ESTIMATES OF THE PRINCIPAL EIGENVALUE OF THE p-LAPLACIAN AND THE p-BIHARMONIC OPERATOR

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Abstract. We survey recent results concerning estimates of the principal eigenvalue of the Dirichlet p-Laplacian and the Navier p-biharmonic operator on a ball of radius R in $\mathbb{R}^N$ and its asymptotics for p approaching 1 and ∞.

Let p tend to ∞. There is a critical radius R_C of the ball such that the principal eigenvalue goes to ∞ for $0 < R \leqslant R_C$ and to 0 for $R > R_C$. The critical radius is $R_C = 1$ for any $N \in \mathbb{N}$ for the p-Laplacian and $R_C = \sqrt{2N}$ in the case of the p-biharmonic operator.

When p approaches 1, the principal eigenvalue of the Dirichlet p-Laplacian is $NR^{-1} \times (1-(p-1)\log R(p-1)) + o(p-1)$ while the asymptotics for the principal eigenvalue of the Navier p-biharmonic operator reads $2N/R^2 + O(-(p-1)\log(p-1))$.

Keywords: eigenvalue problem for p-Laplacian; eigenvalue problem for p-biharmonic operator; estimates of principal eigenvalue; asymptotic analysis

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1. p-Laplacian

Let us consider the eigenvalue problem for the Dirichlet p-Laplacian

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $p > 1$ and Ω is a bounded open subset of $\mathbb{R}^N$, $N \geqslant 1$. It is well-known that the principal eigenvalue of (1.1) is

$$\lambda_1(\Omega, p) \stackrel{\text{def}}{=} \min\Big(\int_\Omega |\nabla u|^p \,\mathrm{d}x \Big/ \int_\Omega |u|^p \,\mathrm{d}x\Big) \tag{1.2}$$

where the minimum is taken over all $u \in W_0^{1,p}(\Omega)$, $u \neq 0$.

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In the one dimensional case $N = 1$ the precise formula

$$\lambda_1((-R,R),p) = \frac{1}{R^p}(p-1)\Big(\frac{\pi}{p\sin(\pi/p)}\Big)^p, \quad p > 1 \tag{1.3}$$

is known (see, e.g., [7], page 244). It implies

$$\lim_{p\to 1+} \lambda_1((-R,R),p) = \frac{1}{R}, \quad \lim_{p\to 1+} \frac{\lambda_1((-R,R),p) - 1/R}{p-1} = \infty,$$

and

$$0 < R \leqslant 1 \Rightarrow \lim_{p\to\infty} \lambda_1((-R,R),p) = \infty,$$
$$R > 1 \Rightarrow \lim_{p\to\infty} \lambda_1((-R,R),p) = 0$$

(see Figure 1).

When $N \geqslant 2$, an explicit formula for $\lambda_1(\Omega,p)$ is not known even in the case when $\Omega = B_N(0,R)$, the open ball of radius $R > 0$ and centered at the origin. Using the Cheeger constant, Kawohl and Fridman [14], Remark 5, proved the lower estimate

$$\lambda_1(B_N(0,R),p) \geqslant \Big(\frac{N}{Rp}\Big)^p, \quad p > 1 \tag{1.4}$$

which together with (1.2) implies (see [14], Corollary 6)

$$\lim_{p\to 1+} \lambda_1(B_N(0,R),p) = \frac{N}{R}.$$

A more precise asymptotics for $\lambda_1(B_N(0,R),p)$ as $p \to 1+$ follows from the estimates

$$\frac{N}{R}\Big(\frac{p'}{R}\Big)^{p-1} \leqslant \lambda_1(B_N(0,R),p) \leqslant \frac{N}{R}\Big(\frac{p'}{R}\Big)^{p-1} \frac{\Gamma(p+1+N/p')}{\Gamma(p+1)\Gamma(2+N/p')}, \quad p > 1 \tag{1.5}$$

where Γ is the Gamma function and $p' \stackrel{\text{def}}{=} p/(p-1)$. The estimate from below was proved in ([8], (8.10) on page 332) and both the estimates from below and from above in [3]. The proof of the estimate from below is based on the Picone identity [1], the estimate from above follows from (1.2) by choosing an appropriate function u.

