\gamma > \frac{N}{2}$

Euler system

- E.Chiodaroli, EF [2014, 2015] - global in time existence of *infinitely many* weak solutions for any smooth initial data
- E.Chiodaroli, EF [2014, 2015] - global in time existence of infinitely many dissipative weak solutions for special initial data
- E.Chiodaroli, C.De Lellis, O.Kreml [2016] - global in time existence of admissible entropy solutions for Lipschitz initial data

What is a good weak solution?

Desired properties

• A weak solution exists globally in time for “any” choice of the initial state
• A weak solution can be identified as a limit of suitable approximate problems, e.g. by adding artificial viscosity
• The set of weak solutions is closed; a limit of a family of weak solutions is another weak solution
• A weak solution can be identified as a limit of a numerical scheme
• A weak solution is the most general object that enjoys the weak–strong uniqueness property

Weak strong uniqueness

A weak solution coincides with a strong (classical) solution as long as the latter exists

Measure–valued solutions

Derivatives

Partial derivatives replaced by distributional derivatives

Oscillations

A parameterized measure (Young measure)

$$\nu_{t,x} \in \mathcal{P}(F),\ \ t - \text{time},\ \ x - \text{spatial variable},\ \ F - \text{phase space}$$

$$\mathbf{U} : Q \to F,\ \ f(\mathbf{U})(t,x) \text{ replaced by expectations } \langle \nu_{t,x}; f(\mathbf{U}) \rangle$$

Concentrations

Concentration measure $\mathcal{C} \in \mathcal{M}(Q)$

Measure valued solutions

Equation of continuity

$$\int_0^T \int_{\mathcal{T}^N} \langle \nu_{t,x}; \varrho \rangle \, \partial_t \varphi + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi \, \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathcal{T}^N} \nabla_x \varphi \cdot \mathrm{d}\mathcal{C}_1$$

for all $\varphi \in C_c^\infty((0,T) \times \mathcal{T}^N)$

Momentum equation

$$\int_0^T \int_{\mathcal{T}^N} \langle \nu_{t,x}; \mathbf{m} \rangle \, \partial_t \varphi + \left\langle \nu_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \varphi + \langle \nu_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \varphi \, \mathrm{d}x\mathrm{d}t$$

$$= \int_0^T \int_{\mathcal{T}^N} \nabla_x \varphi : \mathrm{d}\mathcal{C}_2$$

for all $\varphi \in C_c^\infty((0,T) \times \mathcal{T}^N; R^N)$

Energy dissipation

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{T}^N}\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\ \mathrm{d}x \boxed{\leq} 0$$

Measure-valued energy inequality

$$\int_{\mathcal{T}^N}\left\langle \nu_{\tau,x}; \left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right)\right\rangle\ \mathrm{d}x + \boxed{\mathcal{D}(\tau)}$$
$$\leq \int_{\mathcal{T}^N}\left\langle \nu_0; \left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right)\right\rangle\ \mathrm{d}x$$

Dissipation defect - compatibility

$$\left|\mathcal{C}_1[0,\tau]\times\mathcal{T}^N\right| + \left|\mathcal{C}_2[0,\tau]\times\mathcal{T}^N\right| \leq \xi(\tau)\mathcal{D}(\tau),\ \xi\in L^1(0,T)$$

Truly measure-valued solutions

Truly measure-valued solutions for the Euler system (with E.Chiodaroli, O.Kreml, E. Wiedemann)

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded L^p weak solutions to the Euler system.

Weak (mv) - strong uniqueness

Theorem - EF, P.Gwiazda, A.Świerczewska-Gwiazda, E. Wiedemann [2015]

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

Relative energy (entropy)

