EQUIVARIANT MAPPINGS FROM VECTOR PRODUCT INTO G-SPACE OF VECTORS AND ε-VECTORS WITH $G = O(n, 1, \mathbb{R})$

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Abstract. In this note all vectors and ε-vectors of a system of $m \leqslant n$ linearly independent contravariant vectors in the n-dimensional pseudo-Euclidean geometry of index one are determined. The problem is resolved by finding the general solution of the functional equation $F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{m}{u}) = (\det A)^{\lambda} \cdot A \cdot F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$ with $\lambda = 0$ and $\lambda = 1$, for an arbitrary pseudo-orthogonal matrix A of index one and given vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}$.

Keywords: G-space, equivariant map, pseudo-Euclidean geometry

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1. Introduction

For $n \geqslant 2$ consider the matrix $E_1 = [e_{i,j}] \in GL(n, \mathbb{R})$ where

$$e_{i,j} = \begin{cases} 0 & \text{for } i \neq j, \\ +1 & \text{for } i = j \neq n, \\ -1 & \text{for } i = j = n. \end{cases}$$

Definition 1. A pseudo-orthogonal group of index one is a subgroup of the group $GL(n, \mathbb{R})$ satisfying the condition

$$G = 0(n, 1, \mathbb{R}) = \{A\colon\ A \in GL(n, \mathbb{R}) \wedge A^T \cdot E_1 \cdot A = E_1\}.$$

Denoting $\varepsilon(A) = \operatorname{sign}(\det A) = \det A$ we have $\varepsilon(A \cdot B) = \varepsilon(A) \cdot \varepsilon(B)$.

The class of G-spaces (M_α, G, f_α), where f_α is an action of G on the space M_α, constitutes a category if we take as morphisms equivariant maps $F_{\alpha,\beta}\colon\ M_\alpha \longrightarrow M_\beta$, i.e. the maps which satisfy the condition

$$\bigwedge_{\alpha,\beta} \bigwedge_{x\in M_\alpha} \bigwedge_{A\in G} F_{\alpha,\beta}(f_\alpha(x,A)) = f_\beta(F_{\alpha,\beta}(x),A). \tag{1.1}$$

This category is called a geometry of the group G. In particular, among the objects of this category are:

the G-spaces of contravariant vectors and ε-vectors

$$(\mathbb{R}^n, G, f), \text{ where } \bigwedge_{u\in\mathbb{R}^n} \bigwedge_{A\in G} f(u,A) = \begin{cases} A\cdot u & \text{for vectors,} \\ \varepsilon(A)\cdot A\cdot u & \text{for } \varepsilon\text{-vectors,} \end{cases} \tag{1.2}$$

the G-spaces of scalars and ε-scalars

$$(\mathbb{R}, G, f), \text{ where } \bigwedge_{x\in\mathbb{R}} \bigwedge_{A\in G} f(x,A) = \begin{cases} x & \text{for scalars,} \\ \varepsilon(A)\cdot x & \text{for } \varepsilon\text{-scalars.} \end{cases} \tag{1.3}$$

For $m = 1,2,\ldots,n$ let a system of linearly independent vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}$ be given. Every equivariant mapping F of this system into G-spaces of scalars, ε-scalars, vectors, ε-vectors satisfies the equality (1.1) which, applying the transformation rules (1.2) and (1.3), may be rewritten into the form

$$\bigwedge_{A\in G} F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{m}{u}) = F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) \qquad \text{for scalars,} \tag{1.4}$$

$$\bigwedge_{A\in G} F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{m}{u}) = \varepsilon(A)\cdot F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) \qquad \text{for } \varepsilon\text{-scalars,} \tag{1.5}$$

$$\bigwedge_{A\in G} F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{m}{u}) = A\cdot F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) \qquad \text{for vectors,} \tag{1.6}$$

$$\bigwedge_{A\in G} F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{m}{u}) = \varepsilon(A)\cdot A\cdot F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) \qquad \text{for } \varepsilon\text{-vectors.} \tag{1.7}$$

