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Abstract

The equation $\Delta u + \lambda u + g(\lambda, u)u = 0$ is considered in a bounded domain in $\mathbb{R}^2$ with a Signorini condition on a straight part of the boundary and with mixed boundary conditions on the rest of the boundary. It is assumed that $g(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$, λ is a bifurcation parameter. A given eigenvalue of the linearized equation with the same boundary conditions is considered. A smooth local bifurcation branch of non-trivial solutions emanating at λ_0 from trivial solutions is studied. We show that to know a direction of the bifurcating branch it is sufficient to determine the sign of a simple expression envolving the corresponding eigenfunction u_0. In the case when λ_0 is the first eigenvalue and the branch goes to the right, we show that the bifurcating solutions are asymptotically stable in $W^{1,2}$-norm. The stability of the trivial solution is also studied and an exchange of stability is obtained.

Keywords: Signorini problem; variational inequality; bifurcation direction; stability of bifurcating solutions; exchange of stability.

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1 Introduction

This paper concerns the questions of a direction of bifurcation branches and of stability of solutions to Signorini boundary value problems of the type

$$\Delta u + \lambda u + g(\lambda, u)u = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_D, \quad \partial_\nu u = 0 \quad \text{on } \Gamma_N, \tag{1.2}$$

$$u \le 0, \quad \partial_\nu u \le 0, \quad u\partial_\nu u = 0 \quad \text{on } \Gamma_U. \tag{1.3}$$

Here Ω is a bounded domain in $\mathbb{R}^2$, Γ_D, Γ_N, Γ_U are parts of its boundary, Γ_U being a flat segment (see Section 2) and ∂_ν denotes the outer normal derivative. It will be always assumed that $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-smooth function such that

$$g(\lambda, 0) = 0 \quad \text{for all } \lambda \in \mathbb{R}. \tag{1.4}$$

We suppose that there is given an eigenvalue $\lambda_0 > 0$ and a corresponding eigenfunction u_0 to the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega \tag{1.5}$$

with the nonlinear boundary conditions (1.2), (1.3).

All solutions are understood in the weak sense as solutions of a variational inequality on the cone $K := \{u \in W^{1,2}(\Omega) : u \le 0 \text{ on } \Gamma_U,\ u = 0 \text{ on } \Gamma_D\}$. See Section 2, which summarizes basic assumptions and notation used. Main results are given in Sections 3 and 4 (Theorems 3.1, 4.2 and 4.12).

In Section 3 we suppose that there is a smooth branch $\lambda = \hat{\lambda}(s)$, $u = \hat{u}(s)$ (parametrized by $s \in [0, s_0)$ with $s_0 > 0$) of solutions to (1.1), (1.2), (1.3) with $\hat{\lambda}(0) = \lambda_0$, $\hat{u}(0) = 0$ and $\hat{u}'(0) = u_0$, see Assumption (ESB). In [6] we have proved the existence of such smooth branches for a particular case when Ω is a rectangle and under certain "activity conditions" on the eigenfunction u_0. Such smooth branches exist also in more general situations, but it is complicated to formulate and verify sufficient conditions for their existence. However, we do not need such assumptions for our study of bifurcation direction, and therefore we simply assume that a smooth branch exists. Roughly speaking, we show that it is sufficient to verify the inequalities (3.12) or (3.13) in order to know if $\hat{\lambda}'(0) > 0$ or $\hat{\lambda}'(0) < 0$ (Theorem 3.1).

Section 4 concerns the question of stability of solutions to (1.1), (1.2), (1.3) as stationary solutions to the corresponding evolution problem

$$\partial_t u = \Delta u + \lambda u + g(\lambda, u)u \tag{1.6}$$

with the Signorini boundary conditions (1.2), (1.3) (where ∂_t denotes the partial derivative with respect to time t). We consider only the smallest eigenvalue λ_0 of (1.5), (1.2), (1.3). In this case $u_0 < 0$ on Γ_U, therefore λ_0 is simultaneously the smallest eigenvalue of the problem (1.5) with the classical boundary conditions

$$u = 0 \text{ on } \Gamma_D, \quad \partial_\nu u = 0 \text{ on } \Gamma_N \cup \Gamma_U. \tag{1.7}$$

Due to Crandall-Rabinowitz bifurcation theorem there exists a smooth local branch of nontrivial solutions to (1.1), (1.7) emanating at λ_0 from trivial solutions. It consists of two half-branches bifurcating in the direction u_0 and $-u_0$. The half-branch bifurcating in the direction u_0 is simultaneously a branch of solutions to the Signorini boundary value problem (1.1), (1.2), (1.3). The well-known principle of exchange of stability (see, e.g. [3, Theorem 1.16], [13, Section II.8] and [15, Section I.7]) yields that if this half-branch goes to the right from λ_0 then it consists of solutions which are stable as stationary solutions to (1.6) with the classical boundary conditions (1.7). In general, stability in $W^{1,2}(\Omega)$ of a stationary solution u_* to the classical problem (1.6), (1.7) does not imply stability in $W^{1,2}(\Omega)$ of u_* as a stationary solution of the unilateral problem to (1.6), (1.2), (1.3). Indeed, if $u(t)$ is a time-dependent solution of the problem (1.6), (1.7) with the initial condition $u(0) \in K$ (in particular $u(0) \in K$ arbitrarily close to u_*) then it can happen, in general, that $u(t) \notin K$ for arbitrarily small times $t > 0$. Therefore the solution of the unilateral problem (1.6), (1.2), (1.3) with the same initial condition (which must satisfy $u(t) \in K$ for all t) can differ from that of the classical problem (1.6), (1.7) (even for initial conditions close to u_*). However, we show by using the stability criterion for variational inequalities [20] that in our particular situation, the $W^{1,2}(\Omega)$-stability of $\hat{u}(s)$ as a stationary solution to (1.6), (1.7) implies $W^{1,2}(\Omega)$-stability of $\hat{u}(s)$ as a stationary solution to (1.6), (1.2), (1.3). In particular, we obtain an exchange of stability for Signorini problem. To our best knowledge, up to now no analoga of the principle of exchange of stability for variational inequalities are known, with the exception of some special cases (for example obstacle problems with finitely many obstacles, see [4, 5]).

We do not know any example of a $W^{1,2}(\Omega)$-stable stationary solution u to the evolutionary Signorini problem (1.6), (1.2), (1.3) which does not simultaneously satisfy the classical boundary conditions $u = 0$ on Γ_U or $\partial_\nu u = 0$ on Γ_U. In particular, we do not know any result of the type of exchange of stability in the case when a bifurcation branch of nontrivial solutions to the Signorini problem (1.1), (1.2), (1.3) is not simultaneously a branch of solutions to the corresponding classical boundary value problem (1.1), (1.7). Let us remark that in [4] we have shown an example of a supercritical bifurcation for a variational inequality when the bifurcating non-trivial solutions are stable although they are bifurcating not from the first eigenvalue but from a higher eigenvalue – a certain surprising non-standard case of exchange of stability.

2 Basic Assumptions and Notation

We will consider a bounded domain Ω in $\mathbb{R}^2$ with a boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_U}$, where Γ_D, Γ_N, Γ_U are pairwise disjoint relatively open subsets of $\partial\Omega$, $\Gamma_D \neq \emptyset$, $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ is finite,

$$\Gamma_U = \{(x,0) \,:\, x \in (\gamma_1, \gamma_2)\}$$

with some $\gamma_1 < \gamma_2$. We will assume that

$$\text{there is } \mu_0 > 0 \text{ such that } \Gamma_{N,\mu_0} := \{(x,0) \,:\, x \in (\gamma_1 - \mu_0, \gamma_1) \cup (\gamma_2, \gamma_2 + \mu_0)\} \subset \Gamma_N. \tag{2.1}$$

In particular, Γ_U and its μ_0-neighbourhood in $\partial\Omega$ are supposed to be flat. We introduce a real Hilbert space H with the scalar product $\langle \cdot, \cdot \rangle$, defined by

$$H := \{u \in W^{1,2}(\Omega) \,:\, u = 0 \text{ on } \Gamma_D\}, \quad \langle u, \varphi \rangle := \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}y \text{ for } u, \varphi \in H,$$

and with the corresponding norm $\| \cdot \|$ which is equivalent on our space H to the usual Sobolev norm.

We will assume that the function g is such that

$$\text{the map } (\lambda, u) \mapsto g(\lambda, u) \text{ is } C^1\text{-smooth from } \mathbb{R} \times W^{1,2}(\Omega) \text{ into } L^r(\Omega) \text{ for some } r > 1. \tag{2.2}$$

For example, the assumption (2.2) is satisfied with any $r > 1$ if g is λ-independent and C^1-smooth and if there exist positive constants c, p and q such that

$$|g'(u) - g'(v)| \leq c|u - v|^p (1 + |u|^q + |v|^q)$$

for all $u, v \in \mathbb{R}$ (cf. [1, Proposition I.1.4]). In particular, (2.2) is true if g is a polynomial in u.

Let us consider the closed convex cone

$$K := \{u \in H \,:\, u \leq 0 \text{ on } \Gamma_U\}$$

and introduce the weak formulation of (1.1)–(1.3) and (1.5), (1.2), (1.3) in terms of the variational inequalities

$$u \in K : \int_\Omega (\nabla u \cdot \nabla(\varphi - u) - (\lambda u + g(\lambda, u)u)(\varphi - u)) \, \mathrm{d}x \, \mathrm{d}y \geq 0 \text{ for all } \varphi \in K \tag{2.3}$$

and

$$u \in K : \int_\Omega (\nabla u \cdot \nabla(\varphi - u) - \lambda u(\varphi - u)) \, \mathrm{d}x \, \mathrm{d}y \geq 0 \text{ for all } \varphi \in K, \tag{2.4}$$

respectively.

3 Direction of the bifurcation branch

Let us define the parts of Γ_U

$$I_{\alpha,\beta} := \{(x,0) \in \Gamma_U : \alpha < x < \beta\} = (\alpha,\beta) \times \{0\},$$
$$E_{\alpha,\beta} := \{(x,0) \in \Gamma_U : \gamma_1 < x < \alpha \text{ or } \beta < x < \gamma_2\} = \Gamma_U \setminus \overline{I_{\alpha,\beta}},$$

where α and β are parameters with $\gamma_1 < \alpha < \beta < \gamma_2$. For a continuous function $u \in H$ we will denote by

$$\mathcal{A}(u) := \{x \in (\gamma_1, \gamma_2) \, : \, u(x,0) = 0\}$$

the contact set of the function u. We assume that there is a given solution (λ_0, u_0) to (2.4). Moreover, we assume that the contact set of u_0 is a closed subinterval of (γ_1, γ_2), i.e.

$$\mathcal{A}(u_0) = [\alpha_0, \beta_0] \text{ with } \gamma_1 < \alpha_0 < \beta_0 < \gamma_2. \tag{3.1}$$

This implies

$$u_0 \in H_0 \, : \, \int_\Omega (\nabla u_0 \cdot \nabla \varphi - \lambda_0 u_0 \varphi) \, \mathrm{d}x \, \mathrm{d}y = 0 \quad \text{ for all } \varphi \in H_0, \tag{3.2}$$

where the subspace H_0 is defined by

$$H_0 := \{u \in H \, : \, u = 0 \text{ on } I_{\alpha_0,\beta_0}\}.$$

Let us remark that (3.2) is a weak form of (1.5), (1.2),

$$u = 0 \text{ on } I_{\alpha,\beta}, \quad \partial_\nu u = 0 \text{ on } E_{\alpha,\beta}, \tag{3.3}$$

with $(\alpha, \beta) = (\alpha_0, \beta_0)$. An essential part of our considerations will be related to mixed boundary value problems of this type.

