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Preprint No. 51-2017

PRAHA 2017

Eigenvalues and bifurcation for problems with positively homogeneous operators and reaction-diffusion systems with unilateral terms

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Abstract

Reaction-diffusion systems satisfying assumptions guaranteeing Turing's instability and supplemented by unilateral terms of type v^- and v^+ are studied. Existence of critical points and sometimes also bifurcation of stationary spatially non-homogeneous solutions is proved for rates of diffusions for which it is excluded without any unilateral term. The main tool is a general result giving a variational characterization of the largest eigenvalue for positively homogeneous operators in a Hilbert space satisfying a condition related to potentiality, and existence of bifurcation for equations with such operators. The originally non-variational (non-symmetric) system is reduced to a single equation with a positively homogeneous potential operator and the abstract results mentioned are used.

Keywords: positively homogeneous operators, maximal eigenvalue, variational characterization, global bifurcation, reaction-diffusion systems, unilateral sources

1. Introduction

The original goal of this paper was a study of an influence of unilateral terms of type v^-, v^+ to bifurcation of stationary spatially non-homogeneous solutions of reaction-diffusion systems exhibiting Turing's diffusion driven instability. The systems discussed have the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= d_1\triangle u + b_{11}u + b_{12}v + n_1(u,v)\\ \frac{\partial v}{\partial t} &= d_2\triangle v + b_{21}u + b_{22}v + n_2(u,v) + \hat{g}_-(x,v^-) - \hat{g}_+(x,v^+)\end{aligned} \quad \text{in } \Omega\times[0,\infty), \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^m$ with a Lipschitz boundary, d_1, d_2 are positive parameters, b_{ij} are real constants, $n_1, n_2 : \mathbb{R}^2 \to \mathbb{R}$ are small nonlinear perturbations, v^+, v^- denote the positive and negative part of v, respectively, and $\hat{g}_-, \hat{g}_+ : \Omega\times[0,\infty) \to \mathbb{R}$ are functions describing certain unilateral sources and sinks, see below for more details. However, for our approach we needed a variational characterization of the largest eigenvalue of a compact positively homogeneous operator and existence of bifurcation for equations of type

$$\lambda u - Su + B(u) - N(u) = 0, \tag{2}$$

where S is a linear compact symmetric operator in a Hilbert space, B is a compact positively homogeneous operator and N is a small compact nonlinear perturbation. These results perhaps

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can be of a separate interest and therefore they are given in an abstract form in a separate self-contained Section 4. For a variational characterization of the largest eigenvalue of a compact positively homogeneous operator B we need a certain additional assumption, namely the condition (51), which is related to potentiality. For the proof of existence of bifurcation for the equation mentioned above we need an odd multiplicity of the largest eigenvalue of S and B is supposed to be small.

Reaction diffusion system (1) will be always supplemented by mixed boundary conditions

$$\begin{aligned} &\frac{\partial u}{\partial \vec{\nu}} = \frac{\partial v}{\partial \vec{\nu}} = 0 \quad \text{on } \Gamma_N \\ &u = v = 0 \quad \text{on } \Gamma_D, \end{aligned} \tag{3}$$

where $\vec{\nu}$ is the outer unit normal to the boundary $\partial\Omega$, $\Gamma_N, \Gamma_D \subset \partial\Omega$ are disjoint subsets of $\partial\Omega$ satisfying

$$\text{meas}_{m-1}\, \Gamma_D > 0, \quad \text{meas}_{m-1}(\partial\Omega \backslash (\Gamma_D \cup \Gamma_N)) = 0 \tag{4}$$

(the $(m-1)$-dimensional Lebesgue measure). In fact, the original model should describe a biochemical reaction of two morphogens having a positive constant equilibrium $\bar{u}$, $\bar{v}$. Shifting this positive steady state to zero, we can write the equations in the form (1), where u and v denote deviations of concentrations of the morphogens from the values $\bar{u}$, $\bar{v}$, not concentrations themselves. We will always suppose that n_j are continuously differentiable and

$$n_j(0,0) = \frac{\partial n_j}{\partial u}(0,0) = \frac{\partial n_j}{\partial v}(0,0) = 0, \ j = 1, 2, \tag{5}$$

$$\begin{aligned} \det B := b_{11}b_{22} - b_{12}b_{21} > 0, \quad b_{11} + b_{22} < 0, \\ b_{11} > 0, \quad b_{22} < 0, \quad b_{12}b_{21} < 0. \end{aligned} \tag{6}$$