For a pair u, v of contravariant vectors the mapping $p(u,v) = u^T E_1 v$ satisfies (1.4), namely

$$p(Au, Av) = (Au)^T E_1 (Av) = u^T (A^T E_1 A) v = u^T E_1 v = p(u,v).$$

In [5] it was proved that the general solution of the equation (1.4) is of the form

$$F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = \Theta(p(\underset{i}{u}, \underset{j}{u})) = \Theta(p_{ij}) \quad \text{for } i \leqslant j = 1,2,\ldots,m \leqslant n \tag{1.8}$$

where Θ is an arbitrary function of $\frac{1}{2}m(m+1)$ variables p_{ij}. The general solution of the equation (1.5) was found in [4]. Before presenting the explicit formula for it, let us denote by $L_m = L(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$ the linear subspace generated by the vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}$ and by $p|L_m$ the restriction of the form p to the subspace L_m.

Definition 2. The subspace L_m is called

(1) an Euclidean subspace if the form $p|L_m$ is positively definite,

(2) a pseudo-Euclidean subspace if the form $p|L_m$ is regular and indefinite,

(3) a singular subspace if the form $p|L_m$ is singular.

If we denote

$$P(m) = P(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}) = \begin{vmatrix} p_{11} & p_{12} & \dots & p_{1m} \\ p_{21} & p_{22} & \dots & p_{2m} \\ \dots & \dots & \dots & \dots \\ p_{m1} & p_{m2} & \dots & p_{mm} \end{vmatrix} = \det[p(\underset{i}{u}, \underset{j}{u})]_1^m = \det[p_{ij}]_1^m$$

then the above three cases are equivalent to $P(m) > 0, P(m) < 0$ and $P(m) = 0$, respectively. Let $\overset{m}{P}_{ij}$ denote the cofactor of the element p_{ij} of the matrix $[p_{ij}]_1^m$ and let $\overset{1}{P}_{11} = 1, P(0) = 1$ by definition.

Let us consider an isotropic cone $K_0 = \{u\colon u \in \mathbb{R}^n \wedge p(u,u) = 0 \wedge u \neq 0\}$. It is an invariant and transitive subset. Every isotropic vector $v \in K_0$ determines an isotropic direction which, by virtue of $v^n \neq 0$ and $v = v^n[\frac{v^1}{v^n}, \frac{v^2}{v^n}, \dots, \frac{v^{n-1}}{v^n}, 1]^T = v^n[q^1, q^2, \dots, q^{n-1}, 1]^T$ with $\sum\limits_{i=1}^{n-1} (q^i)^2 = 1 = q^n$, is equivalent to the point q belonging to the sphere S^{n-2}.

In two cases we get particular solutions of the equation (1.5). In the case $m = n$ that equation is fulfilled by the mapping det. For $A \in G$ we have

$$W' = \det(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{n}{u}) = \varepsilon(A) \cdot \det(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}) = \varepsilon(A) \cdot W.$$

If $m = n-1$ and $P(n-1) = 0$ then the singular subspace $L(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n-1}{u})$ determines exactly one isotropic direction $q \in S^{n-2}$ whose representative, if $P(n-2) \neq 0$, is of the form

$$v = \frac{1}{2P(n-2)} \sum_{i=1}^{n-1} \overset{n-1}{P}_{n-1,i} \cdot \underset{i}{u} = v^n[q^1, q^2, \dots, q^{n-1}, 1]^T \in K_0 \cap L_{n-1}. \tag{1.9}$$

From $p(\underset{i}{u}, v) = 0$ for $i = 1, 2, \dots, n-1$ it follows that each vector $\underset{i}{u}$ is of the form

$$\underset{i}{u} = \left[\underset{i}{u}^1, \underset{i}{u}^2, \dots, \underset{i}{u}^{n-1}, \sum_{k=1}^{n-1} \underset{i}{u}^k q^k\right]^T \quad \text{where } \Delta = \det[\underset{i}{u}^j]_1^{n-1} \neq 0. \tag{1.10}$$