Relative energy functional

$$\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right)(\tau)$$

$$= \int_{\mathcal{T}^N} \left\langle \nu_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m} - r\mathbf{U}|^2}{\varrho} + P(\varrho) - P'(r)(\varrho - r) - P(r) \right\rangle \, \mathrm{d}x$$

$$= \int_{\mathcal{T}^N} \left\langle \nu_{\tau,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \, \mathrm{d}x - \int_\Omega \langle \nu_{\tau,x}; \mathbf{m} \rangle \cdot \mathbf{U} \, \mathrm{d}x$$

$$+ \int_\Omega \frac{1}{2} \langle \nu_{\tau,x}; \varrho \rangle \, |\mathbf{U}|^2 \, \mathrm{d}x$$

$$- \int_\Omega \langle \nu_{\tau,x}; \varrho \rangle \, P'(r) \, \mathrm{d}x + \int_\Omega p(r) \, \mathrm{d}x$$

Relative energy (entropy) inequality

Relative energy inequality

$$\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right)(\tau)$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2}\frac{|\mathbf{m} - r\mathbf{U}_0|^2}{\varrho} + P(\varrho) - P'(r_0)(\varrho - r_0) - P(r_0) \right\rangle \, \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) \, \mathrm{d}t$$

Remainder

$$\mathcal{R}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right)$$

$$= - \int_0^\tau \int_\Omega \langle \nu_{t,x}, \mathbf{m} \rangle \cdot \partial_t \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_{\overline{\Omega}} \left[\left\langle \nu_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(\varrho) \rangle \, \mathrm{div}_x \mathbf{U} \right] \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; \varrho \rangle \, \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left[\left\langle \nu_{t,x}; \left(1 - \frac{\varrho}{r} \right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_{\mathcal{T}^N} \frac{1}{2} \nabla_x \left(|\mathbf{U}|^2 - P'(r) \right) \ \mathrm{d}\mathcal{C}_1 - \int_0^\tau \int_{\mathcal{T}^N} \nabla_x \mathbf{U} \ \mathrm{d}\mathcal{C}_2$$

Convergence of a numerical scheme

EF, M. Lukáčová–Medviďová [2016]

Let $\Omega \subset R^3$ be a smooth bounded domain. Let

$$1 < \gamma < 2, \ \Delta t \approx h, \ 0 < \alpha < 2(\gamma - 1).$$

Suppose that the initial data are smooth and that the compressible **Navier-Stokes system** admits a smooth solution in $[0, T]$ in the class

$$\varrho, \ \nabla_x \varrho, \ \mathbf{u}, \nabla_x \mathbf{u} \in C([0, T] \times \overline{\Omega})$$

$$\partial_t \mathbf{u} \in L^2(0, T; C(\overline{\Omega}; R^3)), \ \varrho > 0, \ \mathbf{u}|_{\partial\Omega} = 0.$$

Then the numerical solutions resulting from Karlsen-Karper FV-FE scheme converge unconditionally,

$$\varrho_h \to \varrho \text{ (strongly) in } L^\gamma((0, T) \times K)$$

$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2((0, T) \times K; R^3)$$

for any compact $K \subset \Omega$.

General strategy

Basic properties of numerical scheme

Show stability, consistency, discrete energy inequality

Measure valued solutions

Show convergence of the scheme to a

$$\boxed{\textbf{dissipative measure} - \textbf{valued solution}}$$

Weak-strong uniqueness

Use the weak-strong uniqueness principle in the class of measure-valued solutions. Strong and measure valued solutions emanating from the same initial data coincide as long as the latter exists

Singular limit problem

Scaled Euler system

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = 0$$

Incompressible (low Mach) limit - EF, Ch.Klingenberg, S.Markfelder[2017]

Convergence to the limit system

$$\mathrm{div}_x \mathbf{v} = 0, \;\; \partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

for well/ill prepared initial data.

Complete Euler system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0$$

$$\partial_t \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] + \mathrm{div}_x \left(\left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \mathbf{u} \right)$$

$$+ \mathrm{div}_x(p(\varrho, \vartheta)\mathbf{u}) = 0$$

Entropy inequality (admissibility)

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) \geq 0$$

Constitutive relations

$$p = \varrho \vartheta, \ \ e = c_v \vartheta, \ \ s = \log(\vartheta^{c_v}) - \log(\varrho)$$

