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Cesàro bounded operators in Banach spaces

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Abstract

We study several notions of boundedness for operators. It is known that any power bounded operator is absolutely Cesàro bounded and strong Kreiss bounded (in particular, uniformly Kreiss bounded). The converses do not hold in general. In this note, we give examples of topologically mixing absolutely Cesàro bounded operators on $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, which are not power bounded, and provide examples of uniformly Kreiss bounded operators which are not absolutely Cesàro bounded. These results complement very limited number of known examples (see [25] and [3]). In [3] Aleman and Suciu ask if every uniformly Kreiss bounded operator T on a Banach spaces satisfies that $\lim_n \|\frac{T^n}{n}\| = 0$. We solve this question for Hilbert space operators and, moreover, we prove that, if T is absolutely Cesàro bounded on a Banach (Hilbert) space, then $\|T^n\| = o(n)$ ($\|T^n\| = o(n^{\frac{1}{2}})$, respectively). As a consequence, every absolutely Cesàro bounded operator on a reflexive Banach space is mean ergodic, and there exist mixing mean ergodic operators on $\ell^p(\mathbb{N})$, $1 < p < \infty$. Finally, we give new examples of weakly ergodic 3-isometries and study numerically hypercyclic m-isometries on finite or infinite dimensional Hilbert spaces. In particular, all weakly ergodic strict 3-isometries on a Hilbert space are weakly numerically hypercyclic. Adjoints of unilateral forward weighted shifts which are strict m-isometries on $\ell^2(\mathbb{N})$ are shown to be hypercyclic.

1 Introduction

Throughout this article X stands for a Banach space, the symbol $B(X)$ denotes the space of bounded linear operators defined on X, and X^* is the space of continuous linear functionals on X.

Given $T \in B(X)$, we denote the Cesàro mean by

$$M_n(T)x := \frac{1}{n+1}\sum_{k=0}^{n} T^k x$$

for all $x \in X$.

We need to recall some definitions concerning the behaviour of the sequence of Cesàro means $(M_n(T))_{n\in\mathbb{N}}$.

Definition 1.1. A linear operator T on a Banach space X is called

1. *Uniformly ergodic* if $M_n(T)$ converges uniformly.

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2. *Mean ergodic* if $M_n(T)$ converges in the strong topology of X.

3. *Weakly ergodic* if $M_n(T)$ converges in the weak topology of X.

4. *Absolutely Cesàro bounded* if there exists a constant $C > 0$ such that

$$\sup_{N \in \mathbb{N}} \frac{1}{N} \sum_{j=1}^{N} \|T^j x\| \leq C\|x\| \ ,$$

for all $x \in X$.

5. *Cesàro bounded* if the sequence $(M_n(T))_{n\in\mathbb{N}}$ is bounded.

An operator T is said *power bounded* if there is a $C > 0$ such that $\|T^n\| < C$ for all n.

The class of absolutely Cesàro bounded operators was introduced by Hou and Luo in [17].

Definition 1.2. An operator T is said

1. *Uniformly Kreiss bounded* if there is a $C > 0$ such that

$$\left\| \sum_{k=0}^{n} \lambda^{-k-1} T^k \right\| \leq \frac{C}{|\lambda| - 1} \quad \text{for all } |\lambda| > 1 \text{ and } n = 0, 1, 2, \cdots$$

2. *Strongly Kreiss bounded* if there is a $C > 0$ such that

$$\|(\lambda I - T)^{-k}\| \leq \frac{C}{(|\lambda| - 1)^k} \quad \text{for all } |\lambda| > 1 \text{ and } k = 1, 2, \cdots$$

3. Kreiss bounded if there is a $C > 0$ such that

$$\|(\lambda I - T)^{-1}\| \leq \frac{C}{|\lambda| - 1} \quad \text{for all } |\lambda| > 1.$$

Remark 1.1. 1. In [22], it is proved that an operator T is uniformly Kreiss bounded if and only if there is a C such that

$$\|M_n(\lambda T)\| \leq C \quad \text{for } |\lambda| = 1 \text{ and } n = 0, 1, 2, \cdots .$$

2. We recall that T is strongly Kreiss bounded if and only if

$$\|e^{zT}\| \leq M e^{|z|}, \text{ for all } z \in \mathbb{C}.$$

3. In [15], it is shown that every strong Kreiss bounded operator is uniformly Kreiss bounded. MacCarthy (see [25]) proved that if T is strong Kreiss bounded then $\|T^n\| \leq C n^{\frac{1}{2}}$.

4. There exist Kreiss bounded operators which are not Cesàro bounded, and conversely [29].

5. On finite-dimensional Hilbert spaces, the classes of uniformly Kreiss bounded, strong Kreiss bounded, Kreiss bounded and power bounded operators are equal.

6. Any absolutely Cesàro bounded operator is uniformly Kreiss bounded.

Let X be the space of all bounded analytic functions f on the unit disk of the complex plane such that their derivatives f' belong to the Hardy space H^1, endowed with the norm

$$\|f\| = \|f\|_\infty + \|f\|_{H^1} \ .$$

Then the multiplication operator, M_z, acting on X is Kreiss bounded but it fails to be power bounded. Moreover, this operator is not uniformly Kreiss bounded (see [27]).

Furthermore, for the Volterra operator V acting on $L^p[0,1]$, $1 \leq p \leq \infty$, we have that $I - V$ is uniformly Kreiss bounded, for $p = 2$ it is power bounded (see [22]), and it is asked if every uniformly Kreiss bounded operator on a Hilbert space is power bounded. This is related to the following question in [3, page 279] (see also, [28]):

Question 1.1. If T is a uniformly Kreiss bounded operator on a Banach space, does it follow that $\lim_n \|\frac{T^n}{n}\| = 0$?

Graphically, we show the implications between the above definitions.

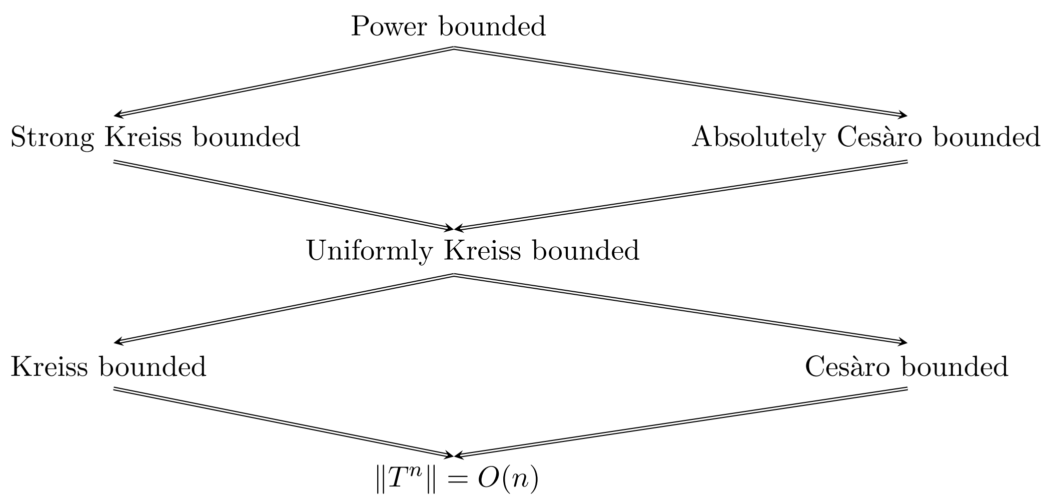

Figure 1: Implications among different definitions related with Kreiss bounded and Cesàro bounded operators in Banach spaces.

We recall some definitions that allow us to study some properties of orbits related to the behavior of the sequence $(M_n(T))_{n\in\mathbb{N}}$.

Definition 1.3. Let $T \in B(X)$. T is *topologically mixing* if for any pair U, V of non-empty open subsets of X, there exists some $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$.

Examples of absolutely Cesàro bounded mixing operators on $\ell^p(\mathbb{N})$ are given in [21], [17], [9] and [10].

Let H be a Hilbert space. For a positive integer m, an operator $T \in B(H)$ is called an *m-isometry* if for any $x \in H$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \|T^k x\|^2 = 0 \ .$$

We say that T is a *strict m-isometry* if T is an m-isometry but it is not an $(m-1)$-isometry.