The two 1-forms $\det(\underset{1}{u}, \dots, \underset{r-1}{u}, v, \underset{r+1}{u}, \dots, \underset{n-1}{u}, x)$ and $p(v,x)$ vanish on the subspace $L(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n-1}{u})$, and consequently there exist uniquely determined numbers $B_r =$

$B_r(\underset{1}{u},\underset{2}{u},\dots,\underset{r}{u},\dots,\underset{n-1}{u})$ such that

$$\det(\underset{1}{u},\dots,\underset{r-1}{u},v,\underset{r+1}{u},\dots,\underset{n-1}{u},x) = -B_r(\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u})\cdot p(v,x). \tag{1.11}$$

As det is an ε-scalar, p is a scalar as well, so it follows from (1.11) that each B_r is an ε-scalar. Taking any given $A \in G$ we have

$$B_r^{/} = B_r(A\underset{1}{u},\dots,A\underset{r}{u},\dots,A\underset{n-1}{u}) = \varepsilon(A)\cdot B_r(\underset{1}{u},\dots,\underset{r}{u},\dots,\underset{n-1}{u}) = \varepsilon(A)\cdot B_r.$$

From (1.9), (1.10) and (1.11) we get in terms of coordinates the formula

$$B_r(\underset{1}{u},\dots,\underset{r}{u},\dots,\underset{n-1}{u}) = \begin{vmatrix} \underset{1}{u}^1 & \dots & \underset{1}{u}^{n-1} \\ \dots & \dots & \dots \\ \underset{r-1}{u}^{1} & \dots & \underset{r-1}{u}^{n-1} \\ q^1 & \dots & q^{n-1} \\ \underset{r+1}{u}^{1} & \dots & \underset{r+1}{u}^{n-1} \\ \dots & \dots & \dots \\ \underset{n-1}{u}^{1} & \dots & \underset{n-1}{u}^{n-1} \end{vmatrix} \qquad \text{for } r = 1,2,\dots,n-1. \tag{1.12}$$

We have $B_r \cdot B_k = \overset{n-1}{P}{}_{rk}$ and in particular $B_r^2 = P(\underset{1}{u},\dots,\underset{r-1}{u},\underset{r+1}{u},\dots,\underset{n-1}{u})$, so at least one of the ε-scalars B_r is different from zero.

In [4] it was proved that the general solution of the equation (1.5) is of the form

$$F(\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u}) = \begin{cases} 0 & \text{if } m < n-1, \\ 0 & \text{if } m = n-1,\ P(m) \neq 0, \\ \sum\limits_{k=1}^{n-1} \Theta^k(p_{ij})\cdot B_k & \text{if } m = n-1,\ P(m) = 0, \\ \Theta(p_{ij})\cdot \det(\underset{1}{u},\underset{2}{u},\dots,\underset{n}{u}) & \text{if } m = n \end{cases} \tag{1.13}$$

where Θ, $\Theta^1,\dots,\Theta^{n-1}$ are arbitrary functions of $\frac{1}{2}m(m+1)$ variables.

In this work we find the general solution of the functional equations (1.6) and (1.7).

Definition 3. Two vectors $u \neq 0$ and $v \neq 0$ satisfying the condition $p(u,v)=0$ are called orthogonal and write $u \perp v$.

Definition 4. We say that a vector u is
(1) a versor, if $p(u,u)=+1$,
(2) a pseudo-versor, if $p(u,u)=-1$.

Definition 5. We say that a system of vectors $\underset{1}{e}, \underset{2}{e}, \ldots, \underset{n}{e}$ constitutes a pseudo-orthonormal base if $[p(\underset{i}{e}, \underset{j}{e})]_1^n = E_1$.

Let a sequence of linearly independent vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}, \ldots, \underset{n}{u}$ be given. This sequence generates a sequence of linear subspaces $L_1 = L(\underset{1}{u})$, $L_2 = L(\underset{1}{u}, \underset{2}{u})$, …, $L_s = L(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}), \ldots, L_n$. Let us denote $\varepsilon_s = \operatorname{sign} P(s)$. Apparently $\varepsilon_n = -1$ and from the definition $\varepsilon_0 = +1$.