It is known that under the assumptions (5), (6), in the case $g_\pm = 0$ the trivial solution of the corresponding system without diffusion, i.e. ODE's obtained from (1) for $d_1 = d_2 = 0$, is asymptotically stable, but the trivial solution of the full system (1), (3) is unstable for d_1, d_2 from a certain open subset D_U of the positive quadrant $\mathbb{R}^2_+$ (Turing instability), and stable only for $(d_1, d_2) \in D_S = \mathbb{R}^2_+ \setminus \overline{D}_U$. See e.g. [13], [14], [5]. Our goal is to prove that for the problem with non-trivial $g_\pm$, there exist global bifurcations of spatially non-homogeneous stationary solutions in the domain D_S, where this is impossible in the case $g_\pm = 0$.

The unilateral terms $g_-(x, v^-)$ and $g_+(x, v^+)$ can model a unilateral source and sink, which is active only in points x and times t where $v(t, x) < 0$ and $v(t, x) > 0$, that means where the concentration of the second morphogen is less and larger, respectively, than $\bar{v}$. We will assume in the whole paper that $\hat{g}_-, \hat{g}_+ : \Omega \times [0, \infty) \to \mathbb{R}$ are functions satisfying Carathéodory conditions, having a derivative with respect to the second variable at zero for a.a. x $\in \Omega$ and

$$\hat{g}_\pm(x, 0) \equiv 0, \qquad \left.\frac{\partial \hat{g}_\pm(x, \xi)}{\partial \xi}\right|_{\xi=0} = s_\pm(x) \quad \text{for a.a. } x \in \Omega, \tag{7}$$

where

$$\begin{aligned} &s_+ \in L^\infty(\Omega), \ s_+(x) \geq 0 \text{ for a.a. } x \in \Omega, \ s_+(x) > 0 \text{ for a.a. } x \in \Omega_+, \\ &s_- \in L^\infty(\Omega), \ s_-(x) \geq 0 \text{ for a.a. } x \in \Omega, \ s_-(x) > 0 \text{ for a.a. } x \in \Omega_-, \\ &\Omega_+, \Omega_- \text{ being open subsets of } \Omega, \text{ not both coinciding with } \Omega. \end{aligned} \tag{8}$$

A typical example is $\hat{g}_\pm(x, \xi) = s_\pm(x)\xi/(1 + \xi)$, i.e.

$$\hat{g}_-(x, v^-) = s_-(x)\frac{v^-}{1 + v^-}, \quad \hat{g}_+(x, v^-) = s_+(x)\frac{v^+}{1 + v^+}.$$

A similar problem was discussed in [4]. However, our current results are in some sense complementary to those from [4], as explained in Section 2, Remark 5. Let us note that an influence of unilateral sources or sinks described by variational or quasi-variational inequalities was studied already in a series of papers, see e.g. [2], [11], [15], [10], [7], [12], [8] and the references therein. However, variational inequalities model sources and sinks with an infinite power which prevent any decrease and increase below and above, respectively, the value mentioned, which is hardly to imagine in a nature. The unilateral terms considered in the current paper and in [4] seem to be more natural.

A basic step is the proof of existence of critical points, that means couples $d_1,\ d_2$ for which the homogenized stationary problem

$$\begin{aligned} &d_1\triangle u{+}b_{11}u + b_{12}v = 0,\\ &d_2\triangle v{+}b_{21}u + b_{22}v + s_-(x)v^- - s_+(x)v^+ = 0 \end{aligned} \tag{9}$$

with (3) has a nontrivial solution lying in D_S. The main idea is to write the weak formulation in the form of a system of operator equations in a subspace of the Sobolev space, to express u from the first equation, substitute it to the second equation, to obtain in this way a single equation with a positively homogeneous operator, and to use abstract results mentioned above. This idea was used for the case of unilateral sources described by variational inequalities already in [11], [10], [1], [8]. For a study of bifurcation we combine these ideas with a use of Leray-Schauder degree (a jump of a degree implies bifurcation).

Let us note that unilateral sources and sinks for u (an activator) have an opposite influence, they reduce the region where bifurcation of stationary solutions can arise, see [10] for the case of quasi-variational inequalities.

The results concerning critical and bifurcation points of reaction diffusion systems with unilateral sources and sinks are formulated and discussed in Section 2. Section 3 is devoted to the reduction of a reaction-diffusion system to a single operator equation. This makes possible to use abstract results of Section 4 concerning eigenvalues and bifurcations for equations with positively homogeneous operators to the proofs of the main results of Section 2, which are given in Section 5. For the completeness, in Appendix we prove the C^1 smoothness of a map N_1 important for the use of Implicit Function Theorem in Section 3 and explain a general global bifurcation result used in Section 4.