Remark 1.2. 1. For $m \geq 2$, the strict m-isometries are not power bounded. Moreover, $\|T^n\| = O(n)$ for 3-isometries and $\|T^n\| = O(n^{\frac{1}{2}})$ for 2-isometries.

2. There are no strict m-isometries on finite dimensional spaces for m even. See [2, Proposition 1.23].

3. An example of weak ergodic 3-isometry is provided in [3].

The paper is organized as follows: In Section 2, we prove the optimal asymptotic behavior of $\|T^n\|$ for absolutely Cesàro bounded operators and for uniformly Kreiss bounded operators. In particular, we prove that, for any $0 < \varepsilon < \frac{1}{p}$, there exists an absolutely Cesàro bounded mixing operator T on $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, with $\|T^n\| = (n+1)^{\frac{1}{p}-\varepsilon}$. Moreover, we show that any absolutely Cesàro bounded operator on a Banach space, and any uniformly Kreiss bounded operator on a Hilbert space, satisfies that $\|T^n\| = o(n)$. For absolutely Cesàro bounded operators T on Hilbert spaces we get $\|T^n\| = o(n^{\frac{1}{2}})$. Section 3 studies ergodic properties of m-isometries on finite or infinite dimensional Hilbert spaces. For example, strict m-isometries with $m > 3$ are not Cesàro bounded, and we give new examples of weakly ergodic 3-isometries. In Section 4 we analyze numerical hypercyclicity of m-isometries. In particular, we obtain that the adjoint of any strict m-isometry unilateral forward weighted shift on $\ell^2(\mathbb{N})$ is hypercyclic. Moreover, we prove that weakly ergodic 3-isometries are weakly numerically hypercyclic.

2 Absolutely Cesàro bounded operators

It is immediate that any power bounded operator is absolutely Cesàro bounded. In general, the converse is not true.

By $e_n, n \in \mathbb{N}$, $e_n = (\delta_{n\,k})_{k\in\mathbb{N}} := (0, \ldots, 0, 1, 0, \ldots)$, we denote the standard canonical basis in $\ell^p(\mathbb{N})$ for $1 \leq p < \infty$.

The following theorem gives a variety of absolutely Cesàro bounded operators with different behavior on $\ell^p(\mathbb{N})$.

Theorem 2.1. *Let T be the unilateral weighted backward shift on $\ell^p(\mathbb{N})$ with $1 \leq p < \infty$ defined by $Te_1 := 0$ and $Te_k := w_k e_{k-1}$ for $k > 1$. If $w_k := \left(\dfrac{k}{k-1}\right)^{\alpha}$ with $0 < \alpha < \frac{1}{p}$, then T is absolutely Cesàro bounded on $\ell^p(\mathbb{N})$.*

Proof. Denote $\varepsilon := 1 - \alpha p$. Then $\varepsilon > 0$ and $\alpha = \frac{1-\varepsilon}{p}$. Fix $x \in \ell^p(\mathbb{N})$ with $||x|| = 1$ given by $x := \displaystyle\sum_{j=1}^{\infty} \alpha_j e_j$ and $N \in \mathbb{N}$. Then

$$
\begin{aligned}
\sum_{n=1}^{N}\|T^n x\|_p^p &= \sum_{n=1}^{N}\sum_{j=n+1}^{\infty}|\alpha_j|^p\Big(\frac{j}{j-n}\Big)^{1-\varepsilon} \\
&= \sum_{j=2}^{\infty}|\alpha_j|^p\, j^{1-\varepsilon}\sum_{n=1}^{\min\{N,\, j-1\}}(j-n)^{\varepsilon-1} \\
&= \sum_{j=2}^{2N}|\alpha_j|^p\, j^{1-\varepsilon}\sum_{n=1}^{\min\{N,\, j-1\}}(j-n)^{\varepsilon-1}+\sum_{j=2N+1}^{\infty}|\alpha_j|^p\sum_{n=1}^{N}\Big(\frac{j}{j-n}\Big)^{1-\varepsilon} \\
&\le \sum_{j=2}^{2N}|\alpha_j|^p\, j^{1-\varepsilon}\sum_{n=1}^{j-1}(j-1)^{\varepsilon-1}+\sum_{j=2N+1}^{\infty}|\alpha_j|^p\sum_{n=1}^{N}\Big(\frac{j}{j-n}\Big)^{1-\varepsilon}. \qquad (1)
\end{aligned}
$$

Notice that for $j > 2N$ and $n \le N$, we have that

$$
\left(\frac{j}{j-n}\right)^{1-\varepsilon} \le 2^{1-\varepsilon} < 2\,.
$$

Hence

$$
\sum_{j=2N+1}^{\infty}|\alpha_j|^p\sum_{n=1}^{N}\Big(\frac{j}{j-n}\Big)^{1-\varepsilon} < 2N\sum_{j=2N+1}^{\infty}|\alpha_j|^p \le 2N\,.
$$

We can estimate the first term of (1) in the following way:

$$
\begin{aligned}
\sum_{n=1}^{j-1}(j-n)^{\varepsilon-1} &= \sum_{n=1}^{j-1}n^{\varepsilon-1} < 1+\int_1^{j-1}t^{\varepsilon-1}dt \\
&\le \frac{(j-1)^{\varepsilon}}{\varepsilon} < \frac{j^{\varepsilon}}{\varepsilon}\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{n=1}^{N}\|T^n x\|_p^p &\le \sum_{j=2}^{2N}|\alpha_j|^p j^{1-\varepsilon}\frac{j^{\varepsilon}}{\varepsilon}+\sum_{j=2N+1}^{\infty}|\alpha_j|^p 2N \\
&= \sum_{j=2}^{2N}|\alpha_j|^p\frac{j}{\varepsilon}+2N\sum_{j=2N+1}^{\infty}|\alpha_j|^p \\
&\le \frac{2N}{\varepsilon}\sum_{j=2}^{2N}|\alpha_j|^p+2N\sum_{j=2N+1}^{\infty}|\alpha_j|^p \\
&\le 2N\left(\frac{1}{\varepsilon}+1\right)\,.
\end{aligned}
$$

By Jensen's inequality

$$
\left(\frac{1}{N}\sum_{n=1}^{N}\|T^n x\|_p\right)^p \le \frac{1}{N}\sum_{n=1}^{N}\|T^n x\|_p^p \le 2\left(\frac{1}{\varepsilon}+1\right)\,,
$$

which yields the result. □

As consequence of above theorem, we obtain

Corollary 2.1. *There exist absolutely Cesàro bounded operators which are not power bounded.*

Proof. It is an immediate consequence of Theorem 2.1. □

Corollary 2.2. *For $1 < p < 2$, there exist absolutely Cesàro bounded operators which are not strongly Kreiss bounded on $\ell^p(\mathbb{N})$.*

Proof. In view of [25, Remark 3], if T is a strong Kreiss bounded operator then $\|T^n\| \leq Cn^{\frac{1}{2}}$. The conclusion follows from part (1) of Theorem 2.1. □

Corollary 2.3. *Let $1 \leq p < \infty$ and $\varepsilon > 0$. Then there exists an absolutely Cesàro bounded operators T on ℓ^p which is mixing and $\|T^n\| = (n+1)^{\frac{(1-\varepsilon)}{p}}$ for all $n \in \mathbb{N}$.*

Proof. By part (1) of Theorem 2.1 we have that T is absolutely Cesàro bounded and

$$\|T^n\| = (n+1)^{\frac{(1-\varepsilon)}{p}} . \tag{2}$$

Moreover by [16, Theorem 4.8] we have that T is mixing if $\left(\prod_{k=1}^n w_k\right)^{-1} \to 0$ as $n \to \infty$. Indeed

$$\left(\prod_{k=1}^n w_k\right)^{-1} = \frac{1}{n^\alpha} \to 0 ,$$

hence T is mixing. □

Further consequences can be obtained for operators on Hilbert spaces.