Definition 6. The sequence $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_s, \ldots, \varepsilon_n) = (+1, \varepsilon_1, \ldots, \varepsilon_s, \ldots, \varepsilon_{n-1}, -1)$ will be called the signature of the sequence of subspaces $L_1, L_2, \ldots, L_s, \ldots, L_n$, or the signature of the sequence of vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}, \ldots, \underset{n}{u}$.

In [5] it was proved that the only restriction is $\varepsilon_i \geqslant \varepsilon_{i+1}$ and that any given system of n linearly independent vectors can be arranged in the sequence $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}, \ldots, \underset{n}{u}$ with the signature either
(1) $\varepsilon_0 = \ldots = \varepsilon_{s-1} = +1$, $\varepsilon_s = \ldots = \varepsilon_n = -1$ for $s \in \{1, 2, \ldots, n\}$ or
(2) $\varepsilon_0 = \ldots = \varepsilon_{s-1} = +1$, $\varepsilon_s = 0$, $\varepsilon_{s+1}, = \ldots = \varepsilon_n = -1$ for $s \in \{1, 2, \ldots, n-1\}$.

In both these cases we construct a pseudo-orthonormal base $\underset{1}{e}, \ldots, \underset{s-1}{e}, \underset{n}{e}, \underset{s+1}{e}, \ldots, \underset{n-1}{e}, \underset{s}{e}$. In the former case the vectors

$$\underset{k}{e} = \frac{\sum\limits_{i=1}^{k} \overset{k}{P}_{ki} \cdot \underset{i}{u}}{\sqrt{|P(k-1)P(k)|}} \qquad \text{for } k = 1, 2, \ldots, n \tag{2.1}$$

form a pseudo-orthonormal base such that

$$\underset{k}{e} = \underset{k}{e}(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}) \qquad \text{and} \qquad p(\underset{k}{e}, \underset{r}{u}) = \begin{cases} 0 & \text{for } r < k, \\ \Theta(p_{ij}) & \text{for } r \geqslant k. \end{cases} \tag{2.2}$$

In the latter case we determine vectors $\underset{1}{e}, \ldots, \underset{s-1}{e}, \underset{s+2}{e}, \ldots, \underset{n}{e}$ constituting a pseudo-orthonormal base using (2.1). Since $P(s) = 0$ we have

$$(\overset{s+1}{P}_{s+1,s})^2 = -P(s-1)P(s+1) \neq 0.$$

There exists only one isotropic direction, determined by the vector

$$(2.3) \qquad v = \frac{1}{2P(s-1)} \sum_{i=1}^{s} \overset{s}{P}_{si} \cdot \underset{i}{u} \perp \underset{1}{u}, \underset{2}{u}, \dots, \underset{s-1}{u}, \underset{s}{u},$$

in the singular space $L(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u})$. In the pseudo-Euclidean space $L(\underset{1}{u}, \dots, \underset{s}{u}, \underset{s+1}{u})$ there exists one more isotropic direction, which is orthogonal to $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s-1}{u}$, determined by the vector

$$(2.4) \qquad \underset{1}{v} = \frac{1}{2\overset{s+1}{P}_{s+1,s} P(s+1)} \sum_{i=1}^{s+1} (2\overset{s+1}{P}_{s+1,s} \cdot \overset{s+1}{P}_{si} - \overset{s+1}{P}_{ss} \cdot \overset{s+1}{P}_{s+1,i}) \cdot \underset{i}{u}.$$

We have $p(\underset{1}{v}, \underset{s}{u}) = 1$ contrary to $p(v, \underset{s}{u}) = 0$. The vectors

$$(2.5) \qquad \underset{s}{e} = \underset{1}{v} - v \qquad \text{and} \qquad \underset{s+1}{e} = \underset{1}{v} + v$$

complement the pseudo-orthonormal base. This base fulfils conditions (2.2) with only two exceptions,

$$(2.6) \qquad \underset{s}{e} = \underset{s}{e}(\underset{1}{u}, \dots, \underset{s}{u}, \underset{s+1}{u}) \qquad \text{and} \qquad p(\underset{s+1}{e}, \underset{s}{u}) = 1.$$