Let us introduce coordinate transformations in $\overline{\Omega}$ which map $I_{\alpha,\beta}$ onto I_{α_0,β_0} and $E_{\alpha,\beta}$ onto E_{α_0,β_0}. They will be used to transform the mixed boundary value problem (1.1), (1.2), (3.3), which has (α,β)-independent coefficients in the equation but (α,β)-dependent boundary conditions, into a mixed boundary value problem with (α,β)-dependent coefficients in the equation but (α,β)-independent boundary conditions. Let $\gamma_1 < \alpha_0 < \beta_0 < \gamma_2$ be the parameters from the assumption (3.1) and set

$$\delta := \frac{1}{3} \min\{\alpha_0 - \gamma_1, \beta_0 - \alpha_0, \gamma_2 - \beta_0\}, \; \mathcal{D} := \{(\alpha,\beta) \, : \, |\alpha - \alpha_0| < \delta, \, |\beta - \beta_0| < \delta\}.$$

For any $(\alpha,\beta) \in \mathcal{D}$ let $\xi_{\alpha,\beta} = \left(\xi^{(1)}_{\alpha,\beta}, \xi^{(2)}_{\alpha,\beta}\right) : \overline{\Omega} \to \overline{\Omega}$ be a function such that

$$\left.\begin{array}{l} \text{the map } (\alpha,\beta,x,y) \in \mathcal{D} \times \overline{\Omega} \mapsto \xi_{\alpha,\beta}(x,y) \in \overline{\Omega} \text{ is } C^1\text{-smooth,} \\ \xi_{\alpha,\beta} \text{ is a diffeomorphism of } \overline{\Omega} \text{ onto } \overline{\Omega}, \end{array}\right\} \tag{3.4}$$

$$\xi_{\alpha,\beta} = id \text{ on a neighbourhood } \mathcal{U} \text{ of } \overline{\Gamma_N} \cup \overline{\Gamma_D}, \tag{3.5}$$

$$\xi_{\alpha_0,\beta_0} = id \text{ on } \overline{\Omega}, \tag{3.6}$$

$$\xi^{(2)}_{\alpha,\beta}(x,y) = y \text{ in } [\alpha_0 - \delta, \beta_0 + \delta] \times [0,\delta], \tag{3.7}$$

$$\xi_{\alpha,\beta}(\Gamma_U) = \Gamma_U, \quad \xi_{\alpha,\beta}(\Gamma_D) = \Gamma_D, \tag{3.8}$$

$$\left.\begin{array}{ll} \xi_{\alpha,\beta}(x,y) = (x + \alpha_0 - \alpha, y) & \text{for } |x-\alpha_0| \le \delta,\ |y| \le \delta, \\ \xi_{\alpha,\beta}(x,y) = (x + \beta_0 - \beta, y) & \text{for } |x-\beta_0| \le \delta,\ |y| \le \delta. \end{array}\right\} \tag{3.9}$$

For $(\alpha,\beta) \in \mathcal{D}$ let us define the linear bounded operator $\Phi_{\alpha,\beta} : L^2(\Omega) \to L^2(\Omega)$ by

$$(\Phi_{\alpha,\beta} f)(x,y) := f(\xi_{\alpha,\beta}(x,y)) \quad \text{for any } f \in L^2(\Omega). \tag{3.10}$$

Now we are ready to formulate our basic Assumption (ESB) about existence of a smooth branch of non-trivial solutions to (2.3) bifurcating in λ_0 from the trivial solution. For a particular case when Ω is a rectangle, the existence of a branch with properties described in Assumption (ESB) is proved in [6, Theorem 2.3] under certain assumptions concerning u_0 (certain "activity conditions"). It is not hard to see that this result can be shown for more general cases but a formulation of the assumptions and their verification in concrete examples is then even more complicated than in [6].

Assumption (ESB): There exist $s_0 > 0$ and C^1-smooth mappings $\hat{\lambda}, \hat{\alpha}, \hat{\beta} : [0, s_0) \to \mathbb{R}$ and $\hat{v} : [0,s_0) \to H_0$ with $\hat{\lambda}(0) = \lambda_0$, $\hat{v}(0) = 0$, $\hat{\alpha}(0) = \alpha_0$ and $\hat{\beta}(0) = \beta_0$ such that for all $s \in (0, s_0)$

$$\hat{u}(s) := s\Phi_{\hat{\alpha}(s),\hat{\beta}(s)}\left(u_0 + \hat{v}(s)\right) \tag{3.11}$$

is a solution to (2.3) with $\lambda = \hat{\lambda}(s)$, $\hat{u}$ is continuous from $[0,s_0)$ into H and C^1-smooth from $[0,s_0)$ into $\in L^2(\Omega)$ and $\hat{u}(s) \in W^{2,p}(\Omega \backslash \mathcal{U})$ for some $p > 2$, where $\mathcal{U}$ is from the assumption (3.5).

In this section we will show that the conditions

$$\int_\Omega \partial_u g(\lambda_0, 0) u_0^3 \,\mathrm{d}x\,\mathrm{d}y < 0 \tag{3.12}$$

or

$$\int_\Omega \partial_u g(\lambda_0, 0) u_0^3 \,\mathrm{d}x\,\mathrm{d}y > 0 \tag{3.13}$$

determine the direction of the branch of non-trivial bifurcating solutions introduced in Assumption (ESB).

Theorem 3.1 *Let (λ_0, u_0) satisfy (2.4), (3.1). Let g be C^1-smooth and satisfy (1.4), (2.2). Let us assume (ESB). Then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\hat{\lambda}(s)\right|_{s=0} = -\left(\int_\Omega u_0^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{-1} \int_\Omega \partial_u g(\lambda_0,0) u_0^3 \,\mathrm{d}x\,\mathrm{d}y. \tag{3.14}$$

In particular, if s_0 is chosen sufficiently small then under the assumption (3.12) or (3.13) we have $\hat{\lambda}(s) > \lambda_0$ or $\hat{\lambda}(s) < \lambda_0$, respectively, for all $s \in (0, s_0)$.

Proof. Let us denote $\Phi_s := \Phi_{\hat{\alpha}(s),\hat{\beta}(s)}$, $\xi_s(x,y) := \xi_{\hat{\alpha}(s),\hat{\beta}(s)}(x,y)$, $\lambda_s := \hat{\lambda}(s)$ and $v_s := \hat{v}(s)$. Since $(\lambda_s, \hat{u}(s))$ satisfies (2.3), it follows (by the choice $\varphi = 2\hat{u}(s)$ and $\varphi = 0$) that

$$\int_\Omega \left(|\nabla \hat{u}(s)|^2 - \lambda_s \hat{u}(s)^2 - g(\lambda_s, \hat{u}(s))\hat{u}(s)^2\right) \mathrm{d}x\,\mathrm{d}y = 0.$$

Realizing (3.11) and dividing by s^2 we get

$$\int_\Omega \left(|\nabla \Phi_s(u_0+v_s)|^2 - \lambda_s(\Phi_s(u_0+v_s))^2 - g(\lambda_s, s\Phi_s(u_0+v_s))(\Phi_s(u_0+v_s))^2\right) \mathrm{d}x\,\mathrm{d}y = 0. \tag{3.15}$$

Let us denote

$$\begin{aligned} &\dot{\lambda}_s := \tfrac{\mathrm{d}}{\mathrm{d}s}\hat{\lambda}(s), \quad \dot{v}_s := \tfrac{\mathrm{d}}{\mathrm{d}s}\hat{v}(s), \quad \dot{\xi}_s(x,y) := \tfrac{\mathrm{d}}{\mathrm{d}s}\xi_s(x,y),\\ &w_0(x,y) := \nabla u_0(x,y)\cdot\dot{\xi}_0(x,y). \end{aligned} \tag{3.16}$$

Formal calculation using the commutativity of integration and differentiation and the chain rule gives

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega \left(|\nabla\Phi_s(u_0+v_s)|^2 - \lambda_s\Phi_s(u_0+v_s)^2 - g(\lambda_s, s\Phi_s(u_0+v_s))\Phi_s(u_0+v_s)^2\right)\mathrm{d}x\,\mathrm{d}y\right]_{s=0}\\ &\qquad = \int_\Omega 2\left(\nabla u_0\cdot\nabla(w_0+\dot{v}_0) - \lambda_0 u_0(w_0+\dot{v}_0)\right) - \dot{\lambda}_0|u_0|^2 - \partial_u g(\lambda_0,0)u_0^3\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{3.17}$$

but the integrand is not smooth enough to enable us to use these considerations. We will prove (3.17) in details later.

First, let us assume that (3.17) is true. Then we obtain by using (3.15) that

$$2\int_\Omega \left(\nabla u_0\cdot\nabla(w_0+\dot{v}_0) - \lambda_0 u_0(w_0+\dot{v}_0)\right)\mathrm{d}x\,\mathrm{d}y = \int_\Omega \left(\dot{\lambda}_0|u_0|^2 + \partial_u g(\lambda_0,0)u_0^3\right)\mathrm{d}x\,\mathrm{d}y. \tag{3.18}$$

Because of (3.5) and Assumption (ESB) we have

$$\dot{\xi}_0(x,y) = (0,0) \text{ for all } (x,y)\in\mathcal{U} \text{ and } u_0 \in W^{2,2}(\Omega\setminus\mathcal{U}), \tag{3.19}$$

where $\mathcal{U}$ is the neighbourhood of $\overline{\Gamma_N}\cup\overline{\Gamma_D}$ from (3.5). Hence, $\nabla u_0\cdot\dot{\xi}_0 \in H$. We have even $\nabla u_0\cdot\dot{\xi}_0 \in H_0$ because the second component of $\dot{\xi}_0(x,0)$ vanishes for all $x\in[\alpha_0,\beta_0]$ by (3.7), and the first component of $\nabla u_0(x,0)$ vanishes for all $x\in[\alpha_0,\beta_0]$ because $u_0\in H_0\cap W^{2,2}(\Omega\setminus\mathcal{U})$. Hence, in terms of the notation (3.16) we have

$$w_0 \in H_0. \tag{3.20}$$

Moreover, since $v_s\in H_0$ for any $s\in(0,s_0)$, we have $\dot{v}_0\in H_0$. It follows that the choice $\varphi = w_0+\dot{v}_0\in H_0$ in (3.2) implies that the left-hand side in (3.18) vanishes. Hence, the right-hand side vanishes as well, and (3.14) follows.