Moreover, it is proved in [3] that the estimates (1.5) yield the asymptotics

$$\lambda_1(B_N(0,R),p) = \frac{N}{R}(1-(p-1)\log R(p-1)) + o(p-1) \quad \text{as } p \to 1+.$$

This follows from the fact that both the lower and the upper bound in (1.5) are subject to the same asymptotics.

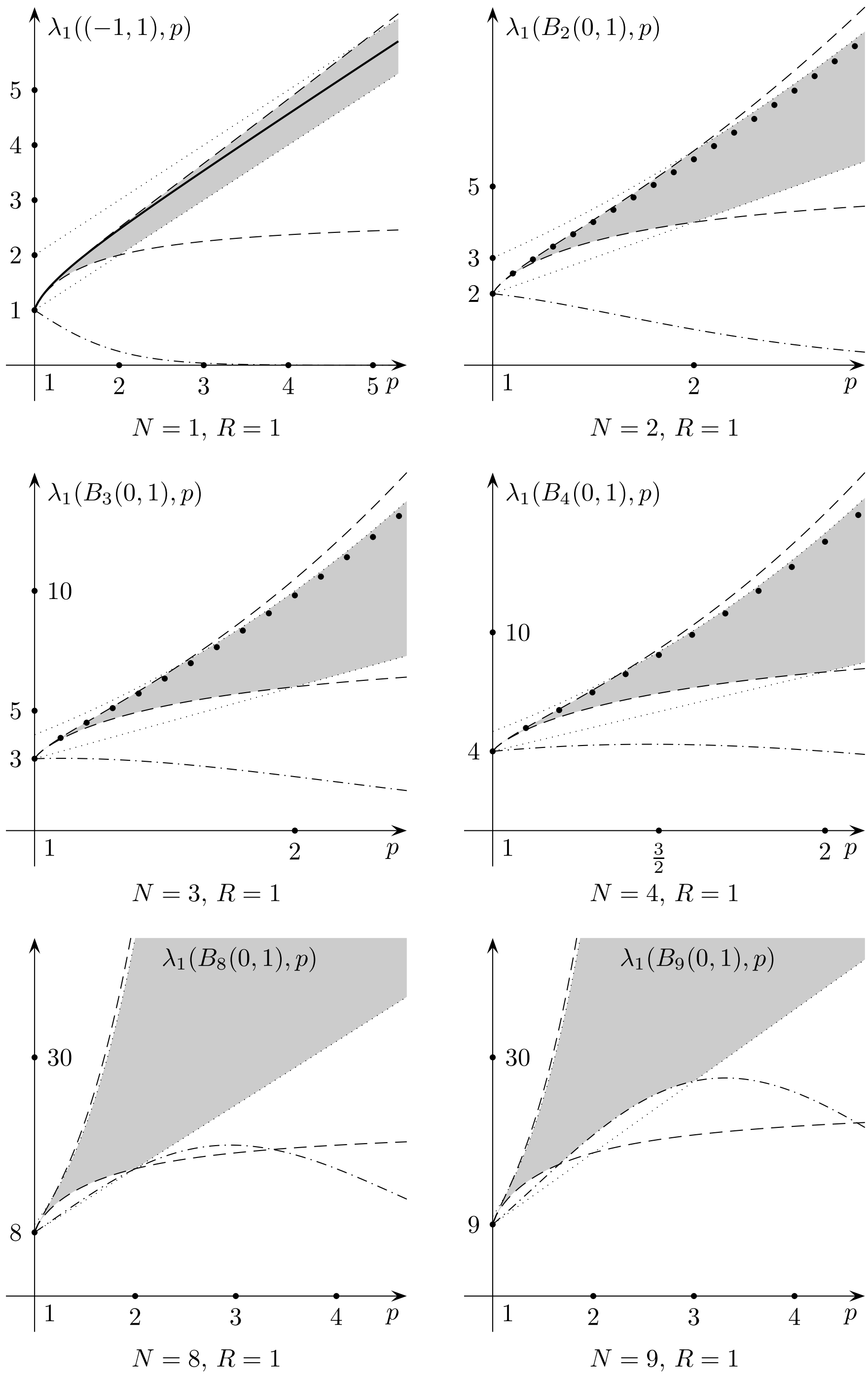

Figure 1. Dependence of λ_1 on p—second-order case.