Corollary 2.4. *There exists a uniformly Kreiss bounded Hilbert space operator that is not absolutely Cesàro bounded.*

Proof. Let H be a separable infinite-dimensional Hilbert space with an orthonormal basis $(u_k)_{k\in\mathbb{N}}$. Let $0 < \alpha < 1/2$. Let $T \in B(H)$ be defined by $Tu_k := \left(\frac{k+1}{k}\right)^\alpha u_{k+1}$. A straightforward computation gives that T is not absolutely Cesàro bounded since $\|T^n u_1\| = (n+1)^\alpha \to \infty$. Note that its adjoint T^* is given by $T^* u_k = \left(\frac{k+1}{k}\right)^\alpha u_{k-1}$ for $k > 1$ and $T^* u_1 = 0$. By Theorem 2.1, T^* is absolutely Cesàro bounded, and hence uniformly Kreiss bounded. Since the uniform Kreiss boundedness is preserved by taking the adjoints, we deduce that T is uniformly Kreiss bounded. □

It is easy to check that

$$\frac{T^n}{n+1} = M_n(T) - \frac{n}{n+1} M_{n-1}(T) . \tag{3}$$

We notice that Cesàro bounded operators satisfy that $\|T^n\| = O(n)$. Moreover, Theorem 2.1 gives an example of a uniformly Kreiss bounded operator on $\ell^1(\mathbb{N})$ such that $\|T^n\| = (n+1)^{1-\varepsilon}$ with $0 < \varepsilon < 1$.

We concentrate now on Question 1.1 for operators on Hilbert spaces.

Theorem 2.2. *Let T be a uniformly Kreiss bounded operator on a Hilbert space H. Then $\lim_{n\to\infty} n^{-1}\|T^n\| = 0$.*

Proof. Let $C > 0$ satisfy $\Big\|\sum_{j=0}^{N-1}(\lambda T)^j\Big\| \le CN$ for all $\lambda, |\lambda| = 1$ and all N. We need several claims.

Claim 1. Let $x \in H$, $\|x\| = 1$ and $N \in \mathbb{N}$. Then

$$\sum_{j=0}^{N-1} \|T^j x\|^2 \le C^2 N^2.$$

Proof. Consider the normalized Lebesgue measure on the unit circle. We have

$$C^2 N^2 \ge \int_{|\lambda|=1} \big\|(I + \lambda T + \cdots + (\lambda T)^{N-1})x\big\|^2 d\lambda$$

$$= \sum_{j,k=0}^{N-1} \int_{|\lambda|=1} \big\langle(\lambda T)^j x, (\lambda T)^k x\big\rangle d\lambda = \sum_{j=0}^{N-1} \int_{|\lambda|=1} \big\langle(\lambda T)^j x, (\lambda T)^j x\big\rangle d\lambda = \sum_{j=0}^{N-1} \|T^j x\|^2.$$

$\square$

Claim 2. Let $0 < M < N$ and $x \in H$, $\|x\| = 1$. Then

$$\sum_{j=0}^{M-1} \frac{\|T^N x\|^2}{\|T^{N-j}x\|^2} \le C^2 M^2.$$

Proof. Set $y = T^N x$. Since T^* is also uniformly Kreiss bounded, we have

$$\int_{|\lambda|=1} \big\|(I + (\bar\lambda T^*) + \cdots + (\bar\lambda T^*)^{M-1})y\big\|^2 d\lambda \le C^2 M^2 \|y\|^2.$$

On the other hand, as in Claim 1 we have

$$\int_{|\lambda|=1} \big\|(I + (\bar\lambda T^*) + \cdots + (\bar\lambda T^*)^{M-1})y\big\|^2 d\lambda = \sum_{j=0}^{M-1} \|T^{*j} y\|^2$$

$$\ge \sum_{j=0}^{M-1} \Big|\Big\langle T^{*j}y, \frac{T^{N-j}x}{\|T^{N-j}x\|}\Big\rangle\Big|^2 = \sum_{j=0}^{M-1} \Big|\Big\langle y, \frac{T^N x}{\|T^{N-j}x\|}\Big\rangle\Big|^2 \ge \|y\|^2 \sum_{j=0}^{M-1} \frac{\|T^N x\|^2}{\|T^{N-j}x\|^2}.$$

Hence

$$\sum_{j=0}^{M-1} \frac{\|T^N x\|^2}{\|T^{N-j}x\|^2} \le C^2 M^2.$$

$\square$

Claim 3. Let $x \in H$, $\|x\| = 1$ and $N \in \mathbb{N}$. Then

$$\sum_{j=0}^{N-1} \frac{1}{\|T^j x\|} \ge \frac{\sqrt{N}}{C}.$$

Proof. Let $a_j = \|T^j x\|$. By Claim 1, $\sum_{j=0}^{N-1} a_j^2 \le C^2 N^2$. So

$$\sum_{j=1}^{N-1} a_j \le \Big(\sum_{j=0}^{N-1} a_j^2\Big)^{1/2} \cdot \sqrt{N} \le C N^{3/2}.$$

Let $B = N\Big(\sum_{j=0}^{N-1} \frac{1}{a_j}\Big)^{-1}$ and $A = N^{-1}\sum_{j=0}^{N-1} a_j$ be the harmonic and arithmetic means of a_j's for $j \in \{0,\dots,N-1\}$, respectively. By the well-known inequality between these two means, we have

$$\sum_{j=0}^{N-1} \frac{1}{\|T^j x\|} = \frac{N}{B} \ge \frac{N}{A} = N^2\Big(\sum_{j=0}^{N-1} a_j\Big)^{-1} \ge \frac{N^2}{CN^{3/2}} = \frac{\sqrt{N}}{C}.$$

□

Claim 4. Let $0 < M_1 < M_2 < N$ and $\|x\| = 1$. Then

$$\sum_{j=M_1}^{M_2-1} \frac{\|T^{N-j}x\|^2}{\|T^N x\|^2} \ge \frac{(M_2 - M_1)^2}{C^2 M_2^2}.$$

Proof. Let $a_j = \frac{\|T^{N-j}x\|^2}{\|T^N x\|^2}$. By Claim 2,

$$\sum_{j=M_1}^{M_2-1} \frac{1}{a_j} \le \sum_{j=0}^{M_2-1} \frac{1}{a_j} \le C^2 M_2^2.$$

Let A and B be the arithmetic and harmonic mean of a_j's for $j \in \{M_1,\dots,M_2-1\}$, respectively. We have

$$\sum_{j=M_1}^{M_2-1} a_j = (M_2 - M_1)A \ge (M_2 - M_1)B = (M_2 - M_1)^2\Big(\sum_{j=M_1}^{M_2-1} \frac{1}{a_j}\Big)^{-1} \ge \frac{(M_2 - M_1)^2}{C^2 M_2^2}.$$

□

Proof of Theorem 2.2. Suppose on the contrary that $\limsup_{n\to\infty} n^{-1}\|T^n\| > c > 0$.

Choose $K > 8C^6 c^{-2}$. Find $N > 2^{K+1}$ with $\|T^N\| > cN$ and $x \in H$, $\|x\| = 1$ with

$$\|T^N x\| > cN.$$

For $|\lambda| = 1$ let $y_\lambda = \sum_{j=0}^{N-1} \frac{(\lambda T)^j x}{\|T^j x\|}$. Then

$$\int_{|\lambda|=1} \|y_\lambda\|^2 d\lambda = N$$

and

$$\int_{|\lambda|=1} \big\|(I + \lambda T + \cdots + (\lambda T)^{N-1})y_\lambda\big\|^2 d\lambda \le C^2 N^2 \int_{|\lambda|=1} \|y_\lambda\|^2 d\lambda = C^2 N^3.$$

On the other hand,

$$\int_{|\lambda|=1} \big\|(I + \lambda T + \cdots + (\lambda T)^{N-1})y_\lambda\big\|^2 d\lambda$$

$$= \int_{|\lambda|=1} \Big\| \sum_{j=0}^{2N-2} (\lambda T)^j x \sum_{r=0}^{\min\{N-1,j\}} \frac{1}{\|T^r x\|} \Big\|^2 d\lambda$$

$$= \sum_{j=0}^{2N-2} \|T^j x\|^2 \Big(\sum_{r=0}^{\min\{N-1,j\}} \frac{1}{\|T^r x\|} \Big)^2 \geq \sum_{j=N-2^K}^{N} \|T^j x\|^2 \Big(\sum_{r=0}^{N-2^K} \frac{1}{\|T^r x\|} \Big)^2,$$

where

$$\sum_{r=0}^{N-2^K} \frac{1}{\|T^r x\|} \geq \frac{\sqrt{N-2^K}}{C} \geq \frac{\sqrt{N}}{C\sqrt{2}}$$

and

$$\sum_{j=N-2^K}^{N} \|T^j x\|^2 \geq \|T^N x\|^2 \sum_{k=0}^{K-1} \sum_{j=N-2^{k+1}}^{N-2^k-1} \frac{\|T^j x\|^2}{\|T^N x\|^2} \geq c^2 N^2 \sum_{k=0}^{K-1} \frac{2^{2k}}{C^2 2^{2k+2}} = \frac{c^2 N^2 K}{4C^2}.$$

Hence

$$\int_{|\lambda|=1} \big\| (I + \lambda T + \cdots + (\lambda T)^{N-1}) y_\lambda \big\|^2 d\lambda \geq \frac{c^2 N^2 K}{4C^2} \cdot \frac{N}{2C^2} = \frac{c^2 K N^3}{8C^4} > C^2 N^3,$$

a contradiction. This finishes the proof.