It remains to prove (3.17). Let us focus on the leading term, which can be rewritten as

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega |\nabla\Phi_s(u_0)|^2\,\mathrm{d}x\,\mathrm{d}y\right]_{s=0} + 2\tfrac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega \nabla\Phi_s(u_0)\cdot\nabla\Phi_s(v_s)\,\mathrm{d}x\,\mathrm{d}y\right]_{s=0} +\\ &\qquad + \tfrac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega |\nabla\Phi_s(v_s)|^2\,\mathrm{d}x\,\mathrm{d}y\right]_{s=0}. \end{aligned} \tag{3.21}$$

Let $\eta_s : \overline{\Omega} \to \overline{\Omega}$ be the inverse of the differemorphism ξ_s. Then clearly

$$\xi_s(\eta_s(x,y)) = (x,y) \quad \text{for all } (s,x,y) \in [0,s_0) \times \overline{\Omega}. \tag{3.22}$$

Denoting ξ_s' and η_s' the Jacobian matrix to ξ_s and η_s, respectively, we get from (3.22) that

$$\xi_s'(\eta_s(x,y))\eta_s'(x,y) = I \quad \text{for all } (s,x,y) \in [0,s_0) \times \overline{\Omega}. \tag{3.23}$$

Moreover, (3.6) implies that

$$\xi_0'(x,y) = \eta_0'(x,y) = I \quad \text{for all } (x,y) \in \overline{\Omega}. \tag{3.24}$$

Furthermore, because of the chain rule we have

$$(\nabla \Phi_s u)(x,y) = \nabla(u \circ \xi_s)(x,y) = \xi_s'(x,y)(\nabla u)(\xi_s(x,y)) \quad \text{for all } u \in H. \tag{3.25}$$

Therefore the change of integration variables $(x,y) = \eta_s(\tilde{x},\tilde{y})$ in the first and the second integral of (3.21) yields

$$\begin{aligned}
\int_\Omega |\nabla \Phi_s u_0|^2 \, \mathrm{d}x \, \mathrm{d}y &= \int_\Omega |\xi_s'(\eta_s(\tilde{x},\tilde{y}))\nabla u_0(\tilde{x},\tilde{y})|^2 \det \eta_s'(\tilde{x},\tilde{y}) \, \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y} \\
&= \int_\Omega (\xi_s'(\eta_s(\tilde{x},\tilde{y}))\nabla u_0(\tilde{x},\tilde{y}) \cdot \xi_s'(\eta_s(\tilde{x},\tilde{y}))\nabla u_0(\tilde{x},\tilde{y})) \det \eta_s'(\tilde{x},\tilde{y}) \, \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y} \\
&= \int_\Omega \big(\xi_s'(\eta_s(\tilde{x},\tilde{y}))^T \xi_s'(\eta_s(\tilde{x},\tilde{y}))\nabla u_0(\tilde{x},\tilde{y}) \cdot \nabla u_0(\tilde{x},\tilde{y})\big) \det \eta_s'(\tilde{x},\tilde{y}) \, \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y}
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
&\int_\Omega \nabla \Phi_s u_0 \cdot \nabla \Phi_s v_s \, \mathrm{d}x \, \mathrm{d}y \\
&\quad = \int_\Omega \xi_s'(\eta_s(\tilde{x},\tilde{y}))\nabla u_0(\tilde{x},\tilde{y}) \cdot \xi_s'(\eta_s(\tilde{x},\tilde{y}))\nabla v_s(\tilde{x},\tilde{y}) \det \eta_s'(\tilde{x},\tilde{y}) \, \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y} \\
&\quad = \int_\Omega \xi_s'(\eta_s(\tilde{x},\tilde{y}))^T \xi_s'(\eta_s(\tilde{x},\tilde{y}))\nabla u_0(\tilde{x},\tilde{y}) \cdot \nabla v_s(\tilde{x},\tilde{y}) \det \eta_s'(\tilde{x},\tilde{y}) \, \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y}.
\end{aligned} \tag{3.27}$$

Denote by $w_s(x,y)$ the integrand in (3.27). Due to Assumptions (ESB) and (3.4) the map $s \in [0,s_0) \mapsto w_s \in L^2(\Omega)$ is C^1-smooth. It follows that the derivative $\frac{d}{ds}(w_s(x,y))$ exists for all $s \in (0,s_0)$ and a.a. $(x,y) \in \Omega$, and as a function of s,x,y it is in $L^1((0,s_0) \times \Omega)$. Due to known results concerning differentiation of an integral with respect to a parameter (see e.g. [21, Theorem 8.4]) we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega \nabla \Phi_s u_0 \cdot \nabla \Phi_s v_s \, \mathrm{d}x \, \mathrm{d}y\right]_{s=0} = \int_\Omega \left[\frac{d}{ds}(w_s(\tilde{x},\tilde{y}))\right]_{s=0} \\
&\quad = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}s}\left[\xi_s'(\eta_s(\tilde{x},\tilde{y}))^T \xi_s'(\eta_s(\tilde{x},\tilde{y}))\nabla u_0(\tilde{x},\tilde{y}) \cdot \nabla v_s(\tilde{x},\tilde{y}) \det \eta_s'(\tilde{x},\tilde{y})\right]_{s=0} \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y}.
\end{aligned} \tag{3.28}$$

Similarly one shows that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega |\nabla \Phi_s u_0|^2 \, \mathrm{d}x \, \mathrm{d}y\right]_{s=0} \\
&\quad = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}s}\left[\big(\xi_s'(\eta_s(\tilde{x},\tilde{y}))^T \xi_s'(\eta_s(\tilde{x},\tilde{y}))\nabla u_0(\tilde{x},\tilde{y}) \cdot \nabla u_0(\tilde{x},\tilde{y})\big) \det \eta_s'(\tilde{x},\tilde{y})\right]_{s=0} \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y}.
\end{aligned} \tag{3.29}$$

Now, we will prove that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega |\nabla \Phi_s u_0|^2 \, \mathrm{d}x \, \mathrm{d}y\right]_{s=0} = 2\int_\Omega \nabla u_0 \cdot \nabla w_0 \, \mathrm{d}x \, \mathrm{d}y. \tag{3.30}$$

Denoting $\dot{\eta}_s := \frac{\mathrm{d}}{\mathrm{d}s}\eta_s$ and differentiating (3.22) by s we obtain

$$\dot{\xi}_s(\eta_s(x,y)) + \xi_s'(\eta_s(x,y))\dot{\eta}_s(x,y) = (0,0) \quad \text{for all } (s,x,y) \in [0,s_0) \times \overline{\Omega}. \tag{3.31}$$

Expressing (3.31) at $s=0$ and realizing (3.6) and (3.24) gives

$$\dot{\xi}_0(x,y) + \dot{\eta}_0(x,y) = (0,0) \quad \text{for all } (x,y) \in \overline{\Omega}. \tag{3.32}$$

We obtain for the Jacobian matrices corresponding to $\dot{\xi}_0(x,y)$ and $\dot{\eta}_0(x,y)$ that

$$\dot{\xi}_0'(x,y) = -\dot{\eta}_0'(x,y) \quad \text{for all } (x,y) \in \overline{\Omega}. \tag{3.33}$$

Differentiating (3.23) by s we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\xi_s'(\eta_s(x,y))\right]\eta_s'(x,y) + \xi_s'(\eta_s(x,y))\frac{\mathrm{d}}{\mathrm{d}s}\left[\eta_s'(x,y)\right] = 0.$$

Expressing this at $s=0$ and realizing (3.6), (3.24) and (3.33) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\xi_s'(\eta_s(x,y))\right]_{s=0} = -\left.\xi_s'(\eta_s(x,y))\right|_{s=0}\frac{\mathrm{d}}{\mathrm{d}s}\left[\eta_s'(x,y)\right]_{s=0} = -\dot{\eta}_0'(x,y) = \dot{\xi}_0'(x,y).$$

Moreover, a simple calculation yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\det \eta_s'(x,y)\right]_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\left[(\det \xi_s'(x,y))^{-1}\right]_{s=0} = -\mathrm{tr}\,\dot{\xi}_0'(x,y),$$

where $\mathrm{tr}\,B$ stays for the trace of the matrix B. Summarizing, we get by the product rule

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\xi_s'(\eta_s(x,y))^T\xi_s'(\eta_s(x,y))\det\eta_s'(x,y)\right]_{s=0} = \dot{\xi}_0'(x,y)^T + \dot{\xi}_0'(x,y) - \left(\mathrm{tr}\,\dot{\xi}_0'(x,y)\right)I. \tag{3.34}$$

For any $h \in \mathbb{R}^2$ we have $h\cdot\dot{\xi}_0'(x,y)^T h = \dot{\xi}_0'(x,y)^T h\cdot h = h\cdot\dot{\xi}_0'(x,y)h$ and we obtain by the choice $h = \nabla u_0$ from (3.34) and (3.29) that

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}s}\textstyle\int_\Omega |\nabla\Phi_s u_0|^2\,\mathrm{d}x\,\mathrm{d}y\Big|_{s=0} \\ &\quad = \textstyle\int_\Omega \nabla u_0(\tilde{x},\tilde{y})\cdot\left(2\dot{\xi}_0'(\tilde{x},\tilde{y}) - \left(\mathrm{tr}\,\dot{\xi}_0'(\tilde{x},\tilde{y})\right)I\right)\nabla u_0(\tilde{x},\tilde{y})\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}. \end{aligned} \tag{3.35}$$

Integration by parts gives

$$\int_\Omega |\nabla u_0|^2 \mathrm{tr}\,\dot{\xi}_0'\,\mathrm{d}x\,\mathrm{d}y = -\int_\Omega \nabla|\nabla u_0|^2\cdot\dot{\xi}_0\,\mathrm{d}x\,\mathrm{d}y + \int_{\partial\Omega}|\nabla u_0|^2\dot{\xi}_0\cdot\nu\,\mathrm{d}\Gamma$$

where the last boundary integral vanishes because the vector $\dot{\xi}_0$ is zero on the domain $\mathcal{U}$ due to (3.5) and tangential to $\partial\Omega$ on Γ_U due to (3.8), and consequently $\dot{\xi}_0\cdot\nu = 0$ on $\partial\Omega$. We obtain by using the notation (3.16), direct calculation and the integration by parts discussed above that

$$\begin{aligned} \textstyle\int_\Omega \nabla u_0\cdot\nabla w_0\,\mathrm{d}x\,\mathrm{d}y &= \textstyle\int_\Omega \nabla u_0\cdot\nabla\left(\nabla u_0\cdot\dot{\xi}_0\right)\,\mathrm{d}x\,\mathrm{d}y \\ &= \textstyle\int_\Omega\left(\frac{1}{2}\nabla|\nabla u_0|^2\cdot\dot{\xi}_0 + \nabla u_0\cdot\dot{\xi}_0'\nabla u_0\right)\,\mathrm{d}x\,\mathrm{d}y \\ &= \textstyle\int_\Omega\left(-\frac{1}{2}|\nabla u_0|^2\mathrm{tr}\,\dot{\xi}_0' + \nabla u_0\cdot\dot{\xi}_0'\nabla u_0\right)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

and (3.30) follows from (3.35).