On the other hand, it follows from [12], Lemma 1.5, that

$$0 < R < 1 \Rightarrow \lim_{p\to\infty} \lambda_1(B_N(0,R),p) = \infty,$$
$$R > 1 \Rightarrow \lim_{p\to\infty} \lambda_1(B_N(0,R),p) = 0.$$

The critical case $R = R_C \stackrel{\text{def}}{=} 1$ is not covered. In [5] we proved the estimates

$$\frac{Np}{R^p} \leqslant \lambda_1(B_N(0,R),p) \leqslant \frac{(p+1)(p+2)\dots(p+N)}{N!R^p}, \quad p > 1 \tag{1.6}$$

which imply that, similarly to the one dimension,

$$0 < R \leqslant 1 \Rightarrow \lim_{p\to\infty} \lambda_1(B_N(0,R),p) = \infty,$$
$$R > 1 \Rightarrow \lim_{p\to\infty} \lambda_1(B_N(0,R),p) = 0.$$

The estimates (1.6) can also be generalized to domains Ω other than a ball. Since the variational characterization (1.2) implies that $\lambda_1(\Omega,p)$ is decreasing with respect to Ω (in the sense of the set inclusion), the upper estimate in (1.6) applies to any bounded open subset of $\mathbb{R}^N$ that contains an inscribed ball of radius $R > 0$ as well. On the other hand, it follows from the Schwarz symmetrization (see [13]) that the lower estimate in (1.6) holds also for any Ω such that $|\Omega| = |B_N(0,R)|$. Moreover, it is proved in [5] that

$$\lambda_1(\Omega,p) \geqslant \frac{kp}{R^p}$$

for any $\Omega \subset B_k(0,R) \times \mathbb{R}^{N-k}$ where $B_k(0,R)$ is the open ball in $\mathbb{R}^k$ of radius $R > 0$ and centered at the origin, $k \in \{1,2,\dots,N\}$. In particular, for $k = 1$ and $R = 1$ it implies $\lim\limits_{p\to\infty} \lambda_1(\Omega,p) = \infty$ for any Ω situated between two parallel hyperplanes of distance 2. However, if Ω cannot be squeezed between two parallel hyperplanes of distance 2 but the radius of the largest inscribed ball has the radius $R \leqslant 1$, the asymptotic behavior of $\lambda_1(\Omega,p)$ as $p \to \infty$ is an *open problem.* A concrete example of such Ω in the plane is the open equilateral triangle with the largest inscribed disc of the radius 1.

In Figure 1 we present estimates of the principal eigenvalue $\lambda_1(B_N(0,R),p)$ in different dimensions $N = 1,2,3,4,8,9$. The solid curve for $N = 1$ depicts the exact value (1.3). For $N = 2$, 3 and 4 the thick dots represent approximate values of λ_1 for certain discrete values of p, which were evaluated in [6]. The dashed curves represent lower and upper estimates from (1.5), the dotted curves visualize those from (1.6). Finally, the dash-dotted curves illustrate the lower estimate (1.4). The shaded regions reflect all the above mentioned estimates for $\lambda_1(B_N(0,R),p)$.

The well-known continuous embedding $W_0^{1,p}(B_N(0,R)) \hookrightarrow L^p(B_N(0,R))$ and the Rellich-Kondrachov Theorem (e.g., [9], Theorem 1.2.28) imply the existence of the minimal constant $C = C(p,N,R) = \lambda_1^{-1/p}(B_N(0,R),p)$ such that for all $u \in W_0^{1,p}(B_N(0,R))$

$$\|u\|_p \leqslant C(p,N,R)\,\|u\|_{1,p}$$

where

$$\|u\|_p \stackrel{\text{def}}{=} \Big(\int_{B_N(0,R)} |u|^p \mathrm{d}x\Big)^{1/p}$$