□

Corollary 2.5. *Any uniformly Kreiss bounded operator on a Hilbert space is mean ergodic.*

We are interested on the behavior of $\frac{\|T^n\|}{n}$ when T is an absolutely Cesàro bounded operator. The following result provides an answer.

Theorem 2.3. *Let X be a Banach space, $C > 0$ and let $T \in B(X)$ satisfy $\|T^n\| \leq Cn$ for all $n \in \mathbb{N}$. Then either $\lim_{n\to\infty} n^{-1}\|T^n\| = 0$ or the set*

$$\Big\{ x \in X : \sup_N N^{-1} \sum_{n=1}^{N} \|T^n x\| = \infty \Big\}$$

is residual in X.

Proof. Suppose that $\frac{\|T^n\|}{n} \not\to 0$. So there exists $c > 0$ such that

$$\limsup_{n\to\infty} n^{-1} \|T^n\| > c.$$

For $s \in \mathbb{N}$ let

$$M_s = \Big\{ x \in X : \sup_N N^{-1} \sum_{n=1}^{N} \|T^n x\| > s \Big\}.$$

Clearly M_s is open.

We show first that each M_s contains a unit vector. Let $s \in \mathbb{N}$. Find $N > \exp\left(\frac{Cs}{c}\right) + 1$ with $\|T^N\| > cN$. Find a unit vector $x \in X$ such that $\|T^N x\| > cN$.

For $k = 1, \dots, N-1$ we have $\|T^N x\| \leq \|T^k\| \cdot \|T^{N-k} x\|$, and so

$$\|T^{N-k} x\| \geq \frac{\|T^N x\|}{\|T^k\|} \geq \frac{cN}{Ck}.$$

Thus

$$N^{-1}\sum_{k=1}^{N}\|T^j x\| \geq \sum_{k=1}^{N-1}\frac{c}{Ck} \geq \frac{c}{C}\ln(N-1) > s,$$

and so $x \in M_s$.

We show that in fact each M_s is dense. Fix $s \in \mathbb{N}$, $y \in X$ and $\varepsilon > 0$. Let $s' > \frac{s}{\varepsilon}$. Find $x \in M_{s'}$, $\|x\| = 1$. For each $j \in \mathbb{N}$ we have

$$\|T^j(y+\varepsilon x)\| + \|T^j(y-\varepsilon x)\| \geq 2\varepsilon\|T^j x\|.$$

So

$$\sup_N N^{-1}\sum_{j=1}^{N}\|T^j(y+\varepsilon x)\| + \sup_N N^{-1}\sum_{j=1}^{N}\|T^j(y-\varepsilon x)\| \geq \sup_N \frac{2\varepsilon}{N}\sum_{j=1}^{N}\|T^j x\| > 2\varepsilon s' > 2s.$$

Hence either $y+\varepsilon x \in M_s$ or $y - \varepsilon x \in M_s$. Since $\varepsilon > 0$ was arbitrary, M_s is dense.

By the Baire category theorem,

$$\bigcap_{s+1}^{\infty} M_s = \left\{x \in X : \sup_N N^{-1}\sum_{j=1}^{N}\|T^j x\| = \infty\right\}$$

is a residual set. □

Corollary 2.6. *Let $T \in B(X)$ be an absolutely Cesàro bounded operator. Then $\lim_{n\to\infty}\frac{\|T^n\|}{n} = 0$.*

Proof. There exists $C > 0$ such that

$$\|T^n x\| \leq \sum_{k=1}^{n}\|T^k x\| \leq Cn\|x\|$$

for all $x \in X$. By Theorem 2.3, we have that $\lim_{n\to\infty}\frac{\|T^n\|}{n} = 0$, since the second possibility in Theorem 2.3 contradicts to the assumption that T is absolutely Cesàro bounded. □

As consequence, we obtain

Corollary 2.7. *Any absolutely Cesàro bounded operator on a reflexive Banach space is mean ergodic.*

Hence by Corollary 2.3, we have that

Corollary 2.8. *There exist mean ergodic and mixing operators on $\ell^p(\mathbb{N})$ for $1 < p < \infty$.*

For $0 < \varepsilon < 1$, by Theorem 2.1 we have an example of absolutely Cesàro bounded operators on $\ell^2(\mathbb{N})$ such that $\|T^n\| = (n+1)^{\frac{1}{2}-\varepsilon}$. On the other hand, if there exists $\varepsilon > 0$ such that $\|T^n\| \geq Cn^{\frac{1}{2}+\varepsilon}$ for all n in a Hilbert space, then by [23, Theorem 3], there exists $x \in X$ such that $\|T^n x\| \to \infty$, thus T is not absolutely Cesàro bounded. Hence it is natural to ask: does every absolutely Cesàro bounded operator on a Hilbert space satisfy $\lim_{n\to\infty} n^{-1/2}\|T^n\| = 0$?

Theorem 2.4. *Let H be a Hilbert space and let $T \in B(H)$ be an absolutely Cesàro bounded operator. Then* $\lim_{n\to\infty} \frac{\|T^n\|}{n^{1/2}} = 0$.

Proof. Let $C > 0$ satisfy $N^{-1}\sum_{n=0}^{N-1} \|T^n x\| < C\|x\|$ for all $N \in \mathbb{N}$ and $x \in H$.

Suppose on the contrary that $\limsup_{n\to\infty} N^{-1/2}\|T^n\| > 0$. We distinguish two cases:

Case I. Suppose that $\limsup_{n\to\infty} n^{-1/2}\|T^n\| = \infty$.

Then there exist positive integers $N_1 < N_2 < \cdots$ and positive constants $K_1 < K_2 < \cdots$ with $\lim_{m\to\infty} K_m = \infty$ such that $\|T^{N_m}\| > K_m N_m^{1/2}$ and

$$\|T^j\| \le 2K_m j^{1/2} \qquad (j \le N_m).$$

Let $x_m \in H$ be a unit vector satisfying $\|T^{N_m} x_m\| > K_m N_m^{1/2}$.

Let $N'_m = \left[\frac{N_m}{6}\right]$ (the integer part). Consider the set

$$\{\|T^j x_m\| : 2N'_m \le j < 4N'_m\}.$$

Let A be the median of this set. More precisely, we have

$$\operatorname{card}\{j : 2N'_m \le j < 4N'_m, \|T^j x_m\| \ge A\} \ge N'_m \qquad \text{and}$$

$$\operatorname{card}\{j : 2N'_m \le j < 4N'_m, \|T^j x_m\| \le A\} \ge N'_m.$$

We have

$$4N'_m C \ge \sum_{j=0}^{4N'_m-1} \|T^j x_m\| \ge \sum_{j=2N'_m}^{4N'_m-1} \|T^j x_m\| \ge N'_m A.$$

So $A \le 4C$ (note that this estimate does not depend on m).