Differentiating the integrand in (3.28), using (3.19), (3.24) and realizing in addition that $v_0 = 0$ we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega \nabla\Phi_s u_0 \cdot \nabla\Phi_s v_s \,\mathrm{d}x\,\mathrm{d}y\right]_{s=0} = \int_\Omega \nabla u_0 \cdot \nabla \dot{v}_0 \,\mathrm{d}x\,\mathrm{d}y. \tag{3.36}$$

It remains to express the last integral in (3.21). The map $s \in [0, s_0) \mapsto v_s \in H$ is C^1-smooth by Assumption (ESB), hence $v_0 = 0$ yields that $\|v_s\|^2 = o(s)$ for $s \to 0$. Therefore, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_\Omega |\nabla\Phi_s v_s|^2 \,\mathrm{d}x\,\mathrm{d}y\right]_{s=0} = \lim_{s\to 0}\frac{\|\Phi_s v_s\|^2}{s} \le \lim_{s\to 0}\frac{\|\Phi_s\|^2\,\|v_s\|^2}{s} = 0. \tag{3.37}$$

The lower order terms in (3.17) are such that we can directly exchange differentiation and integration and we get by using (1.4), the form (3.11) and (3.24) that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}s}&\left[\textstyle\int_\Omega (\lambda_s\Phi_s(u_0+v_s)^2 + g(\lambda_s, s\Phi_s(u_0+v_s))\Phi_s(u_0+v_s)^2)\,\mathrm{d}x\,\mathrm{d}y\right]_{s=0}\\ &= \textstyle\int_\Omega \left(\dot{\lambda}_0|u_0|^2 + 2\lambda_0 u_0(w_0+\dot{v}_0) + \partial_u g(\lambda_0,0)u_0^3\right)\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Hence, (3.30), (3.35) and (3.37) yield that (3.17) is true. This finishes the proof. ∎

Example 3.2 *We will consider the same situation as in [6, Example 2.7]. Let* $\Omega := (0,1)\times(0,\ell)$*,* $\Gamma_D := (\{0\}\times(0,\ell)) \cup (\{1\}\times(0,\ell))$*,* $\Gamma_U := ((\gamma_1,\gamma_2)\times\{0\})$ *with* $0 < \gamma_1 < \gamma_2 < 1$ *and* $\Gamma_N := \partial\Omega \setminus \left(\overline{\Gamma_D}\cup\overline{\Gamma_U}\right)$*. First, let us consider the eigenvalue problem (1.5) with the boundary conditions (1.7). The eigenvalues and eigenfunctions of this problem are*

$$\lambda_{m,n} = (m\pi)^2 + \left(\frac{n\pi}{\ell}\right)^2, \quad u_{m,n}(x,y) = \sin m\pi x \cdot \cos\frac{n\pi}{\ell}y, \quad m = 1,2,\ldots,\ n = 0,1,2,\ldots,$$

respectively. If $1/\sqrt{15} < \ell < 1/\sqrt{8}$ *then the first four eigenvalues are* $\lambda_{1,0} < \lambda_{2,0} < \lambda_{3,0} < \lambda_{1,1}$*. Let us assume that* $1/3 < \gamma_1 < \gamma_2 < 2/3$*. Then*

$$u_{3,0} < 0,\ -u_{1,1} < 0 \ \text{ on } \overline{\Gamma_U}.$$

The method developed in [16] shows that there is at least one eigenvalue $\lambda_0 \in (\lambda_{3,0},\lambda_{1,1})$ *of the variational inequality (2.4). This eigenvalue and the corresponding eigenfunction can be calculated numerically and (3.1) is seen as well as the activity condition (2.19) from [6], see Fig. 1. (Everything coincides with ideas coming from the method [16] mentioned). It follows also that* λ_0 *is simple as an eigenvalue of the problem (1.5), (3.3) (with* $\alpha = \alpha_0$*,* $\beta = \beta_0$*) because standard numerical approaches usually fail in the case of multiple eigenvalues. The condition (2.21) from [6], which is fulfilled generically, can be also verified numerically for a given* ℓ*. It follows from [6, Theorem 2.3] that a branch from our assumption (ESB) exists. Now, our Theorem 3.1 guarantees that this branch bifurcates to the right or to the left from* λ_0 *under the assumption (3.12) or (3.13), respectively.*

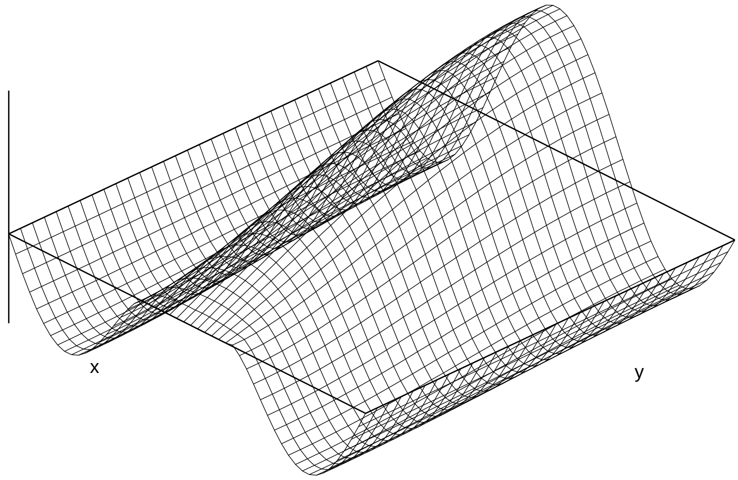

Figure 1: The eigenfunction u_0 from Example 3.2 with $\ell = 0.27$, $\lambda_0 = 99.8$, $\alpha_0 = 0.38$ and $\beta_0 = 0.62$.

4 Stability of bifurcating solutions, exchange of stability

In this section we will study the stability of the trivial solution and of the nontrivial solutions bifurcating from the first eigenvalue to (2.4) (i.e. to (1.5), (1.2), (1.3)) as of stationary solutions of the corresponding evolution variational inequality

$$\begin{aligned} &u(t) \in K = \{u \in H \,:\, u \leq 0 \text{ on } \Gamma_U\}: \\ &\int_\Omega \left(\tfrac{\mathrm{d}u}{\mathrm{d}t}(\varphi - u) + \nabla u \cdot \nabla(\varphi - u) - (\lambda u + g(\lambda, u)u)(\varphi - u)\right) \mathrm{d}x\, \mathrm{d}y \geq 0 \\ &\text{for all } \varphi \in K, \text{ a.a. } t \in (0, T). \end{aligned} \tag{4.1}$$

Instead of the assumption (2.2) we will need the stronger condition

$$\text{the map } (\lambda, u) \mapsto g(\lambda, u) \text{ is } C^2\text{-smooth from } \mathbb{R} \times W^{1,2}(\Omega) \text{ into } L^r(\Omega) \text{ for some } r > 2. \tag{4.2}$$

Let us remark that (4.2) is true if, for example, $g(\lambda, \cdot)$ is a polynom in u.

By a strong solution we mean $u \in C([0, T], K)$ such that $u : (0, T) \to L^2(\Omega)$ is differentiable a.e. on $(0, T)$ and satisfies (4.1). By stability of a stationary solution u_s in $W^{1,2}$ we mean that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_{ic} \in K$, $\|u_{ic} - u_s\| < \delta$ then there is a unique strong solution u of (4.1) on $[0, \infty)$ satisfying the initial condition $u(0) = u_{ic}$ and we have $\|u(t) - u_s\| < \varepsilon$ for all $t \geq 0$. By asymptotical stability of a stationary solution u_s we mean that u_s is stable and $\lim_{t\to\infty} \|u(t) - u_s\| = 0$.

Simultaneously, we will discuss the problem (1.1), (1.7) having the weak formulation

$$u \in H \,:\, \int_\Omega (\nabla u \nabla \varphi - \lambda u\varphi - g(\lambda, u)u\varphi)\,\mathrm{d}x\,\mathrm{d}y = 0 \text{ for all } \varphi \in H, \tag{4.3}$$

and the eigenvalue problem (1.5) with (1.7) having the weak formulation

$$u \in H \,:\, \int_\Omega (\nabla u \cdot \nabla \varphi - \lambda u\varphi)\,\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{for all } \varphi \in H. \tag{4.4}$$

We will introduce operators $A : H \to H$ and $G : \mathbb{R} \times H \to H$ by

$$\begin{aligned} \langle Au, \varphi\rangle &:= \textstyle\int_\Omega u\varphi\,\mathrm{d}x\,\mathrm{d}y, \\ \langle G(\lambda, u), \varphi\rangle &:= \textstyle\int_\Omega g(\lambda, u)u\varphi\,\mathrm{d}x\,\mathrm{d}y \quad \text{ for all } \varphi \in H. \end{aligned} \tag{4.5}$$

Under the assumption (4.2), these operators are well-defined, A is linear, symmmetric, positive and compact,

$$G \text{ is } C^2\text{-smooth}, \quad G(\lambda, 0) = \partial_u G(\lambda, 0) = 0 \text{ for all } \lambda. \tag{4.6}$$

For fixed λ, u, the linear map $\partial_u G(\lambda, u) : H \to H$ is symmetric and compact. An abstract form of (2.3) is

$$\lambda \in \mathbb{R},\, u \in K \,:\, \langle u - \lambda Au - G(\lambda, u), \varphi - u\rangle \geq 0 \quad \text{ for all } \varphi \in K. \tag{4.7}$$

Lemma 4.1 *The smallest eigenvalue λ_0 of the variational inequality (2.4) is simultaneously the smallest and simple eigenvalue of the problem (4.4). There is only one normalized eigenfunction u_0 of (2.4) corresponding to λ_0 and it coincides with the negative eigenfunction of (4.4) corresponding to λ_0. We have*

$$u_0 < 0 \quad on \quad [\gamma_1, \gamma_2] \times \{0\} = \overline{\Gamma_U}. \tag{4.8}$$

Proof. It is known that the smallest eigenvalue $\lambda^I_{\min}$ of the variational inequality (2.4) is characterized by

$$1/\lambda^I_{\min} = \max_{u \in K, \|u\|=1} \int_\Omega u^2\,\mathrm{d}x\,\mathrm{d}y \tag{4.9}$$

(see e.g. [22]), and the smallest eigenvalue $\lambda_{\min}$ of the problem (4.4) is characterized by

$$1/\lambda_{\min} = \max_{u \in H, \|u\|=1} \int_\Omega u^2\,\mathrm{d}x\,\mathrm{d}y. \tag{4.10}$$

The eigenfunctions of (2.4) or (4.4) correponding to $\lambda^I_{\min}$ or $\lambda_{\min}$ are exactly all maximizers of (4.9) or (4.10), respectively. It is well-known that the smallest eigenvalue $\lambda_{\min}$ of (4.4) is simple and there is a corresponding eigenfunction u^E_1 of (4.4) with $u^E_1 < 0$ in $\Omega \cup \Gamma_N \cup \Gamma_U$. In particular, $u^E_1 \in K$. It follows from the variational characterization above that $\lambda_0 = \lambda^I_{\min} = \lambda_{\min}$ and that u^E_1 is simultaneously the corresponding eigenfunction of (2.4). If u_0 is an arbitrary normalized eigenfunction of (2.4) then it is a maximizer of (4.9) and also of (4.10). That means u_0 is simultaneously an eigenfunction of (4.4), i.e. $u_0 = u^E_1$ because of $-u^E_1 \notin K$. ■

In the sequel we will assume that (λ_0, u_0) is the couple from Lemma 4.1.