To each vector $\underset{i}{e}$ of the pseudo-orthonormal base we assign the covector $\underset{i}{\overset{*}{e}} = \underset{i}{e}^T \cdot E_1$ and then

$$p(\underset{i}{e}, \underset{r}{u}) = \underset{i}{e}^T E_1 \underset{r}{u} = \underset{i}{\overset{*}{e}} \cdot \underset{r}{u}.$$

Definition 7. We say that a pseudo-orthogonal matrix A whose successive rows consist of successive coordinates of covectors $\underset{1}{\overset{*}{e}}, \dots, \underset{s-1}{\overset{*}{e}}, \underset{n}{\overset{*}{e}}, \underset{s+1}{\overset{*}{e}}, \dots, \underset{n-1}{\overset{*}{e}}, \underset{s}{\overset{*}{e}}$ corresponds to the pseudo-orthonormal base $\underset{1}{e}, \dots, \underset{s-1}{e}, \underset{n}{e}, \underset{s+1}{e}, \dots, \underset{n-1}{e}, \underset{s}{e}$, or corresponds to the sequence of vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$.

The matrix $A = A(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u})$ allows us to solve functional equations (1.6) and (1.7).

3. Solution of the equation $F(A\underset{1}{u},\dots,A\underset{m}{u}) = A\cdot F(\underset{1}{u},\dots,\underset{m}{u})$

We arrange a given system of $1 \leqslant m \leqslant n$ linearly independent vectors into a sequence $\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u}$ whose signature up to ε_m must be in one of the forms

1. $(+1,\dots,+1)$ for $m \in \{1,2,\dots,n-1\}$
2. $(+1,\dots,+1,-1,\dots,-1)$ for $m \in \{1,2,\dots,n\}$
3. $(+1,\dots,+1,0,-1,\dots,-1)$ for $m \in \{1,2,\dots,n\}$
4. $(+1,\dots,+1,0)$ for $m \in \{1,2,\dots,n-1\}$.

We solve the equation (1.6) in the first three cases. We construct the vectors $\underset{1}{e},\underset{2}{e},\dots,\underset{m}{e}$ of a pseudo-orthonormal base using formulas (2.1) or (2.1) and (2.5). The other vectors of the base $\underset{m+1}{e},\dots,\underset{n}{e}$, if there is lack of them, are built in the orthogonal complement $L^{\perp}(\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u})$. To simplify the following argument we consider only the first case. Inserting the matrix $\underset{0}{A}$, which corresponds to the base $\underset{1}{e},\underset{2}{e},\dots,\underset{n}{e}$ and then the matrix $\underset{m+1}{A}$, which corresponds to the base $\underset{1}{e},\dots,\underset{m}{e},-\underset{m+1}{e},\underset{m+2}{e},\dots,\underset{n}{e}$ into equation (1.6) we get

$$
\begin{aligned}
F(\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u}) &= \underset{0}{A}^{-1}F(\underset{0}{A}\underset{1}{u},\underset{0}{A}\underset{2}{u},\dots,\underset{0}{A}\underset{m}{u}) = (E_1\underset{0}{A}^{T}E_1)F(\underset{0}{A}\underset{1}{u},\underset{0}{A}\underset{2}{u},\dots,\underset{0}{A}\underset{m}{u})\\
&= E_1\underset{0}{A}^{T}F_0(p_{ij}) = E_1\underset{m+1}{A}^{T}F_0(p_{ij}).
\end{aligned}
\tag{3.1}
$$

The constant vector F_0 is the same in both cases and from the last equation we conclude that its $(m+1)$ component is zero. Moreover, it is obvious that $F_0^{m+1} = F_0^{m+2} = \ldots = F_0^{n} = 0$. We get further from (3.1) that

$$
F(\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u}) = E_1\underset{0}{A}^{T}F_0(p_{ij}) = \sum_{k=1}^{n} F_0^k\cdot\underset{k}{e} = \sum_{k=1}^{m} F_0^k\cdot\underset{k}{e} = \sum_{k=1}^{m}\Theta^k(p_{ij})\cdot\underset{k}{u},
\tag{3.2}
$$

where $\Theta^1,\Theta^2,\dots,\Theta^m$ are arbitrary functions of $\frac{1}{2}m(m+1)$ variables. The same result we get in the cases 2 and 3.