For $\lambda \in \mathbb{C}$, $|\lambda| = 1$ let

$$y_{m,\lambda} = \sum_{j=1}^{N_m} \frac{(\lambda T)^j x_m}{\|T^j x_m\|}.$$

Then

$$\int \|y_{m,\lambda}\|^2 d\lambda = \int \sum_{j,j'=1}^{N_m} \frac{\langle \lambda^j T^j x_m, \lambda^{j'} T^{j'} x_m\rangle}{\|T^j x_m\| \cdot \|T^{j'} x_m\|} d\lambda$$

$$= \int \sum_{j=1}^{N_m} \frac{\langle T^j x_m, T^j x_m\rangle}{\|T^j x_m\|^2} d\lambda = N_m.$$

Let

$$u_{m,\lambda} = (I + \lambda T + \cdots + (\lambda T)^{N_m - 1}) y_{m,\lambda}.$$

Then $\|u_{m,\lambda}\| \le C N_m \|y_{m,\lambda}\|$ and

$$\int \|u_{m,\lambda}\|^2 d\lambda \le C^2 N_m^2 \int \|y_{m,\lambda}\|^2 d\lambda = C^2 N_m^3.$$

On the other hand,

$$u_{m,\lambda} = \sum_{j=1}^{N_m} (\lambda T)^j x_m \sum_{k=1}^{j} \frac{1}{\|T^k x_m\|} + \sum_{j=N_m+1}^{2N_m-1} (\lambda T)^j x_m \sum_{k=j-N_m+1}^{N_m} \frac{1}{\|T^k x_m\|}.$$

As above,

$$\int \|u_{m,\lambda}\|^2 d\lambda \ge \sum_{j=1}^{N_m} \|T^j x_m\|^2 \Big(\sum_{k=1}^{j} \frac{1}{\|T^k x_m\|}\Big)^2 \ge \|T^{N_m} x_m\|^2 \Big(\sum_{k=2N'_m}^{4N'_m-1} \frac{1}{\|T^k x_m\|}\Big)^2$$

$$\ge K_m^2 N_m \cdot \Big(\frac{N'_m}{A}\Big)^2 \ge K_m^2 \cdot \mathrm{const} \cdot N_m^3.$$

Since $K_m \to \infty$, this is a contradiction.

Case II. Let K satisfy $0 < K < \limsup_{n\to\infty} n^{-1/2}\|T^n\| < 2K$.

Let N_0 satisfy $n^{-1/2}\|T^n\| \le 2K \quad (n \ge N_0)$. Find an increasing sequence (N_m) of positive integers such that $\|T^{N_m}\| > K N_m^{1/2}$. Find x_m, $\|x_m\| = 1$ such that $\|T^{N_m} x_m\| > K N_m^{1/2}$.

As in case I, let $N'_m = \left[\frac{N_m}{6}\right]$ and let A be the median of the set

$$\{\|T^j x_m\| : 2N'_m \le j < 4N'_m\}.$$

Again one has $A \le 4C$.

As in case I, for $|\lambda| = 1$ let

$$y_{m,\lambda} = \sum_{j=1}^{N_m} \frac{(\lambda T)^j x_m}{\|T^j x_m\|}$$

and

$$u_{m,\lambda} = (I + \lambda T + \cdots + (\lambda T)^{N_m-1}) y_{m,\lambda}.$$

Again we have $\displaystyle\int \|y_{m,\lambda}\|^2 d\lambda = N_m$ and

$$\int \|u_{m,\lambda}\|^2 d\lambda \le C^2 N_m^3.$$

On the other hand,

$$u_{m,\lambda} = \sum_{j=1}^{N_m} (\lambda T)^j x_m \sum_{k=1}^{j} \frac{1}{\|T^k x_m\|} + \sum_{j=N_m+1}^{2N_m-1} (\lambda T)^j x_m \sum_{k=j-N_m+1}^{N_m} \frac{1}{\|T^k x_m\|}$$

and

$$\int \|u_{m,\lambda}\|^2 d\lambda \ge \sum_{j=1}^{N_m} \|T^j x_m\|^2 \Big(\sum_{k=1}^{j} \frac{1}{\|T^k x_m\|}\Big)^2 \ge \sum_{j=4N'_m}^{N_m-1} \|T^j x_m\|^2 \Big(\sum_{k=2N'_m}^{4N'_m-1} \frac{1}{\|T^k x_m\|}\Big)^2$$

$$\ge \sum_{j=4N'_m}^{N_m-1} \|T^j x_m\|^2 \Big(\frac{N'_m}{A}\Big)^2.$$

Moreover, for $4N'_m \le j < N_m$ we have

$$K N_m^{1/2} < \|T^{N_m} x_m\| \le \|T^{N_m-j}\| \cdot \|T^j x_m\| \le 2K(N_m - j)^{1/2} \|T^j x_m\|.$$

So

$$\sum_{j=4N'_m}^{N_m} \|T^j x_m\|^2 \geq \sum_{j=4N'_m}^{N_m-1} \frac{N_m}{4(N_m - j)} \geq \frac{N_m}{4} \sum_{j=1}^{2N'_m} \frac{1}{j} \geq \frac{N_m \ln(2N'_m)}{4}.$$

Hence

$$\int \|u_{m,\lambda}\|^2 d\lambda \geq \text{const} \cdot N_m^3 \ln(2N'_m),$$

a contradiction. $\square$

The following picture summarizes the implications between the properties studied here and the behaviour of $\|T^n\|$.

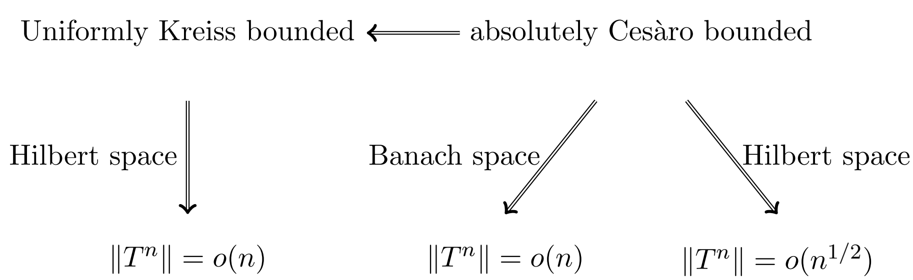

Figure 2: Behavior of $\|T^n\|$ for uniformly Kreiss and Cesàro bounded operators.

We finish this section with a couple of questions.

Question 2.1. Are there absolutely Cesàro bounded operators on Hilbert spaces which are not strongly Kreiss bounded?

Question 2.2. Are there strongly Kreiss bounded operators which are not absolutely Cesàro bounded?

3 Ergodic properties for m-isometries

The following implications for operators on reflexive Banach spaces among various concepts in ergodic theory are a direct consequence of the corresponding definitions:

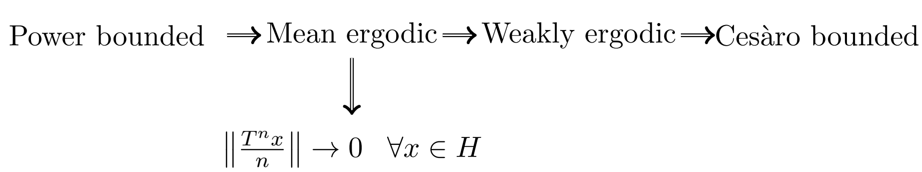

Figure 3: Behavior between different definitions in ergodic theory.

In general, the converse implications of the above figure are not true.

The purpose of this section is to study m-isometries within the framework of these definitions. It is clear that isometries (1-isometries) are power bounded. It is natural to ask about strict m-isometries and the definitions of Figure 3 on finite or infinite Hilbert spaces.

The following example is due to Assani. See [13, page 10] and [3, Theorem 5.4] for more details.

Example 3.1. Let H be $\mathbb{R}^2$ or $\mathbb{C}^2$ and $T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$. It is clear that

$$T^n = \begin{pmatrix} (-1)^n & (-1)^{n-1}2n \\ 0 & (-1)^n \end{pmatrix}$$

and $\sup_{n\in\mathbb{N}} \|M_n(T)\| < \infty$. Then T is Cesàro bounded and $\frac{\|T^n x\|}{n}$ does not converge to 0 for some $x \in H$. Hence T is not mean ergodic. Note that T is a strict 3-isometry.

The above example shows that on a 2-dimensional Hilbert space there exists a 3-isometry which is Cesàro bounded and not mean ergodic. This example could be generalized to any Hilbert space of dimension greater or equal to 2.

Let H a Hilbert space and $T \in B(H)$. Tomilov and Zemánek in [30] considered the Hilbert space $\mathcal{H} = H \oplus H$ with the norm

$$\|x_1 \oplus x_2\|_{H\oplus H} = \sqrt{\|x_1\|^2 + \|x_2\|^2}\ ,$$

and the bounded linear operator $\mathcal{T}$ on $\mathcal{H}$ given by the matrix

$$\mathcal{T} := \begin{pmatrix} T & T-I \\ 0 & T \end{pmatrix} .$$

In fact, they obtained the following relations of ergodic properties between the operators $\mathcal{T}$ and T.