Theorem 4.2 *Let λ_0 be the smallest eigenvalue of the variational inequality (2.4) and let (1.4), (2.1), (4.2) be fulfilled. Then there exist $s_0 > 0$ and C^1-smooth maps $\hat{\lambda} : (-s_0, s_0) \to \mathbb{R}$, $\hat{v} : (-s_0, s_0) \to (u_0)^\perp$ (the orthogonal complement in H) such that $\hat{\lambda}(0) = \lambda_0$, $\hat{v}(0) = 0$ and the couple $(\hat{\lambda}(s), \hat{u}(s))$ with $\hat{u}(s) := s(u_0 + \hat{v}(s))$ for any $s \in (-s_0, s_0)$ satisfies (4.3), and for any $s \in (0, s_0)$ it satisfies simultaneously the variational inequality (2.3). Moreover, there exists $\eta > 0$ such that for any solution $(\lambda, u) \in \mathbb{R} \times (H \setminus \{0\})$ to (2.3) with*

$$|\lambda - \lambda_0| + \|u\| < \eta \tag{4.11}$$

there is $s \in (0, s_0)$ with $u = \hat{u}(s)$ and $\lambda = \hat{\lambda}(s)$.

In addition, let the condition (3.12) be fulfilled. Then

$$\hat{\lambda}(s) > \lambda_0 \text{ for all } s \in (0, s_0), \quad \hat{\lambda}(s) < \lambda_0 \text{ for all } s \in (-s_0, 0). \tag{4.12}$$

For $s \in (0, s_0)$, the solution $\hat{u}(s)$ is asymptotically stable in $W^{1,2}$ as the stationary solution of the evolution equation corresponding to (4.3) as well as the stationary solution of the evolution variational inequality (4.1). For $s \in (-s_0, 0)$, the solution $\hat{u}(s)$ is unstable as the stationary solution of the evolution equation corresponding to (4.3) (and it is no solution of the variational inequality).

With a knowledge of Lemma 4.1, Theorem 4.2 is very natural. The only non-easy problem in the proof is to show that any solution of the variational inequality (2.3) near $(\lambda_0, 0)$ is simultaneously a solution of the equation (4.3). The following series of Lemmas 4.4–4.9, Remarks 4.3, 4.6 and Observation 4.8 will be used just for this part of the proof. They are similar to the corresponding assertions in [6], nevertheless, the situation now is different.

Remark 4.3 *Let (λ, u) satisfy (2.3) (i.e. (1.1), (1.2), (1.3) in the weak sense). Since u is continuous on Γ_U (see e.g. the first part of Lemma 4.4 below), it follows that Γ_U consists of two maximally countable sets $\mathcal{S}_N$, $\mathcal{S}_D$ of unknown open subintervals such that $u < 0$ on Γ for all $\Gamma \in \mathcal{S}_N$ and*

$$u = 0 \text{ on } \Gamma \tag{4.13}$$

for for all $\Gamma \in \mathcal{S}_D$, and some isolated points $(x, 0)$ with $u(x, 0) = 0$. For $\Gamma \in \mathcal{S}_N$ we have

$$\int_\Omega \nabla u \cdot \nabla \varphi - (\lambda u + g(\lambda, u)u)\varphi \, \mathrm{d}x \, \mathrm{d}y = 0 \text{ for all } \varphi \in H \text{ satisfying } \varphi = 0 \text{ on } \partial\Omega \setminus \Gamma, \tag{4.14}$$

i.e. (λ, u) satisfies in a weak sense (1.1)–(1.3) and, in addition, $\partial_\nu u = 0$ on Γ. If $(x_0, 0)$ is an isolated point of the set $\{(x, 0) \in \Gamma_U : u(x, 0) = 0\}$ then (4.14) is fulfilled with an open neighbourhood Γ of $(x_0, 0)$ in $\partial\Omega$.

Lemma 4.4 *Let $\Omega' \subset \Omega$ be a sub-domain with a smooth boundary such that $\overline{\Omega'} \subset \Omega \cup (\Gamma_U \cup \Gamma_{N,\mu_0})$, Γ_{N,μ_0} being from (2.1). Then we have $u \in W^{2,2}(\Omega')$ for any $(\lambda, u) \in \mathbb{R} \times H$ satisfying (2.3). Moreover, for any $\eta_0 > 0$ small enough there is $C > 0$ such that*

$$\left\| \frac{u}{\|u\|} \right\|_{W^{2,2}(\Omega')} \leq C \text{ for all } (\lambda, u) \in \mathbb{R} \times (H \setminus \{0\}) \text{ satisfying } (2.3),\ |\lambda - \lambda_0| + \|u\| < \eta_0. \quad (4.15)$$

If $\Omega' \subset \Omega$, $\overline{\Omega'} \subset \Omega \cup \Gamma$ with some $\Gamma \subset (\gamma_1 - \mu_0, \gamma_2 + \mu_0) \times \{0\}$ ($(\gamma_j, 0)$ are now included), then the same assertion holds for all $(\lambda, u) \in \mathbb{R} \times (H \setminus \{0\})$ satisfying also (4.14) in addition to (2.3).

Proof. First, let Ω' be such that $\overline{\Omega'} \subset \Omega \cup \Gamma_U$. Then there is $C > 0$ such that for any $f \in L^2(\Omega)$, the solution of the variational inequality

$$w \in K : \int_\Omega (\nabla w \cdot \nabla(\varphi - w) - f(\varphi - w))\, \mathrm{d}x\, \mathrm{d}y \geq 0 \text{ for all } \varphi \in K \quad (4.16)$$

satisfies $w \in W^{2,2}(\Omega')$ and

$$\|w\|_{W^{2,2}(\Omega')} \leq C(1 + \|w\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)}) \quad (4.17)$$

(see e.g. [18]). If u satisfies (2.3) then $w := \frac{u}{\|u\|}$ satisfies (4.16) with

$$f = \lambda w + g(\lambda, \|u\| w) w.$$

We have $\|g(\lambda, \|u\| w)\|_{L^r(\Omega)} \leq C$ for all λ, u, w such that $|\lambda - \lambda_0| + \|u\| < \eta_0$, $\|w\| = 1$ (see the assumption (4.2)) and therefore we get $\|g(\lambda, \|u\|w)w\|_{L^2(\Omega)} \leq \|g(\lambda, \|u\|w)\|_{L^r(\Omega)} \|w\|_{L^q(\Omega)} \leq C\|w\| \leq C$ (with $1/r + 1/q = 1$) by using Hölder inequality and embedding theorems. Substituting f from the formula above we get (4.15).

Now, let Ω' be such that $\overline{\Omega'} \subset (\Omega \cup \Gamma_{N,\mu_0})$. For the Neumann problem for the equation $-\Delta w = f$ on a bounded convex smooth domain Ω'', the estimate $\|w\|_{W^{2,2}(\Omega'')} \leq C(\|w\|_{W^{1,2}(\Omega'')} + \|f\|_{L^2(\Omega'')})$ is known (see e.g. [11, Theorem 3.1.2.3]). It is easy to find a suitable $\Omega'' \subset \Omega$ with $\overline{\Omega''} \subset (\Omega \cup \Gamma_{N,\mu_0})$, $\overline{\Omega'} \subset (\Omega'' \cup \operatorname{int}(\overline{\Omega''} \cap \partial\Omega))$ where int denotes the interior in $\partial\Omega$, and a cut-off function $\chi = 1$ in Ω', $\chi = 0$ in $\Omega \setminus \Omega''$ such that $w := \chi \frac{u}{\|u\|}$ is a solution of the homogeneous Neumann problem for the equation $-\Delta w = f$ in Ω'' with

$$f = \lambda \chi \frac{u}{\|u\|} + g\left(\lambda, \|u\| \frac{u}{\|u\|}\right) \chi \frac{u}{\|u\|} - 2\nabla \frac{u}{\|u\|} \cdot \nabla \chi - \frac{u}{\|u\|} \Delta \chi \quad (4.18)$$

if u is a solution of (2.3). By similar estimates of f as above we obtain the estimate for w, and since $w = \frac{u}{\|u\|}$ in Ω' we get also (4.15).

Combining both situations considered we obtain (4.15) for any Ω' from the assumptions of the first part of our Lemma. The case $\overline{\Omega'} \subset \Omega \cup \Gamma$, $\Gamma \subset (\gamma_1 - \mu_0, \gamma_2 + \mu_0) \times \{0\}$ is in fact the same as $\overline{\Omega'} \subset (\Omega \cup \Gamma_{N,\mu_0})$ because it was essential only that the solutions under consideration satisfy Neumann condition in the weak sense on a straight part of $\overline{\Omega'} \cap \partial\Omega$. ■

Remark 4.5 *If $\Gamma \subset (\Gamma_U \cup \Gamma_{N,\mu_0})$ is an open (in $\Gamma_U \cup \Gamma_{N,\mu_0}$) set and $\Omega' \subset \Omega$ is a sub-domain, $\overline{\Omega'} \subset (\Omega \cup \overline{\Gamma})$, then for any (λ, u) satisfying (2.3) and in addition (4.13) we have $u \in C^{1,\gamma}(\overline{\Omega'})$ with some $\gamma \in (0,1)$. In particular, the classical normal derivative $\partial_\nu u$ exists on Γ. Indeed, it follows from Lemma 4.4 and the embedding theorems that $u \in C^0(\overline{\Omega'})$ for any (λ, u) satisfying (2.3). We can prolong it anti-symmetrically by $u(x,y) = -u(x,-y)$ onto the domain $\Omega'_M :=$* $\operatorname{int}\{(x,y):$ *either $(x,y) \in \Omega'$ or $(x,-y) \in \Omega'$ or $(x,0) \in \overline{\Omega'}\}$. Under the assumption (4.13), the prolonged function is a weak solution of $\Delta u = f$ in Ω'_M with $f = \lambda u + g(\lambda, u)u \in C^0(\overline{\Omega'_M})$. Well-known regularity results (e.g. [14, Th. 11.1.2] imply that $u \in C^{1,\gamma}(\overline{\Omega'_M})$.*

If $\Gamma \subset (\overline{\Gamma_U} \cup \Gamma_{N,\mu_0})$ (the points $(\gamma_j, 0)$ can be now included) then we can use analogous considerations for (λ, u) satisfying (2.3) and (4.14) (instead of (4.13)) to get $u \in C^{1,\gamma}(\overline{\Omega'})$. We must only use the symmetrical prolongation $u(x,y) = u(x,-y)$ under the assumption (4.14).