while

$$\|u\|_{1,p} \stackrel{\text{def}}{=} \Big(\int_{B_N(0,R)} |\nabla u|^p \mathrm{d}x\Big)^{1/p}$$

is an equivalent (radially symmetric) norm on $W_0^{1,p}(B_N(0,R))$. It then follows from the estimates (1.5) and (1.6) that

$$\frac{R}{N^{1/p}(p')^{1/p'}}\Big(\frac{\Gamma(p+1)\Gamma(2+N/p')}{\Gamma(p+1+N/p')}\Big)^{1/p} \leqslant C(p,N,R) \leqslant \frac{R}{N^{1/p}(p')^{1/p'}}$$

and

$$R\Big(\frac{N!}{(p+1)(p+2)\dots(p+N)}\Big)^{1/p} \leqslant C(p,N,R) \leqslant \frac{R}{N^{1/p}p^{1/p}},$$

respectively. Consequently, for all $u \in W_0^{1,p}(B_N(0,R))$ we have

$$\|u\|_p \leqslant \frac{R}{N^{1/p}\max\{p^{1/p},(p')^{1/p'}\}}\|u\|_{1,p}.$$

2. p-biharmonic operator

We also study the Navier p-biharmonic (fourth-order) eigenvalue problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda |u|^{p-2}u & \text{in } B_N(0,R), \\ u = \Delta u = 0 & \text{on } \partial B_N(0,R) \end{cases} \tag{2.1}$$

where $p > 1$. The principal eigenvalue of (2.1) is

$$\lambda_1(B_N(0,R),p) \stackrel{\text{def}}{=} \min \frac{\int_{B_N(0,R)} |\Delta u|^p \mathrm{d}x}{\int_{B_N(0,R)} |u|^p \mathrm{d}x} \tag{2.2}$$

where the minimum is taken over all $u \in W^{2,p}(B_N(0,R)) \cap W_0^{1,p}(B_N(0,R))$, $u \neq 0$ (see [10]).

A precise formula for $\lambda_1(B_N(0,R),p)$ is not known even in one dimension. The estimates

$$\tag{2.3} \Big(\frac{2N}{R^2}\Big)^p\Big(\frac{\sqrt{\pi}\Gamma(p')}{\Gamma(p'+1/2)}-\frac{1}{p'}\Big)^{1-p} \leqslant \lambda_1(B_N(0,R),p) \leqslant \Big(\frac{2N}{R^2}\Big)^p\Big(\frac{2\Gamma(p'+1+N/2)}{N\Gamma(N/2)\Gamma(p'+1)}\Big)^{p-1}, \quad p>1$$

were proved in [2] using [4]. These estimates imply the asymptotics

$$\lambda_1(B_N(0,R),p)=\frac{2N}{R^2}+O(-(p-1)\log(p-1)) \quad \text{as } p\to 1+.$$

On the other hand, using the Picone identity for the p-biharmonic operator due to Jaroš [11] and the variational characterization (2.2), respectively, the lower and the upper estimate,

$$\tag{2.4} \Big(\frac{2N}{R^2}\Big)^p\frac{1}{\sqrt{\pi}\Gamma(p)/[\Gamma(p+1/2)]-1/p} \leqslant \lambda_1(B_N(0,R),p) \leqslant \Big(\frac{2N}{R^2}\Big)^p\frac{2\Gamma(p+1+N/2)}{N\Gamma(N/2)\Gamma(p+1)}$$

were proved in [4]. They yield that, similarly to the second-order case, there is a critical radius $R_C=\sqrt{2N}$ such that

$$0<R\leqslant R_C \Rightarrow \lim_{p\to\infty}\lambda_1(B_N(0,R),p)=\infty,$$
$$R>R_C \Rightarrow \lim_{p\to\infty}\lambda_1(B_N(0,R),p)=0.$$

However, here the critical radius does depend on the dimension.

In Figure 2 we present estimates for the principal eigenvalue in different dimensions $N=1$, 2, 3, and 4. The dashed curves represent lower and upper estimates from (2.3), the dotted curves visualize those from (2.4). The shaded regions reflect all the above mentioned estimates for λ_1.