A priori estimates

Energy bounds, total mass conservation

$$\int_{\mathcal{T}^N} \varrho \, \mathrm{d}x = \int_{\mathcal{T}^N} \varrho_0 \, \mathrm{d}x$$

$$\int_{\mathcal{T}^N} \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \, \mathrm{d}x = \int_{\mathcal{T}^N} \frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \, \mathrm{d}x$$

Entropy transport

$$s(\varrho, \vartheta)(\tau, x) \geq \inf s(\varrho_0, \vartheta_0)$$

L^1 estimates

$$\|\varrho\|_{L^1}, \|\varrho \mathbf{u}\|_{L^1}, \ \|\varrho |\mathbf{u}|^2\|_{L^1}, \ \|\varrho \vartheta\|_{L^1}, \ \|\varrho s\|_{L^1}, \ \|p\|_{L^1}, \|\varrho s \mathbf{u}\|_{L^1} \ \text{bounded}$$

MV solutions, I

Basic state variables

$$\text{density } \varrho, \text{ momentum } \mathbf{m}, \text{ internal energy } E = \varrho e(\varrho, \vartheta)$$

$$\nu_{t,x} \in \mathcal{P}([0,\infty) \times R^N \times [0,\infty))$$

Equation of continuity

$$\int_0^T \int_{\mathcal{T}^N} \left[\langle \nu_{t,x}; \varrho \rangle \partial_t \varphi + \langle \nu_{t,x}; \mathbf{m} \rangle \cdot \nabla_x \varphi \right] \ \mathrm{d}x \mathrm{d}t = 0$$

for any $\varphi \in C_c^\infty((0,T) \times \mathcal{T}^N)$

Momentum equation

$$\int_0^T \int_{\mathcal{T}^N} \left[\langle \nu_{t,x}; \mathbf{m} \rangle \cdot \boldsymbol{\varphi} + \left\langle \nu_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \mathrm{d}t$$

$$+ \int_0^T \int_{\mathcal{T}^N} \langle \nu_{t,x}; p(\varrho, E) \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \ \mathrm{d}x \mathrm{d}t = \int_0^T \int_{\mathcal{T}^N} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathcal{C}$$

for any $\varphi \in C_c^\infty((0,T) \times \mathcal{T}^N; R^N)$

MV solutions, II

Entropy balance

$$\int_0^T \int_{\mathcal{T}^N} \left[\langle \nu_{t,x}; \varrho Z(s) \rangle \partial_t \varphi + \langle \nu_{t,x}; Z(s)\mathbf{m} \rangle \cdot \nabla_x \varphi \right] \mathrm{d}x \mathrm{d}t = - \int_0^T \int_{\mathcal{T}^N} \varphi \mathrm{d}\mathcal{D}_1$$

$\mathcal{D}_1 \geq 0$, for any $\varphi \in C_c^\infty((0,T) \times \mathcal{T}^N)$, $\varphi \geq 0$, and any Z concave, $Z' \geq 0$, $\sup Z < \infty$

Total energy balance

$$\left[\int_\Omega \left\langle \nu_{t,x}; \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + E \right\rangle \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \mathcal{D}_2(\tau) = 0$$

Compatibility

$$\|\mathcal{C}\|_{\mathcal{M}([0,\tau)\times\Omega; R^{3\times 3})} \leq c \int_0^\tau \left[\mathcal{D}_1(t) + \mathcal{D}_2(t) \right] \mathrm{d}t$$

Relative energy

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta),$$

Relative energy

$$\mathcal{E}_Z \left(\varrho, \vartheta, \mathbf{u} \Big| r, \Theta, \mathbf{U} \right)$$

$$= \frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \varrho e(\varrho, \vartheta) - \Theta \varrho Z(s(\varrho, \vartheta)) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho} (\varrho - r) - H_\Theta(r, \Theta).$$

Weak strong uniqueness

Hypotheses

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)$$

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0,\ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0 \text{ for all } \varrho,\vartheta > 0$$

$$|p(\varrho,\vartheta)| \leq c(1 + \varrho + \varrho|s(\varrho,\vartheta)| + \varrho e(\varrho,\vartheta))$$

Conclusion [Březina, EF 2016]

Weak(MV)–strong uniqueness holds provided the initial density and temperature are strictly positive