Lemma 3.1. *[30, Lemmma 2.1] Let $T \in B(H)$. Then*

1. *$\mathcal{T}$ is Cesàro bounded if and only if T is power bounded.*
2. *$\mathcal{T}$ is mean ergodic if and only if T^n converges in the strong topology of H.*
3. *$\mathcal{T}$ is weakly ergodic if and only if T^n converges in the weak topology of H.*

Recall some properties of m-isometries.

Lemma 3.2. *Let $T \in B(H)$ and $m \in \mathbb{N}$. Then*

1. *[6, Theorem 2.1] T is a strict m-isometry if and only if $\|T^n x\|^2$ is a polynomial at n of degree less or equal to $m-1$ for all $x \in H$, and there exists $x_m \in H$ such that $\|T^n x_m\|^2$ is a polynomial of degree exactly $m-1$.*
2. *[7, Theorem 2.7] If H is a finite dimensional Hilbert space, then T is a strict m-isometry with odd m if and only if there exist a unitary $U \in B(H)$ and a nilpotent operator $Q \in B(H)$ of order $\frac{m+1}{2}$ such that $UQ = QU$ with $T = U + Q$.*

3. *[7, Theorem 2.2] If $A \in B(H)$ is an isometry and $Q \in B(H)$ is a nilpotent operator of order n such that commutes with A, then $A+Q$ is a strict $(2n-1)$-isometry.*

Example 3.2. Let H be a Hilbert space and $T \in B(H)$ such that $T = I+Q$ where $Q^n = 0$ for some $n \geq 2$ and $Q^{n-1} \neq 0$. Define the Hilbert space $\mathcal{H}$ and the bounded linear operator $\mathcal{T}$ on $\mathcal{H}$ as above. By construction $\mathcal{T} = \mathcal{A} + \mathcal{Q}$ where

$$\mathcal{A} := \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad \mathcal{Q} := \begin{pmatrix} Q & Q \\ 0 & Q \end{pmatrix}$$

where $\mathcal{Q}^n = 0$ and $\mathcal{Q}^{n-1} \neq 0$. By parts (3) and (1) of Lemma 3.2, T is a strict $(2n-1)$-isometry and hence not power bounded. Thus, by Lemma 3.1 we have that $\mathcal{T}$ is not Cesàro bounded. It is also simple to verify that $\mathcal{T}$ is strict $(2n-1)$- isometry by Lemma 3.2.

Example 3.3. Let λ be a unimodular complex number different from 1. Then

$$\mathcal{A} := \begin{pmatrix} \lambda & \lambda - 1 \\ 0 & \lambda \end{pmatrix}$$

is a Cesàro bounded operator (since $\sup_n |\lambda^n| < \infty$), it is not mean ergodic (since $\lambda^n x$ does not converge) and is a 3-isometry on $\mathbb{C}^2$, see Lemmas 3.1 and 3.2.

Now we give some ergodic properties of m-isometries.

Example 3.1 is a Cesàro bounded 3-isometries. However, as a consequence of Theorem 2.2 and Lemma 3.2, we obtain the following.

Corollary 3.1. *There is no uniformly Kreiss bounded strict 3-isometry.*

Theorem 3.1. *Assume that H is a finite n-dimensional Hilbert space. Then*

1. *If $n \geq 2$, then there exists a Cesáro bounded strict 3-isometry.*

2. *The isometries are the only mean ergodic strict m-isometries on H.*

Proof. *(1)* Let

$$\mathcal{A} := \begin{pmatrix} \lambda & \lambda - 1 \\ 0 & \lambda \end{pmatrix}$$

be the operator on $\mathbb{C}^2$ considered in Example 3.3. Write $H = \mathbb{C}^2 \oplus \mathbb{C}^{n-2}$ and let $\mathcal{B} := \mathcal{A} \oplus I_{\mathbb{C}^{n-2}}$. Then $\mathcal{B}$ is a strict 3-isometry which is Cesàro bounded (and not power bounded).

(2) Suppose that T is a strict m-isometry with $m > 1$ on a finite dimensional Hilbert space, then $m \geq 3$. Using part (1) of Lemma 3.2, it is easy to prove that $\frac{\|T^n x\|}{n}$ does not converges to 0 for some $x \in H$. So, T is not mean ergodic. □

In infinite dimensional Hilbert space we can say more.

Theorem 3.2. *Let T be a strict m-isometry. Then*

1. *If $m > 3$, then T is not Cesàro bounded. In particular there is no weakly ergodic strict m-isometry for $m > 3$.*

2. *If $m \geq 3$, then T is not mean ergodic.*

Proof. By part (1) of Lemma 3.2, there exists $x \in H$ such that $\|T^n x\|^2$ is a polynomial at n of order $m-1$ exactly. Thus by equation (3), the proof is complete. □

Theorem 3.3. *There exists a Cesàro bounded and weakly ergodic strict* 3*-isometry.*

Proof. Let U be the bilateral shift. Define

$$\mathcal{M} := \begin{pmatrix} U & U-I \\ 0 & U \end{pmatrix} .$$

First observe that $\mathcal{M}$ is Cesàro bounded, by part (1) of Lemma 3.1. Since $U^n \to 0$ in the weak operator topology, $\mathcal{M}$ is weakly ergodic by part (3) of Lemma 3.1. Therefore, the conclusion is derived by part (3) of Lemma 3.2. $\square$

In [3], it is given an example of a Cesàro bounded strict 3-isometry T on a Hilbert space H for which the sequence $\left(\frac{T^n}{n}\right)_{n\in\mathbb{N}}$ is bounded below for all $x \in H \setminus \{0\}$. In particular, $(M_n(T)x)_{n\in\mathbb{N}}$ diverges for each $x \in H \setminus \{0\}$, and it is weakly ergodic.

We give a characterization of this property.

Given an m-isometry T, the *covariance operator* of T is defined by

$$\Delta_T := \frac{1}{k!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} T^j .$$

Theorem 3.4. *Let T be a strict* 3*-isometry on a Hilbert space H. Then the sequence $\left(\frac{T^n x}{n}\right)_{n\in\mathbb{N}}$ is bounded below for all $x \in H \setminus \{0\}$ if and only if the covariance operator Δ_T is injective.*

Proof. If T is a strict 3-isometry and Δ_T is injective, then $\inf_n \frac{\|T^n x\|}{n} > 0$ for all $x \in H \setminus \{0\}$ (see the proof of [5, Theorem 3.4]).

If Δ_T is not injective, then there exists x such that $\langle \Delta_T x, x\rangle = 0$. By [5, Proposition 2.3], we have that $\inf_n \frac{\|T^n x\|}{n} \to \langle \Delta_T x, x\rangle = 0$, and thus the sequence $\frac{T^n x}{n}$ is not bounded below. $\square$

There exist weakly ergodic strict 3-isometries with the covariance operator Δ_T injective by [3, Section 5.2] and not injective, see the proof of Theorem 3.3.

The Uniform ergodic theorem of Lin [20, Theorem] asserts that if $\frac{\|T^n\|}{n} \to 0$, then T is uniformly ergodic if and only if the range of $I - T$ is closed. On the other hand, T is uniformly ergodic if and only if $\frac{\|T^n\|}{n} \to 0$ and 1 is a pole of the resolvent operator.

Corollary 3.2. *For $m > 1$, there is no uniform ergodic strict m-isometry on a Hilbert space.*

Proof. Since there is no mean ergodic strict m-isometry for $m \geq 3$, the result follows immediately from the fact that any strict 2-isometry T satisfies that the spectrum $\sigma(T) = \overline{\mathbb{D}}$ and, thus, 1 is not an isolated point of $\sigma(T)$. $\square$

There exists a strict 3-isometry T which is weakly ergodic (thus Cesàro bounded), but it is not mean ergodic. For 2-isometries something else can be established.

Corollary 3.3. *Let H be an infinite dimensional Hilbert space and let T be a strict 2-isometry. Then the following assertions are equivalent:*

1. *T is mean ergodic.*

2. *T is weakly ergodic.*

3. *T is Cesàro bounded.*

Proof. It is a consequence of part (1) of Lemma 3.2, since $\frac{T^n x}{n}$ converges to zero for all $x \in H$. $\square$

The following example provides a 2-isometry that is not Cesàro bounded.