Again, the well-known continuous embedding $W^{2,p}(B_N(0,R))\cap W_0^{1,p}(B_N(0,R))\hookrightarrow L^p(B_N(0,R))$ and the Rellich-Kondrachov Theorem imply the existence of the minimal constant $C=C(p,N,R)=\lambda_1^{-1/p}(B_N(0,R),p)$ such that for all $u\in W^{2,p}(B_N(0,R))\cap W_0^{1,p}(B_N(0,R))$

$$\|u\|_p\leqslant C(p,N,R)\,\|u\|_{2,p}$$

where

$$\|u\|_p \stackrel{\text{def}}{=} \Big(\int_{B_N(0,R)}|u|^p\,\mathrm{d}x\Big)^{1/p}$$

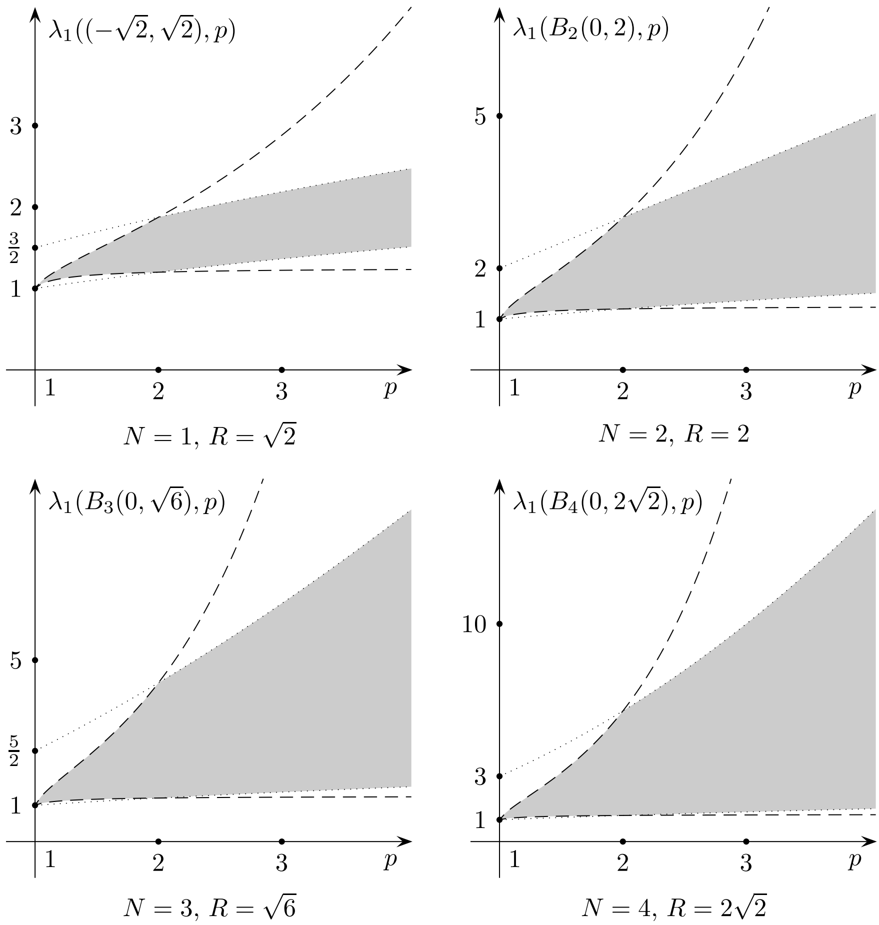

Figure 2. Dependence of λ_1 on p—fourth-order case.