Let us consider the case 4. Now $P(m-1) > 0$ and $P(m) = 0$. In the singular subspace L_m there lies its only isotropic direction $q = [v]$, where the vector v is given by the formula (2.3) for $s = m$. The subspace $L^{\perp}_{m-1}$ is a pseudo-Euclidean space of dimension $n-m+1$. If $n-m+1 = 2$ or equivalently $m = n-1$ then there exists in $L^{\perp}_{m-1}$ exactly one isotropic direction $[\underset{1}{v}] = q_1 \neq q$ such that $p(\underset{1}{v},\underset{m}{u}) = 1$. If $m < n-1$ we find at least two such directions q_1 and q_2 represented by linearly independent vectors $\underset{1}{v}$ and $\underset{2}{v}$. Since

$$
P(\underset{1}{u},\dots,\underset{m-1}{u},\underset{m}{u},\underset{1}{v}) = -P(\underset{1}{u},\dots,\underset{m-1}{u}) < 0
$$

we get the vectors $\underset{1}{e},\dots,\underset{m-1}{e}$ of a pseudo-orthonormal base using formulas (2.1), the vectors $\underset{m}{e},\underset{m+1}{e}$ we get using formulas (2.5) and the vectors $\underset{m+2}{e},\dots,\underset{n}{e}$ we find in the orthogonal complement $L^{\perp}(\underset{1}{u},\dots,\underset{m-1}{u},\underset{m}{u},\underset{1}{v})$. Let C_0 denote the pseudo-orthogonal matrix which corresponds to this base. We get similarly to (3.1) and (3.2)

$$
\begin{aligned}
(3.3)\qquad F(\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u}) &= E_1\underset{0}{C}^T F_0(p_{ij}) = \sum_{k=1}^{n} F_0^k\cdot\underset{k}{e}\\
&= \sum_{k=1}^{m+1} F_0^k\cdot\underset{k}{e} = \sum_{k=1}^{m}\Theta^k(p_{ij})\cdot\underset{k}{u} + \Theta(p_{ij})\cdot\underset{1}{v}.
\end{aligned}
$$

Now, if $m<n-1$ we have at the same time

$$
(3.4)\qquad F(\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u}) = \sum_{k=1}^{m}\Theta^k(p_{ij})\cdot\underset{k}{u} + \Theta(p_{ij})\cdot\underset{2}{v}.
$$

In this case we have $\Theta(p_{ij})\equiv 0$ and analogously to the previous cases we get $F=\sum\limits_{k=1}^{m}\Theta^k\cdot\underset{k}{u}$.

If $m=n-1$ then the direction of the vector $\underset{1}{v}$ is determined unambiguously. As $P(n-2)>0$ we conclude that $L^{\perp}(\underset{1}{u},\underset{2}{u},\dots,\underset{n-2}{u})$ is a two dimensional pseudo-Euclidean space with exactly two isotropic directions $q=[v]$ and $q_1=[\underset{1}{v}]$, where $\underset{1}{v}\notin L(\underset{2}{u},\underset{2}{u},\dots,\underset{n-1}{u})$ contrary to $v\in L_{n-1}$.