Example 3.4. On $\ell^2(\mathbb{N})$ we consider the operator T given by $T(x_1, x_2, \ldots) := (x_1, x_1, x_2, x_3, \ldots)$. Then T is a 2-isometry which is not Cesàro bounded.

Proposition 3.1. *Let T be the weighted backward shift in $\ell^p(\mathbb{N})$ with $1 \leq p < \infty$ defined by $Te_1 := 0$, $Te_j := \left(\frac{j}{j-1}\right)^{1/p} e_{j-1} \quad (j > 1)$. Then T is not Cesàro bounded.*

Proof. Let $x_n := \frac{1}{n^{1/p}} \sum_{s=1}^{n} e_s$ with even n. It is clear that $\|x_n\|_p = 1$. We have

$$\begin{aligned} \Big\| \frac{1}{n} \sum_{j=0}^{n-1} T^j x_n \Big\|_p^p &= \frac{1}{n^{p+1}} \Big\| \sum_{j=0}^{n-1} \sum_{s=1}^{n} T^j e_s \Big\|_p^p = \frac{1}{n^{p+1}} \Big\| \sum_{s=1}^{n} e_s \sum_{j=s}^{n} \Big(\frac{j}{s}\Big)^{1/p} \Big\|_p^p \\ &= \frac{1}{n^{p+1}} \sum_{s=1}^{n} \Big(\sum_{j=s}^{n} \Big(\frac{j}{s}\Big)^{1/p} \Big)^p \geq \frac{1}{n^{p+1}} \sum_{s=1}^{n/2+1} \frac{1}{s} \Big(\sum_{j=n/2+1}^{n} j^{1/p} \Big)^p, \end{aligned}$$

where

$$\sum_{j=n/2+1}^{n} j^{1/p} \geq \int_{n/2}^{n} t^{1/p} dt \geq \frac{1}{p^{-1}+1} \Big(n^{1+p^{-1}} - \Big(\frac{n}{2}\Big)^{1+p^{-1}} \Big) = c n^{1+1/p}$$

with $c = \frac{p}{p+1}(1 - \frac{1}{2^{1+p^{-1}}}) > 0$. So

$$\Big\| n^{-1} \sum_{j=0}^{n-1} T^j x_n \Big\|^p \geq \frac{1}{n^{p+1}} \sum_{s=1}^{n/2} \frac{c^p n^{p+1}}{s} \geq c^p \ln \frac{n}{2} \to \infty$$

as $n \to \infty$. Hence T is not Cesàro bounded. $\square$

Corollary 3.4. *There is no Cesàro bounded weighted forward shift on $\ell^2(\mathbb{N})$, which is a strict 2-isometry.*

Proof. Assume that T is a weighted forward shift with weights $(w_n)_{n \in \mathbb{N}}$. By [1, Theorem 1] (see also [6, Remark 3.9]), if T is a strict 2-isometry, then

$$|w_n|^2 = \frac{p(n+1)}{p(n)} \ ,$$

where p is a polynomial of degree 1, that is, $p(n) := an + b$.

First, suppose that $b = 0$. Then $w_n = \sqrt{\frac{n}{n-1}}$, since $a \neq 0$. Hence $T^* e_n := \sqrt{\frac{n}{n-1}} e_{n-1}$. By Proposition 3.1, T^* is not Cesàro bounded. Since Cesàro boundedness is preserved by taking adjoints, T is not Cesàro bounded.

Now, assume that $b \neq 0$, then $w_n(c) := \sqrt{\frac{cn+1}{c(n-1)+1}}$ with $c \neq 0$. Denote $T_c e_n := w_n(c)e_{n+1}$ and the diagonal operator $Ve_n := \alpha_n e_n$, where $\alpha_n := \sqrt{\frac{c(n-1)+1}{n}}$. Then V is invertible and satisfies that $VT_1 = VT_c$. Moreover, T_1 is not Cesàro bounded, by following an argument as in Proposition 3.1. Using that Cesàro boundedness is preserved by similarities, we obtain that T_c is not Cesàro bounded. □

Corollary 3.5. *There is no absolutely Cesàro bounded strict* 2*-isometry on a Hilbert space.*

Proof. It is immediate by Theorem 2.4 and part (1) of Lemma 3.2. □

Question 3.1. Is it possible to construct a Cesàro bounded strict 2-isometry on an infinite dimensional Hilbert space?

4 Numerically hypercyclic properties of m-isometries

In this section we study numerically hypercyclic m-isometries. For simplicity we discuss only operators on Hilbert spaces.

Definition 4.1. Let H be a Hilbert space. An operator $T \in B(X)$ is called numerically hypercyclic if there exists a unit vector $x \in H$ such that the set $\{\langle T^n x, x\rangle : n \in \mathbb{N}\}$ is dense in $\mathbb{C}$.

Clearly the numerical hypercyclicity is preserved by unitary equivalence but in general not by similarity. This leads to the following definition:

Definition 4.2. Let $T \in B(X)$. We say that T is *weakly numerically hypercyclic* if T is similar to a numerically hypercyclic operator.

In [26, Proposition 1.5], Shkarin proved that $T \in B(H)$ is weakly numerically hypercyclic if and only if there exist $x, y \in H$ such that the set $\{\langle T^n x, y\rangle : n \in \mathbb{N}\}$ is dense in $\mathbb{C}$.

Faghih and Hedayatian proved in [14] that m-isometries on a Hilbert space are not weakly hypercyclic. Moreover, m-isometries on a Banach space are not 1-weakly hypercyclic [4]. However, there are isometries that are weakly supercyclic [24] (in particular cyclic). Thus the first natural question is the following: are there numerically hypercyclic m-isometries?

Theorem 4.1. *There are no weakly numerically hypercyclic m-isometries on $B(\mathbb{C}^n)$ for $n \leq 3$.*

Proof. If $n = 1$, there are not weakly numerically hypercyclic operators. Let $n = 2$. By [26, Theorem 1.13], if $T \in B(\mathbb{C}^2)$ is a weakly numerically hypercyclic operator, then there exists $\lambda \in \sigma(T)$, with $|\lambda| > 1$ and thus T is not an m-isometry. For $n = 3$, it is the same by [26, Theorem 1.14]. □

We discuss the existence of weakly numerically hypercyclic m-isometries on n-dimensional spaces for $n \geq 4$.

We say that $\lambda_1, \lambda_2 \in \mathbb{T}$ are *rationally independent* if $\lambda_1^{m_1}\lambda_2^{m_2} \neq 1$ for every non-zero pair $m = (m_1, m_2) \in \mathbb{Z}^2$, or equivalently if $\lambda_j = e^{i\theta_j}$ with $\theta_j \in \mathbb{R}$ with π, θ_1, θ_2 are linearly independent over the field $\mathbb{Q}$ of rational numbers.

If $T \in B(X)$ and there are rationally independent $\lambda_1, \lambda_2 \in \mathbb{T}$ such that $ker(T-\lambda_j I)^2 \neq ker(T-\lambda_j I)$ for $j \in \{1,2\}$, then T is weakly numerically hypercyclic [26, Theorem 1.9]. Moreover if X is a Hilbert space, then T is numerically hypercyclic [26, Proposition 1.12]. The following result gives an answer to the above question for some m-isometries.

Theorem 4.2. *There exists a numerically hypercyclic strict $(2m-1)$-isometry on $B(\mathbb{C}^n)$, with $n \geq 4$, for $2 \leq m \leq n-2$.*

Proof. Let $\ell \in \{2, 3, \ldots, n-2\}$. We will construct a numerically hypercyclic strict $(2\ell-1)$-isometry. Define D the diagonal operator with diagonal

$$(\underbrace{\lambda_1, \cdots, \lambda_1}_{\ell}, \lambda_2, \lambda_2, \underbrace{1, \cdots, 1}_{k-2\ell})$$

where λ_1 and λ_2 are rationally independent complex numbers with modulus 1 and Q by

$$\begin{aligned} Qe_i : &= e_{i-1} \text{ for } i \in \{2, 3, \cdots, \ell\} \\ Qe_{\ell+2} : &= e_{\ell+1} \text{ and} \\ Qe_i : &= 0 \text{ for } i = 1, i = \ell+1 \text{ and } i \geq \ell+3 \,. \end{aligned}$$

It is clear that $Q^\ell = 0$ and $Q^{\ell-1}e_\ell = e_1 \neq 0$. Moreover,

$$\begin{aligned} QDe_i &= DQe_i = \lambda_1 e_{i-1} \text{ for } 2 \leq i \leq \ell \\ QDe_{\ell+2} &= DQe_{\ell+2} = \lambda_2 e_{\ell+1} \\ QDe_i &= DQe_i = 0 \text{ for } i = 1, \ell+1 \text{ and } \geq i \geq \ell+3 \,. \end{aligned}$$

By part (3) of Lemma 3.2, $T := D+Q$ is a strict $(2\ell-1)$-isometry for any $\ell \in \{2, 3, \cdots, n-2\}$.