Remark 4.6 *Let Ω' be as in Lemma 4.4. The estimate (4.15) guarantees that if $(\lambda, u) \in \mathbb{R} \times H$ is a solution of (2.3) with $\|u\|$ small enough, λ close to λ_0 then $\|u\|_{C(\overline{\Omega'})}$ is small. Due to (1.4) and (4.15), for all $(\lambda, u) \in \mathbb{R} \times H$ satisfying (2.3) with $\|u\|$ small enough and λ close to λ_0 we have $|g(\lambda, u(x,y))u(x,y)| < \lambda |u(x,y)|$, and consequently* $\operatorname{sign}(\lambda u(x,y) + g(\lambda, u(x,y))u(x,y)) =$ $\operatorname{sign} u(x,y)$ *for all $(x,y) \in \Omega'$ with $u(x,y) \neq 0$.*

Lemma 4.7 *Let Ω' be a sub-domain of Ω, $\overline{\Omega'} \subset \Omega$. Then there is $C > 0$ such that if $(\lambda, u) \in \mathbb{R} \times (H \setminus \{0\})$ satisfies (2.3) then $u \in W^{2,2}(\Omega')$ and*

$$\left\| \frac{u}{\|u\|} - u_0 \right\|_{W^{2,2}(\Omega')} \leq C \left(|\lambda - \lambda_0| + \left\| \frac{u}{\|u\|} - u_0 \right\| + \|g(\lambda, u)\|_{L^r(\Omega)} \right), \tag{4.19}$$

where r is from the assumption (4.2).

Proof If $(\lambda, u) \in \mathbb{R} \times (H \setminus \{0\})$ satisfies (2.3) then $w := \frac{u}{\|u\|} - u_0$ satisfies $\Delta w = f$ with

$$f = (\lambda - \lambda_0)\frac{u}{\|u\|} - \lambda_0(\frac{u}{\|u\|} - u_0) - g(\lambda, u)\frac{u}{\|u\|}.$$

It is well-known that for such equation, the estimate

$$\|w\|_{W^{2,2}(\Omega')} \leq C(\|w\| + \|f\|_{L^2(\Omega)})$$

holds (see e.g. [10, Theorem 8.8]). Due to the assumption (4.2) and the embedding theorems we have

$$\left\| g(\lambda, u)\frac{u}{\|u\|} \right\|_{L^2(\Omega)} \leq \|g(\lambda, u)\|_{L^r(\Omega)} \left\| \frac{u}{\|u\|} \right\|_{L^q(\Omega)} \leq C\|g(\lambda, u)\|_{L^r(\Omega)},$$

where $1/r + 1/q = 1$. It follows that $\|f\|_{L^2(\Omega)} \leq C \left(|\lambda - \lambda_0| + \lambda_0 \left\| \frac{u}{\|u\|} - u_0 \right\| + \|g(\lambda, u)\|_{L^r(\Omega)} \right)$, and our assertion follows. ■

Additional Notation. For $\varepsilon > 0$ and $a < b$ we will denote

$$\Omega^\varepsilon_{a,b} = (a,b) \times (0, \varepsilon), \quad \Gamma^\varepsilon_{a,b} = (a,b) \times \{\varepsilon\}.$$

Observation 4.8 *([6, Observation 3.11]) Let $\varepsilon > 0$, $a < b$. The smallest eigenvalue of the problem*

$$\Delta w + \lambda w = 0 \text{ in } \Omega^\varepsilon_{a,b}, \quad w = 0 \text{ on } \Gamma^\varepsilon_{a,b}, \quad \partial_\nu w = 0 \text{ on } \partial\Omega^\varepsilon_{a,b} \setminus \Gamma^\varepsilon_{a,b}$$

is $\lambda^\varepsilon_{\min} = \left(\frac{\pi}{2\varepsilon}\right)^2$. (The corresponding eigenfunction is $w_0(x,y) = \sin\frac{\pi(y-\varepsilon)}{2\varepsilon}$.) It is well known that $\lambda^\varepsilon_{\min} = \min\limits_{w\in W_0} \dfrac{\|\nabla w\|_{L^2(\Omega^\varepsilon_{a,b})}}{\|w\|_{L^2(\Omega^\varepsilon_{a,b})}}$, $W_0 = \left\{w \in W^{1,2}(\Omega^\varepsilon_{a,b}) : w = 0 \text{ on } \Gamma^\varepsilon_{a,b}\right\}$. Hence,

$$\|w\|_{L^2(\Omega^\varepsilon_{a,b})} \le \left(\frac{2\varepsilon}{\pi}\right)^2 \|\nabla w\|_{L^2(\Omega^\varepsilon_{a,b})} \text{ for all } w \in W^{1,2}\left(\Omega^\varepsilon_{a,b}\right), \quad w = 0 \text{ on } \Gamma^\varepsilon_{a,b}. \tag{4.20}$$

Lemma 4.9 *There are $\eta > 0$ and $\mu \in (0, \mu_0/2)$ such that if (λ, u) satisfies (2.3) and*

$$\|u\| \ne 0, \quad \|u\| + \left\|\frac{u}{\|u\|} - u_0\right\| + |\lambda - \lambda_0| < \eta \tag{4.21}$$

then for any subdomain $\Omega' \subset \Omega$, $\overline{\Omega'} \subset (\Omega \cup \overline{\Gamma_U} \cup \Gamma_{N,\mu_0})$ we have $u \in C^{1,\gamma}(\overline{\Omega'})$ with some $\gamma > 0$ and

$$u(x,0) < 0 \quad \text{for all } x \in (\gamma_1 - \mu, \gamma_2 + \mu). \tag{4.22}$$

Proof. Due to (4.8) and the continuity of u_0, there exist $\mu \in (0, \mu_0/2)$, $\varepsilon_0 > 0$ and $\zeta > 0$ such that

$$u_0 < -\zeta \quad \text{in } \overline{\Omega^{\varepsilon_0}_{\gamma_1-2\mu,\gamma_2+2\mu}}. \tag{4.23}$$

Let us set

$$\begin{aligned}\Omega' &:= \Omega^{\varepsilon_0}_{\gamma_1-2\mu,\gamma_2+2\mu} \setminus \left(\overline{\Omega^{\varepsilon_0}_{\gamma_1-\mu,\gamma_1+\mu}} \cup \overline{\Omega^{\varepsilon_0}_{\gamma_2-\mu,\gamma_2+\mu}}\right)\\ &= ((\gamma_1 - 2\mu, \gamma_1 - \mu) \times (0, \varepsilon_0)) \cup ((\gamma_1 + \mu, \gamma_2 - \mu) \times (0, \varepsilon_0)) \cup ((\gamma_2 + \mu, \gamma_2 + 2\mu) \times (0, \varepsilon_0)).\end{aligned}$$

Let us fix a certain $\eta_0 > 0$. We have $\overline{\Omega'} \subset (\Omega \cup \Gamma_{N,\mu_0} \cup \overline{\Gamma_U})$ and it follows by using Lemma 4.4 that there is $C > 0$ such that for all (λ, u) satisfying (2.3) and (4.21) with $\eta = \eta_0$ we have $\|u\|_{W^{2,2}(\Omega')} \le C\|u\|$. Due to the continuous embedding $W^{2,2}(\Omega') \subset C^{0,\gamma}(\overline{\Omega'})$ with some $\gamma > 0$ we have also

$$\|u\|_{C^{0,\gamma}(\overline{\Omega'})} \le C\|u\| \text{ for all } (\lambda, u) \text{ satisfying } (2.3), (4.21) \text{ with } \eta = \eta_0. \tag{4.24}$$

There exists $\rho > 0$ such that if $u \in C^{0,\gamma}(\overline{\Omega'})$, $\|u\| \ne 0$, $u(x_0, y_0) \ge 0$ for some $(x_0, y_0) \in \overline{\Omega'}$ then $\frac{u(x,y)}{\|u\|_{C^{0,\gamma}(\overline{\Omega'})}} > -\zeta/2C$ for all $(x,y) \in \overline{B_\rho(x_0,y_0)} \cap \overline{\Omega'}$, where $B_\rho(x_0, y_0)$ is the open disc with the radius ρ centered at (x_0, y_0). Due to (4.24), we have also $\frac{u(x,y)}{\|u\|} > -\zeta/2$ for all $(x,y) \in \overline{B_\rho(x_0,y_0)} \cap \overline{\Omega'}$ if (2.3), (4.21) with $\eta = \eta_0$ are fulfilled. Hence, we get $\text{meas}\{(x,y) \in \overline{\Omega'} : u(x,y) \ge -\zeta\|u\|/2\} \ge \theta := \text{meas}\left(B_\rho(x_0,y_0) \cap \overline{\Omega'}\right)$. In particular, we have proved the existence of $\theta > 0$ such that

$$\begin{aligned}&\text{if } (\lambda, u) \in \mathbb{R} \times H \text{ satisfies } (2.3), (4.21) \text{ with } \eta = \eta_0\\ &\text{and meas}\left\{(x,y) \in \overline{\Omega'} : \tfrac{u(x,y)}{\|u\|} \ge -\tfrac{\zeta}{2}\right\} < \theta, \text{ then } u < 0 \text{ in } \overline{\Omega'}.\end{aligned} \tag{4.25}$$

Furthermore, let us prove that there is $\eta \in (0, \eta_0)$ such that

$$\operatorname{meas}\left\{(x,y) \in \overline{\Omega'} : \frac{u(x,y)}{\|u\|} \geq -\frac{\zeta}{2}\right\} < \theta \text{ for all } (\lambda, u) \text{ satisfying } (2.3), (4.21). \tag{4.26}$$

Indeed, in the opposite case we would have a sequence u_n such that $\left\|\frac{u_n}{\|u_n\|} - u_0\right\| \to 0$ and $\operatorname{meas}\left\{(x,y) \in \overline{\Omega'} : u_n(x,y) \geq -\zeta\|u_n\|/2\right\} \geq \theta$. However, due to Jegorov Theorem there should be a measurable $M \subset \overline{\Omega'}$ such that $\operatorname{meas} M < \theta$ and $\frac{u_n}{\|u_n\|} \to u_0$ uniformly in $\overline{\Omega'} \setminus M$. This would contradict (4.23).