Let a sequence $\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u}$ of linearly independent vectors with $P(n-2)>0$ and $P(n-1)=0$ be given. Let Δ^i for $i=1,2,\dots,n-1$ denote the cofactors of the elements $\underset{n-1}{u}^{i}$ of the determinant $\Delta(\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u})$ and let by definition $\Delta^n=0$. Let us denote $2D=\sum\limits_{i=1}^{n-1}(\Delta^i)^2$ and $B=B_{n-1}$, where B_r is defined by formula (1.12). $B\neq 0$ because of $B^2=P(n-2)$. Taking these facts into account we have

Theorem 1. *Let the mapping η assign $\eta=\eta(\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u})\in\mathbb{R}^n$ to the sequence $\underset{1}{u},\underset{2}{u},\dots,\underset{n-2}{u},\underset{n-1}{u}$, such that $P(n-2)\neq 0$ and $P(n-1)=0$, by the formula*

$$
(3.5)\qquad \eta^i=\frac{1}{\Delta\cdot B}(B\Delta^i-Dq^i)\qquad \text{for } i=1,2,\dots,n.
$$

Then the equation

$$
(3.6)\qquad \eta(A\underset{1}{u},A\underset{2}{u},\dots,A\underset{n-1}{u}) = A\cdot\eta(\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u})
$$

holds for an arbitrary matrix $A\in G$.

P r o o f. The mapping η is the only solution of the system of n equations

$$\begin{cases} p(\eta, \underset{i}{u}) = 0 \quad \text{for } i = 1, 2, \ldots, n-2, \\ p(\eta, \underset{n-1}{u}) = 1, \\ p(\eta, \eta) = 0. \end{cases}$$

As the right hand sides are scalars so η is a vector, so it fulfils (3.6). The vector η is linearly independent of $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n-1}{u}$ because

$$\det(\underset{1}{u}, \ldots, \underset{n-1}{u}, \eta(\underset{1}{u}, \ldots, \underset{n-1}{u})) = -B(\underset{1}{u}, \ldots, \underset{n-1}{u}) \neq 0.$$

□

The vector $\underset{1}{v}$ from (3.3) and η must be collinear. We have proved

Theorem 2. *Every solution of the functional equation*

$$F(A\underset{1}{u}, A\underset{2}{u}, \ldots, A\underset{m}{u}) = A \cdot F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$$

for given vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}$ and any matrix $A \in G$ is of the form

$$\text{(3.7)} \quad F(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = \begin{cases} \sum\limits_{k=1}^{m} \Theta^k \cdot \underset{k}{u} & \textit{for } m \neq n-1 \textit{ or } m = n-1, P(n-1) \neq 0, \\ \Theta \cdot \eta + \sum\limits_{k=1}^{n-1} \Theta^k \cdot \underset{k}{u} & \textit{for } m = n-1,\ P(n-1) = 0,\ P(n-2) \neq 0 \end{cases}$$

where $\Theta, \Theta^1, \ldots, \Theta^{n-1}$ are arbitrary functions of $\frac{1}{2}m(m+1)$ variables p_{ij}.

4. Solution of the equation $F(A\underset{1}{u}, \ldots, A\underset{m}{u}) = \varepsilon(A) \cdot A \cdot F(\underset{1}{u}, \ldots, \underset{m}{u})$

If $m = n$ then according to (1.13) and (3.7) the general solution of the above equation is of the form

$$F = \det(\underset{1}{u}, \ldots, \underset{n}{u}) \left(\sum_{k=1}^{n} \Theta^k \cdot \underset{k}{u} \right).$$

If $m < n$ and $P(m) \neq 0$ then at least one of the vectors of the required pseudo-orthogonal base, let us say $\underset{r}{e}$, lies in the orthogonal complement $L^{\perp}(\underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$. Let the matrix $\underset{+}{A}$ corresponds to a base which includes $\underset{r}{e}$ while the matrix $\underset{-}{A}$ corresponds to the same base in which $\underset{r}{e}$ is replaced by $-\underset{r}{e}$. We have

$$\begin{aligned}(4.1)\qquad F(\underset{1}{u},\dots,\underset{m}{u}) &= \varepsilon(\underset{+}{A})E_1\underset{+}{A}^T F_0 = \varepsilon(\underset{+}{A})\sum_{k=1}^{n} F_0^k\cdot\underset{k}{e}\\ &= \varepsilon(\underset{+}{A})\Big(F_0^r\cdot\underset{r}{e}+\sum_{k\neq r}F_0^k\cdot\underset{k}{e}\Big) = \varepsilon(\underset{-}{A})\Big(-F_0^r\cdot\underset{r}{e}+\sum_{k\neq r}F_0^k\cdot\underset{k}{e}\Big).\end{aligned}$$

In this case the required ε-vector F must have the direction of the vector $\underset{r}{e}$. It is obvious that if $\underset{r}{e}$ is not uniquely determined by the vectors $\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u}$, then the equation (1.7) has only the trivial solution $F\equiv 0$. It is so for $m<n-1$.