Let us prove that T satisfies that $Ker(\lambda_i - T) \neq Ker(\lambda_i - T)^2$ for $i = 1, 2$. By definition $e_2 \in Ker(\lambda_1 - T)^2 \setminus Ker(\lambda_1 - T)$ and $e_{\ell+1} \in Ker(\lambda_2 - T)^2 \setminus Ker(\lambda_2 - T)$. So by [26, Proposition 1.9], T is numerically hypercyclic. □

As a consequence of the proof of Theorem 4.2, we obtain

Corollary 4.1. *Let H be a complex Hilbert space with dimension at least 4. Then there exists a numerically hypercyclic strict 3-isometry on H.*

Theorem 4.3. *An n-dimensional Hilbert space supports no weakly numerically hypercyclic strict $(2n-3)$ or $(2n-1)$-isometries.*

Proof. Let H be a finite-dimensional Hilbert space, $\dim H = n < \infty$. Suppose on the contrary that $T \in B(H)$ is a weakly numerically hypercyclic $(2n-1)$-isometry. Since $\|T^k x\|^2$ grows polynomially for each $x \in H$ and there exists $u \in H$ such that $\|T^k u\|^2$ is a polynomial of degree $2n-2$, the Jordan form of T has only one block corresponding to an eigenvalue λ with $|\lambda| = 1$. Thus $T = \lambda I + Q$ where $Q^n = 0$. Thus

$$T^k = \sum_{j=0}^{n-1} \binom{k}{j} \lambda^{k-j} Q^j = \lambda^k \sum_{j=0}^{k} \binom{k}{j} \lambda^{-j} Q^j$$

for all $k \in \mathbb{N}$.

Let $x, y \in H$ and suppose that the set $\{\langle T^k x, y\rangle : k \in \mathbb{N}\}$ is dense in $\mathbb{C}$. We have $\langle T^k x, y\rangle = \lambda^k p(k)$ for some polynomial p of degree $\leq n-1$. If $\deg p \geq 1$ then $|\langle T^k x, y\rangle| \to \infty$ so the set $\{\langle T^k x, y\rangle : k \in \mathbb{N}\}$ is not dense in $\mathbb{C}$.

If $\deg p = 0$ then the set $\{\langle T^k x, y\rangle : k \in \mathbb{N}\}$ is bounded and again is not dense in $\mathbb{C}$. Hence T is not weakly numerically hypercyclic.

The case of $(2n-3)$-isometries can be treated similarly. If $T \in B(H)$ is a strict $(2n-3)$-isometry then the Jordan form of T has two blocks: one of dimension $n-1$ corresponding to an eigenvalue λ, $|\lambda| = 1$ and the second one-dimensional block corresponding to an eigenvalue $\mu, |\mu| = 1$. For $x, y \in H$ we have $\langle T^k x, y\rangle = \lambda^k p(k) + a\mu^k$ for some polynomial $p, \deg p \leq n-2$ and a number $a \in \mathbb{C}$. Again one can show easily that the set $\{\langle T^k x, y\rangle : k \in \mathbb{N}\}$ cannot be dense in $\mathbb{C}$. Hence there are no weakly numerically hypercyclic $(2n-3)$-isometries on H. □

Theorem 4.4. *For $m \geq 2$, there exists a numerically hypercyclic strict m-isometry on $\ell^2(\mathbb{N})$.*

Proof. For $m \geq 2$, no strict m-isometry is power bounded [12, Theorem 2]. Also by [1, Theorem 1], there exist forward weighted shifts on $\ell^2(\mathbb{N})$ that are strict m-isometries for $m \geq 2$. Now, using that if $1 < p < \infty$ and T is a forward weighted shift on $\ell^p(\mathbb{N})$, then T is numerically hypercyclic if and only if T is not power bounded ([18] & [26]), we obtain the result. □

Since both numerical hypercyclicity and m-isometricity are properties preserved by unitary equivalence, we have that

Corollary 4.2. *Let H be an infinite dimensional separable complex Hilbert space and $m \geq 2$. Then there exists a numerically hypercyclic m-isometry on H.*

Theorem 4.5. *There exists a numerically hypercyclic Cesàro bounded strict 3-isometry on $\mathbb{C}^4$.*

Proof. Let T be the operator considered in the proof of Theorem 4.2

$$T := \begin{pmatrix} \lambda_1 & \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & \lambda_2 - 1 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{T}$ are rationally independent. By the proof of Theorem 4.2, it is clear that T is numerically hypercyclic.

Since both blocks

$$\begin{pmatrix} \lambda_1 & \lambda_1 - 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda_2 & \lambda_2 - 1 \\ 0 & \lambda_2 \end{pmatrix}$$

are Cesàro bounded by Lemma 3.1, it is easy to see that T is Cesàro bounded.

□

We know that there exist examples of numerically hypercyclic 3-isometries and weakly ergodic 3-isometries. The following result goes further in this direction.

Theorem 4.6. *Any weakly ergodic strict 3-isometry on a Hilbert space is weakly numerically hypercyclic.*

Proof. If T is a weakly ergodic strict 3-isometry, then there exists x such that $\frac{T^n x}{n}$ is weakly convergent but it is not norm convergent. Indeed for a strict 3-isometry T, there exists x such that $\frac{T^n x}{n}$ does not converge to zero in norm.

Then, since $x_n = \frac{T^n x}{n}$ is weakly convergent but it is not norm convergent, by [26, Lemma 6.1] there is $y \in H$ such that $\{n\langle x_n, y\rangle : n \in \mathbb{N}\}$ is dense on $\mathbb{C}$. Hence T is weakly numerically hypercyclic.

□

In particular, the example of a weakly ergodic 3-isometry defined in [3, Section 5.2] is weak numerically hypercyclic.

Question 4.1. Do there exist numerically hypercyclic weakly ergodic 3-isometries?

Let T be an m-isometry. What can we say about dynamical properties of T^*? Some particular classes of operators allow the study of the (chaotic) dynamics of the adjoints.

Theorem 4.7. *Let S_w be a forward weighted shift strict m-isometry on $\ell^2(\mathbb{N})$. Then*

1. *S_w^* is mixing if and only if $m \geq 2$.*
2. *S_w^* is chaotic if and only if $m \geq 3$.*

Proof. By [1, Theorem 1], a unilateral weighted forward shift on a Hilbert space is an m-isometry if and only if there exists a polynomial p of degree at most $m-1$ such that for any integer $n \geq 1$, we have that $p(n) > 0$ and $|w_n|^2 = \frac{p(n+1)}{p(n)}$. Thus for $m \geq 2$, S_w^* satisfies condition ii) of (c) from [16, Theorem 4.8] and S_w^* is mixing. For $m \geq 3$, S_w^* satisfies condition ii) of c) from [16, Theorem 4.8] and S_w^* is chaotic.

□

Notice that, if S_w is a unilateral forward weighted shift and a strict m-isometry on $\ell^2(\mathbb{N})$ with $m \geq 2$, then S_w^* is hypercyclic operator.

Since on $\ell^2(\mathbb{Z})$ there exist bilateral forward weighted shifts which are strict m-isometries only for odd m, then we have

Theorem 4.8. *Let S_w be a bilateral forward weighted shift strict m-isometry on $\ell^2(\mathbb{Z})$ with $m > 1$. Then S_w^* is chaotic.*

Proof. By [1, Theorem 19 & Corollary 20], a bilateral weighted forward shift on a Hilbert space is a strict m-isometry if and only if there exists a polynomial p of degree at most $m-1$ such that for any integer n, we have $p(n) > 0$ and $|w_n|^2 = \frac{p(n+1)}{p(n)}$ and m is an odd integer. Hence, for $m \geq 3$, S_w^* satisfies condition ii) of c) from [16, Theorem 4.13]. Thus S_w^* is chaotic.

□

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