Now, it follows from (4.25) and (4.26) that $u < 0$ in $\overline{\Omega'}$ for all solutions (λ, u) to (2.3) satisfying (4.21). Due to (4.23) and Lemma 4.7 (with $\Omega' = \Omega^{\varepsilon_0}_{\gamma_1-2\mu,\gamma_2+2\mu} \setminus \Omega^{\varepsilon}_{\gamma_1-2\mu,\gamma_2+2\mu}$), for any ε we can choose $\eta \in (0, \eta_0)$ simultaneously so small that also $u < 0$ in $\overline{\Omega^{\varepsilon_0}_{\gamma_1-2\mu,\gamma_2+2\mu}} \setminus \Omega^{\varepsilon}_{\gamma_1-2\mu,\gamma_2+2\mu}$ for (λ, u) considered. Summarizing, for any $\varepsilon \in (0, \varepsilon_0)$ there is $\eta > 0$ such that

$$u < 0 \text{ in } \overline{\Omega^{\varepsilon_0}_{\gamma_1-2\mu,\gamma_2+2\mu}} \setminus (\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_1+\mu} \cup \Omega^{\varepsilon}_{\gamma_2-\mu,\gamma_2+\mu}) \text{ for all } (\lambda, u) \text{ satisfying } (2.3), (4.21). \tag{4.27}$$

Let u^+ denote the positive part of u. Due to (4.27), we have in particular

$$u^+ = 0 \quad \text{in } \Gamma^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu} \text{ for all } (\lambda, u) \text{ satisfying } (2.3), (4.21). \tag{4.28}$$

We will prove below that there is $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\|\nabla u^+\|_{L^2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu}\right)} \leq C\|u^+\|_{L^2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu}\right)} \quad \begin{array}{r}\text{for all } \varepsilon \in (0, \varepsilon_0) \text{ and } (\lambda, u) \in \mathbb{R} \times H \\ \text{satisfying } (2.3), (4.21) \text{ with } \eta \text{ such that } (4.27) \text{ holds.}\end{array} \tag{4.29}$$

First, let us assume that (4.29) is true, choose a fixed $\varepsilon \in \left(0, \min\left\{\frac{\pi}{2C^{1/2}}, \varepsilon_0\right\}\right)$ and the corresponding η such that (4.27) holds and let us show that then

$$u \leq 0 \text{ in } \Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu} \text{ for all } (\lambda, u) \text{ satisfying } (2.3), (4.21). \tag{4.30}$$

(In fact, only the inequality in $\Omega^{\varepsilon}_{\gamma_j-\mu,\gamma_j+\mu}$ is essential, $\Omega^{\varepsilon}_{\gamma_1+\mu,\gamma_2-\mu}$ is included already in (4.27).) Let us consider an arbitrary $(\lambda, u) \in \mathbb{R} \times H$ satisfying (2.3), (4.21) with η such that (4.27) holds. Therefore the inequality in (4.29) is true. On the other hand, due to (4.28) and Observation 4.8, also the inequality in (4.20) holds with $a = \gamma_1 - \mu$, $b = \gamma_2 + \mu$, $w = u^+$. However, both inequalities with $\varepsilon < \frac{\pi}{2C^{1/2}}$ can be simultaneously fulfilled only if $u^+ = 0$. Hence, (4.30) follows.

Now, let us prove (4.22) for all (λ, u) satisfying (2.3), (4.21) if η is small enough, where μ is such that (4.23) holds. (We know that u is continuous on $\Gamma_U \cup \Gamma_{N,\mu_0}$, e.g. Lemma 4.4.) First, we will show that

$$u(x, 0) < 0 \quad \text{for all } x \in (\gamma_1 - \mu, \gamma_1) \cup (\gamma_1, \gamma_2) \cup (\gamma_2, \gamma_2 + \mu) \tag{4.31}$$

for all (λ, u) satisfying (2.3), (4.21) if η is small enough. Let us assume by way of contradiction that there is (λ, u) satisfying (2.3), (4.21) and $u(x_0, 0) = 0$ for some $x_0 \in I := (\gamma_1 - \mu, \gamma_1) \cup$

$(\gamma_1,\gamma_2)\cup(\gamma_2,\gamma_2+\mu)$. Since the set $\{x\in I;\ u(x,0)=0\}$ is closed in I, it consists of a maximally countable set of closed (in I) intervals and isolated points. It follows easily that either it contains a closure of some open interval I_0 or it contains an isolated point x_0. In the former case, the classical normal derivative $\partial_\nu u(x,0)$ exists for any $x\in I_0$ by Remark 4.5 (with $\Gamma=I_0\times\{0\}$). In the latter case we have (4.14) with $\Gamma=I_0\times\{0\}$, I_0 being an open interval containing x_0 (see Remark 4.3). Hence, the normal derivative exists in the classical sense on Γ by the second part of Remark 4.5. Consider a sub-domain $\Omega'\subset\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}$ with $\overline{\Omega'}\subset(\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}\cup\Gamma_U\cup\Gamma_{N,\mu_0})$, $(x_0,0)\in\overline{\Omega'}$. We have $u\le 0$ in Ω' by (4.30). Hence, it follows from (1.1) and Remark 4.6 that $\Delta u\ge 0$ on Ω' if η is small enough, u attains its maximum over $\overline{\Omega'}$ at $(x_0,0)$ and the strong maximum principle implies $\partial_\nu u(x_0,0)>0$, which contradicts (1.2) or (1.3). Hence, (4.31) for all (λ,u) satisfying (2.3), (4.21) is proved.

It follows from (4.31) that (4.14) holds for all solutions under consideration with $\Gamma=(\gamma_1-\mu,\gamma_2+\mu)$, that means in particular $\partial_\nu u=0$ on Γ in the weak sense (cf. Remark 4.3). Now, Remark 4.5 (with $\Gamma=(\gamma_1-\mu_0,\gamma_2+\mu_0)\times\{0\}$) implies that $\partial_\nu u(\gamma_j,0)$ exists in the classical sense. If it were $u(\gamma_j,0)=0$ then it would be $\partial_\nu u(\gamma_j,0)>0$ by the maximum principle as above, which would contradict (1.2) or (1.3) again. Hence, (4.22) for all (λ,u) satisfying (2.3), (4.21) is proved. It follows from Remark 4.5 that $u\in C^{1,\gamma}(\overline{\Omega'})$ for any Ω' considered in the formulation of Lemma 4.9.

It remains to prove (4.29). First, we need to show that

$$\int_{\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}}|\nabla(w^+)|^2\,\mathrm{d}x\,\mathrm{d}y=\int_{\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}}\nabla w\cdot\nabla(w^+)\,\mathrm{d}x\,\mathrm{d}y\quad\text{ for all } w\in H. \tag{4.32}$$

If $w\in H$ is smooth then the set $\{(x,y)\in\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}\ :\ w(x,y)>0\}$ is open and w^+ coincides with w on a neighbourhood of any its point, i.e. also derivatives of w^+ coincide with those of w. Both integrands are zero on $\{(x,y)\in\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}\ :\ w(x,y)<0\}$. Further, $\nabla(w^+)=0$ in the points where $w=0$, $\nabla w=0$. Finally, the set $\{(x,y)\in\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}\ :\ w(x,y)=0,\ \nabla w(x,y)\neq 0\}$ is of measure zero because in a neighbourhood of any its point it forms a smooth curve due to the implicit function theorem. Hence, (4.32) holds for w smooth. For general $w\in H$ we get (4.32) via approximation by smooth functions.

Now, let us consider (λ,u) satisfying (2.3), (4.21) with η such that (4.27) holds. Let us define $\bar u=u^+$ in $\overline{\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}}$ and $\bar u=0$ in $\Omega\setminus\overline{\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}}$. We have $\bar u\in W^{1,2}(\Omega)$ by (4.27). Since $\bar u=u^+=0$ on Γ_U because of $u\in K$, we have $\pm\bar u\in K$ and we get by using (4.32), (2.3) with $\varphi=u\pm\bar u$ that

$$\begin{aligned}\textstyle\int_{\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}}|\nabla u^+|^2\,\mathrm{d}x\,\mathrm{d}y&=\textstyle\int_{\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}}\nabla u\cdot\nabla u^+\,\mathrm{d}x\,\mathrm{d}y=\int_\Omega\nabla u\cdot\nabla\bar u\,\mathrm{d}x\,\mathrm{d}y\\&=\textstyle\int_\Omega(\lambda u+g(\lambda,u)u)\bar u\,\mathrm{d}x\,\mathrm{d}y=\int_{\Omega^\varepsilon_{\gamma_1-\mu,\gamma_2+\mu}}\left(\lambda(u^+)^2+g(\lambda,u)(u^+)^2\right)\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{4.33}$$

There is $C>0$ such that $\|g(\lambda,u)\|_{L^r(\Omega)}\le C$ for all (λ,u), $|\lambda-\lambda_0|+\|u\|\le\eta_0$ with $r>2$ from the assumption (4.2). Let us set $q:=\frac{2r}{r-2}>2$, i.e. $1/r+1/q+1/2=1$. Since we have $u^+=0$

in $\Omega^{\varepsilon_0}_{\gamma_1-\mu,\gamma_2+\mu} \setminus \Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu}$ by (4.27), we obtain by using the embedding $W^{1,2}(\Omega^{\varepsilon_0}_{\gamma_1-\mu,\gamma_2+\mu}) \subset L^q(\Omega^{\varepsilon_0}_{\gamma_1-\mu,\gamma_2+\mu})$ that

$$\begin{aligned}\|u^+\|_{L^q\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_1+\mu}\right)} &= \left(\int_{\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu}} (u^+)^q \,\mathrm{d}x\,\mathrm{d}y\right)^{1/q} = \left(\int_{\Omega^{\varepsilon_0}_{\gamma_1-\mu,\gamma_2+\mu}} (u^+)^q \,\mathrm{d}x\,\mathrm{d}y\right)^{1/q} \\ &\le C\left(\int_{\Omega^{\varepsilon_0}_{\gamma_1-\mu,\gamma_2+\mu}} |\nabla u^+|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} = C\left(\int_{\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu}} |\nabla u^+|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} = C\|\nabla u^+\|_{L^2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_1+\mu}\right)}\end{aligned}$$

with C independent of the choice of ε. These estimates together with Hölder inequality and (4.20) (with $a = \gamma_1 - \mu, b = \gamma_2 + \mu$) imply that the last integral in (4.33) can be estimated by