Let $m = n-1$. The equivalent of the well-known cross product in Euclidean geometry, the ε-vector $\omega(\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u})$ given by the conditions

$$(4.2)\qquad \begin{cases} p(\underset{i}{u},\omega(\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u}))=0 & \text{for } i=1,2,\dots,n-1,\\ \det(\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u},\omega) = -p(\omega,\omega) = P(n-1) & \end{cases}$$

has the direction of the orthogonal complement if $P(n-1)\neq 0$. Then using (4.2) we obtain for $A\in G$

$$\omega(A\underset{1}{u},A\underset{2}{u},\dots,A\underset{n-1}{u}) = \varepsilon(A)\cdot A\cdot\omega(\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u})$$

and in accordance with (4.1) we get $F=\Theta\cdot\omega$. In the case $P(n-1)=0$ we have a decomposition $\omega=\sum_{r=1}^{n-1}B_r\cdot\underset{r}{u}$ and $L^{\perp}(\underset{1}{u},\dots,\underset{n-1}{u})$ is not the orthogonal complement. Starting from linearly independent vectors $\underset{1}{u},\underset{2}{u},\dots,\underset{n-1}{u},\eta(\underset{1}{u},\dots,\underset{n-1}{u})$, whose signature is $(+1,\dots,+1,0,-1)$, we define $\underset{1}{e},\underset{2}{e},\dots,\underset{n-2}{e}$ by formulas (2.1) and additionally by $\underset{n-1}{e}=\eta+v$ and $\underset{n}{e}=\eta-v$. The matrix D corresponding to this base has the determinant $B/\sqrt{P(n-2)}$. Inserting D into equation (1.7) we get

$$\begin{aligned}F(\underset{1}{u},\dots,\underset{n-1}{u}) &= \varepsilon(D)\cdot E_1\cdot D^T\cdot F_0 = \varepsilon(D)\sum_{k=1}^{n}F_0^k\cdot\underset{k}{e}\\ &= \frac{B}{\sqrt{P(n-2)}}\Big(\sum_{k=1}^{n-2}F_0^k\cdot\underset{k}{e}+F_0^{n-1}(\eta+v)+F_0^n(\eta-v)\Big)\\ &= B\Big(\Theta\cdot\eta+\sum_{k=1}^{n-1}\Theta^k\cdot\underset{k}{u}\Big).\end{aligned}$$

Theorem 3. *The general solution of the functional equation*

$$F(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{m}{u}) = \varepsilon(A) \cdot A \cdot F(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u})$$

for given vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}$ *and an arbitrary matrix* $A \in G$ *is of the form*

$$F(\underset{1}{u}, \dots, \underset{m}{u}) = \begin{cases} 0 & \text{for } m < n-1, \\ \Theta \cdot \omega(\underset{1}{u}, \dots, \underset{n-1}{u}) & \text{for } m = n-1,\ P(n-1) \neq 0, \\ B \cdot \Big(\Theta \cdot \eta + \sum\limits_{k=1}^{n-1} \Theta^k \cdot \underset{k}{u}\Big) & \text{for } m = n-1,\ P(m) = 0,\ P(n-2) \neq 0, \\ \det(\underset{1}{u}, \dots, \underset{n}{u}) \sum\limits_{k=1}^{n} \Theta^k \cdot \underset{k}{u} & \text{for } m = n, \end{cases}$$

where Θ, $\Theta^1, \Theta^2, \dots, \Theta^n$ *are arbitrary functions of* $\frac{1}{2}m(m+1)$ *variables* p_{ij}.

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