and

$$\|u\|_{2,p} \stackrel{\text{def}}{=} \left(\int_{B_N(0,R)} |\Delta u|^p \mathrm{d}x\right)^{1/p}$$

is an equivalent (radially symmetric) norm on $W^{2,p}(B_N(0,R)) \cap W_0^{1,p}(B_N(0,R))$. It follows from the estimates (2.3) and (2.4) that

$$\frac{R^2}{2N}\Big(\frac{N\Gamma(N/2)\Gamma(p'+1)}{2\Gamma(p'+1+N/2)}\Big)^{1/p'} \leqslant C(p,N,R) \leqslant \frac{R^2}{2N}\Big(\frac{\sqrt{\pi}\Gamma(p')}{\Gamma(p'+1/2)} - \frac{1}{p'}\Big)^{1/p'}$$

and

$$\frac{R^2}{2N}\Big(\frac{N\Gamma(N/2)\Gamma(p+1)}{2\Gamma(p+1+N/2)}\Big)^{1/p} \leqslant C(p,N,R) \leqslant \frac{R^2}{2N}\Big(\frac{\sqrt{\pi}\Gamma(p)}{\Gamma(p+1/2)} - \frac{1}{p}\Big)^{1/p},$$

respectively. Consequently, for all $u \in W^{2,p}(B_N(0,R)) \cap W_0^{1,p}(B_N(0,R))$ we have

$$\|u\|_p \leqslant \frac{R^2}{2N} \min\Big\{\Big(\frac{\sqrt{\pi}\Gamma(p)}{\Gamma(p+1/2)} - \frac{1}{p}\Big)^{1/p}, \Big(\frac{\sqrt{\pi}\Gamma(p')}{\Gamma(p'+1/2)} - \frac{1}{p'}\Big)^{1/p'}\Big\}\|u\|_{2,p}.$$

References

[1] *W. Allegretto, Y. X. Huang*: A Picone's identity for the p-Laplacian and applications. Nonlinear Anal., Theory Methods Appl. *32* (1998), 819–830. zbl MR

[2] *J. Benedikt, P. Drábek*: Asymptotics for the principal eigenvalue of the p-biharmonic operator on the ball as p approaches 1. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *95* (2014), 735–742. zbl MR

[3] *J. Benedikt, P. Drábek*: Asymptotics for the principal eigenvalue of the p-Laplacian on the ball as p approaches 1. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *93* (2013), 23–29. zbl MR

[4] *J. Benedikt, P. Drábek*: Estimates of the principal eigenvalue of the p-biharmonic operator. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *75* (2012), 5374–5379. zbl MR

[5] *J. Benedikt, P. Drábek*: Estimates of the principal eigenvalue of the p-Laplacian. J. Math. Anal. Appl. *393* (2012), 311–315. zbl MR

[6] *R. J. Biezuner, J. Brown, G. Ercole, E. M. Martins*: Computing the first eigenpair of the p-Laplacian via inverse iteration of sublinear supersolutions. J. Sci. Comput. *52* (2012), 180–201. zbl MR

[7] *R. J. Biezuner, G. Ercole, E. M. Martins*: Computing the first eigenvalue of the p-Laplacian via the inverse power method. J. Funct. Anal. *257* (2009), 243–270. zbl MR

[8] *H. Bueno, G. Ercole, A. Zumpano*: Positive solutions for the p-Laplacian and bounds for its first eigenvalue. Adv. Nonlinear Stud. *9* (2009), 313–338. zbl MR

[9] *P. Drábek, J. Milota*: Methods of Nonlinear Analysis. Applications to Differential Equations. Birkhäuser Advanced Texts: Basel Lehrbücher, Birkhäuser, Basel, 2007. zbl MR

[10] *P. Drábek, M. Ôtani*: Global bifurcation result for the p-biharmonic operator. Electron. J. Differ. Equ. (electronic only) *2001* (2001), No. 48, 19 pages. zbl MR

[11] *J. Jaroš*: Picone's identity for the p-biharmonic operator with applications. Electron. J. Differ. Equ. (electronic only) *2011* (2011), No. 122, 6 pages. zbl MR

[12] *P. Juutinen, P. Lindqvist, J. J. Manfredi*: The ∞-eigenvalue problem. Arch. Ration. Mech. Anal. *148* (1999), 89–105. zbl MR

[13] *B. Kawohl*: Rearrangements and Convexity of Level Sets in PDE. Lecture Notes in Mathematics 1150, Springer, Berlin, 1985. zbl MR

[14] *B. Kawohl, V. Fridman*: Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant. Commentat. Math. Univ. Carol. *44* (2003), 659–667. zbl MR

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