$$\begin{aligned}(\lambda_0 + \eta)\|u^+\|^2_{L^2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_1+\mu}\right)} &+ \|g(\lambda,u)\|_{L^r(\Omega)} \cdot \|u^+\|_{L^q\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_1+\mu}\right)} \cdot \|u^+\|_{L^2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_1+\mu}\right)} \\ &\le C\|\nabla u^+\|_{L^2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_1+\mu}\right)} \cdot \|u^+\|_{L^2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_1+\mu}\right)}\end{aligned}$$

with C independent of $\varepsilon \in (0, \varepsilon_0)$. Hence, we obtain from (4.33) that

$$\|\nabla u^+\|^2_{L_2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu}\right)} \le C\|\nabla u^+\|_{L_2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu}\right)} \cdot \|u^+\|_{L_2\left(\Omega^{\varepsilon}_{\gamma_1-\mu,\gamma_2+\mu}\right)},$$

which means (4.29). ■

Remark 4.10 *For the proof of the last assertion of Theorem 4.2 we will use the stability criterion for variational inequalities [20, Theorem 1]. Formulating it for our concrete problem, we get the following assertion. If u_s is a stationary solution of (4.1) and*

$$\Lambda(s) := \limsup_{w+u_s\in K,\, \|w\|\to 0} \frac{1}{\int_\Omega w^2 \,\mathrm{d}x\,\mathrm{d}y} \langle -w+\lambda_s Aw+\partial_u G(\lambda_s,u_s)w-u_s+\lambda_s Au_s+G(\lambda_s,u_s), w\rangle \quad (4.34)$$

is negative then u_s is asymptotically stable in the $W^{1,2}(\Omega)$ norm. Let us note that in [20, Theorem 1], the opposite signs of all expressions are considered and therefore also lim inf *instead of* lim sup *appears. We consider our notation more natural in our situation.*

Remark 4.11 *It follows from Lemma 4.1 (see in particular (4.10)) that $1/\lambda_0 > 0$ is the largest and simple eigenvalue of the operator A, that means $\mu_0 = 1$ is the largest and simple eigenvalue of the operator $\lambda_0 A$. Let us denote the largest eigenvalue of the operator $\lambda_s A + G_u(\lambda_s, u_s)$ by μ_s ($s \in [0, s_0)$). It follows from the well-known variational characterization of the largest eigenvalue of a symmetric compact operator that*

$$\mu_s = \max_{\varphi\in H,\varphi\neq 0} \frac{\langle \lambda_s A\varphi + G_u(\lambda_s,u_s)\varphi, \varphi\rangle}{\|\varphi\|^2}, \quad (4.35)$$

$$\mu_0 = 1 = \max_{\varphi\in H,\varphi\neq 0} \frac{\lambda_0\langle A\varphi,\varphi\rangle}{\|\varphi\|^2}, \ \textit{i.e.}\ \lambda_0 = \min_{\varphi\in H,\varphi\neq 0} \frac{\|\varphi\|^2}{\langle A\varphi,\varphi\rangle}. \quad (4.36)$$

We have $\mu_s \to \mu_0 = 1$ for $s \to 0$.

Proof of Theorem 4.2. The operator A is linear, symmetric, the operator G satisfies (4.6) under our assumptions and $1/\lambda_0$ is a simple eigenvalue of the operator A (see Remark 4.11). It follows from Crandall-Rabinowitz type theorem (see e.g. [2, Theorem 1.7] or [15, Theorem I.5.1]) that there exists a smooth branch of nontrivial solutions of the problem (4.3) emanating from the trivial solutions at λ_0. More precisely, there exist $s_0 > 0$, $\eta > 0$ and C^1 maps $\hat{\lambda} : (-s_0, s_0) \to \mathbb{R}$, $\hat{v} : (-s_0, s_0) \to (u_0)^\perp$ such that $\hat{\lambda}(0) = \lambda_0$, $\hat{v}(0) = 0$ and the couple (λ, u) satisfies (4.3), (4.11) if and only if $(\lambda, u) = (\hat{\lambda}(s), \hat{u}(s)) =: (\lambda_s, u_s)$ with $\hat{u}(s) := s(u_0 + \hat{v}(s))$ for some $s \in (-s_0, s_0)$. We have $\left\| \frac{u_s}{\|u_s\|} - u_0 \right\| \to 0$ for $s \to 0_+$. Since u_s are solutions of (4.3), which is a weak formulation of a mixed boundary value problem, it follows from [12, Theorem 1] (cf. also [6, Remark 3.8]) that $\left\| \frac{u_s}{\|u_s\|} - u_0 \right\|_{C(\overline{\Omega})} \to 0$ for $s \to 0_+$. Hence, it follows from (4.8) that $u_s < 0$ on $\overline{\Gamma_U}$ and therefore $u_s \in K$ for any $s \in (0, s_0)$, s_0 small enough. It immediately follows that (λ_s, u_s) satisfies (2.3) for any such s.

Let us show that there is no solution $(\lambda, u) \in \mathbb{R} \times (H \setminus \{0\})$ to (2.3) in a neighbourhood of $(\lambda_0, 0)$ except of those (λ_s, u_s) lying on the smooth branch mentioned. By contradiction, let us have $(\lambda_n, u_n) \in \mathbb{R} \times (H \setminus \{0\})$ satisfying (2.3), $(\lambda_n, u_n) \to (\lambda_0, 0)$, $(\lambda_n, u_n) \neq (\lambda_s, u_s)$ for all $s \in (0, s_0)$, $w_n := \frac{u_n}{\|u_n\|} \rightharpoonup w$. Since A is compact and G satisfies (4.6), standard considerations of the bifurcation theory for variational inequalities give $w_n \to w \in K$ and (λ_0, w) satisfies (2.4) (see e.g. [17] or [19]). Due to the simplicity of λ_0 we obtain $w = u_0$, see Lemma 4.1. It follows that $(\lambda, u) = (\lambda_n, u_n)$ for n large satisfy (4.21) (with η from Lemma 4.9). Hence, Lemma 4.9 implies that $u_n(x, 0) < 0$ on $(\gamma_1 - \mu, \gamma_2 + \mu)$ for n large enough, and such (λ_n, u_n) satisfy also (4.3). The properties of the bifurcation branch for the problem (4.3) mentioned at the beginning of the proof gives that for any n large enough there is $s \in (0, s_0)$ such that $(\lambda_n, u_n) = (\lambda_s, u_s)$ (negative s are excluded by the fact that $u_0 \notin (-K)$). This contradiction proves our assertion.

Since $u_s - \lambda_s A u_s - G(\lambda_s, u_s) = 0$ for all $s \in (0, s_0)$, we get for the value $\Lambda(s)$ introduced in (4.34) (see Remark 4.10)

$$\begin{aligned}
\Lambda(s) &= \limsup_{w + u_s \in K,\, \|w\| \to 0} \frac{1}{\int_\Omega w^2 \, \mathrm{d}x \, \mathrm{d}y} \langle -w + \lambda_s A w + \partial_u G(\lambda_s, u_s) w, w \rangle \\
&\le \sup_{\varphi \in H, \varphi \neq 0} \frac{1}{\int_\Omega \varphi^2 \, \mathrm{d}x \, \mathrm{d}y} \langle -\varphi + \lambda_s A \varphi + \partial_u G(\lambda_s, u_s) \varphi, \varphi \rangle \\
&= \sup_{\varphi \in H, \varphi \neq 0} \frac{1}{\langle A\varphi, \varphi \rangle} \langle -\varphi + \lambda_s A \varphi + \partial_u G(\lambda_s, u_s) \varphi, \varphi \rangle \\
&= \sup_{\varphi \in H, \varphi \neq 0} \frac{\langle -\varphi + \lambda_s A \varphi + \partial_u G(\lambda_s, u_s) \varphi, \varphi \rangle}{\|\varphi\|^2} \cdot \frac{\|\varphi\|^2}{\langle A\varphi, \varphi \rangle}.
\end{aligned} \tag{4.37}$$

It follows from well-known bifurcation formulas (see, e.g. [15, Sections I.6, I.7]) that the first eigenvalue $-1 + \mu_s$ of the operator $-I + \lambda_s A + \partial_u G(\lambda_s, u_s)$ is negative for all small $s > 0$ if and only if $\lambda_s > \lambda_0$ for all small $s > 0$. In particular, we have

$$-1 + \mu_s < 0 \tag{4.38}$$

under the assumption (3.12) (cf. Theorem 3.1). It follows from (4.35), (4.38), (4.36) that the last term in (4.37) can be estimated by

$$\sup_{\varphi\in H,\varphi\neq 0}(-1+\mu_s)\cdot\frac{\|\varphi\|^2}{\langle A\varphi,\varphi\rangle}=(-1+\mu_s)\inf_{\varphi\in H,\varphi\neq 0}\frac{\|\varphi\|^2}{\langle A\varphi,\varphi\rangle}=(-1+\mu_s)\lambda_0<0.$$

Hence, $\Lambda(s)<0$ for all small $s>0$, and the asymptotical stability of u_s as a solution of our variational inequality (4.1) follows from Remark 4.10. ■

Theorem 4.12 *Let λ_0 be the smallest eigenvalue of the variational inequality (2.4). Then the trivial solution of (4.1) is asymptotically stable in the norm of the space $W^{1,2}(\Omega)$ for $\lambda<\lambda_0$ and unstable for $\lambda>\lambda_0$.*

Remark 4.13 *We will use again the stability criterion for variational inequalities [20, Theorem 1], which gives for our situation the following assertion (cf. also Remark 4.10). If*

$$\widetilde{\Lambda}(\lambda):=\sup_{u\in K}\frac{1}{\int_\Omega u^2\,\mathrm{d}x\,\mathrm{d}y}\langle -u+\lambda Au,u\rangle=\sup_{u\in K}\left(-\frac{\int_\Omega|\nabla u|^2-\lambda u^2\,\mathrm{d}x\,\mathrm{d}y}{\int_\Omega u^2\,\mathrm{d}x\,\mathrm{d}y}\right)<0 \qquad (4.39)$$

then the trivial solution of (4.1) is asymptotically stable in the $W^{1,2}(\Omega)$ norm.

On the other hand, [20, Theorem 2] implies for our situation that if $\widetilde{\Lambda}(\lambda)>0$ then the trivial solution of (4.1) is unstable. Let us emphasize that in this second assertion it is important that K is the cone with its vertex at the origin, and therefore cannot be used for the proof of instability of u_s, because the corresponding set $K-u_s$ is not a cone with its vertex at the origin.

Proof of Theorem 4.12. Due to the variational characterization of the smallest eigenvalue λ_0 of (2.4) (see (4.9) in the proof of Lemma 4.1) we have

$$\widetilde{\Lambda}(\lambda)=\sup_{u\in K,u\neq 0}\frac{-\|u\|^2}{\langle Au,u\rangle}+\lambda=-\lambda_0+\lambda.$$

Our assertion follows by using Remark 4.13. ■

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