The authors express theirs thanks to Lutz Recke for nice and fruitful discussions during his one week stay in Prague, which substantially improved the final formulation of the condition (51) and Theorem 7 in Appendix.

2. Main results for systems with unilateral sources and sinks

2.1. Systems with unilateral sources and sinks in the interior of the domain

We are interested in stationary solutions of the system (1), (3), i.e. we will study the system

$$\begin{aligned} &d_1\triangle u + b_{11}u + b_{12}v + n_1(u,v) = 0,\\ &d_2\triangle v + b_{21}u + b_{22}v + n_2(u,v) + \hat{g}_-(x,v^-) - \hat{g}_+(x,v^+) = 0 \end{aligned} \tag{10}$$

with boundary conditions (3), n_1, n_2 satisfying (5) and $\hat{g}_\pm$ satisfying Caratheodory conditions and (7), (8). By a solution we will always mean a weak solution $(u,v) \in \mathbb{H}_D \times \mathbb{H}_D$, where

$$\mathbb{H}_D := \{\varphi \in W^{1,2}(\Omega)|\ \varphi|_{\Gamma_D} = 0 \ \text{ in the sense of traces}\},$$

see also Observation 2 in Section 3. The space $\mathbb{H}_D$ will be equipped with the scalar product and the norm

$$\langle v,\varphi\rangle = \int_\Omega \nabla v\cdot\nabla\varphi \ \mathrm{dx} \ \text{ and } \ \|v\| = \left(\int_\Omega |\nabla v|^2 \ \mathrm{dx}\right)^{\frac{1}{2}} \quad \text{for all } v,\varphi\in\mathbb{H}_D. \tag{11}$$

It is well known that under the assumption (4), this norm is equivalent with the usual Sobolev norm. We will assume that there exists $C > 0$ such that

$$\begin{aligned} &|n_i(\chi,\zeta)| \le C(1+|\chi|^{p-1}+|\zeta|^{p-1}) \text{ for } i=1,2 \text{ and all } \zeta,\chi\in\mathbb{R}, \\ &|\hat{g}_\pm(x,\xi)| \le C(1+|\xi|^{p-1}) \text{ for a.a. } x\in\Omega \text{ and all } \xi\in\mathbb{R}, \end{aligned} \tag{12}$$

$$\left|\frac{\partial n_1}{\partial\chi}(\chi,\zeta)\right| + \left|\frac{\partial n_1}{\partial\zeta}(\chi,\zeta)\right| \le C(1+|\chi|^{p-2}+|\zeta|^{p-2}) \text{ for all } \chi,\zeta\in\mathbb{R} \tag{13}$$

with some p satisfying

$$p > 2 \text{ if } m=2 \text{ or } 2 < p < \frac{2m}{m-2} \text{ if } m > 2. \tag{14}$$

For the case $m = 1$ we do not need (12), (13) and we can formally set $p = \infty$. Let us emphasize that (13) is supposed only for n_1, not n_2. The reason will be clear in Section 3. We will also discuss the problem with $\hat{g}_-(x, v^-)$, $\hat{g}_+(x, v^+) \equiv 0$, i.e. the system

$$\begin{aligned} d_1\triangle u + b_{11}u + b_{12}v + n_1(u,v) = 0, \\ d_2\triangle v + b_{21}u + b_{22}v + n_2(u,v) = 0 \end{aligned} \tag{15}$$

with boundary conditions (3), and the corresponding linearization

$$\begin{aligned} d_1\triangle u + b_{11}u + b_{12}v = 0, \\ d_2\triangle v + b_{21}u + b_{22}v = 0, \end{aligned} \tag{16}$$

again with b.c. (3).

Definition 1. *The points $d = (d_1, d_2) \in \mathbb{R}^2$ for which the problem (16), (3) or (9), (3) has a nontrivial (weak) solution are called critical points of (16), (3) or (9), (3), respectively.*

By a bifurcation point of (15), (3) or (10), (3) we mean a point (d_1^b, d_2^b) for which in any neighborhood of $(d_1^b, d_2^b, 0, 0)$ in $\mathbb{R}^2 \times \mathbb{H}_D \times \mathbb{H}_D$ there exists (d_1, d_2, u, v) with $(u, v) \neq 0$ satisfying (in the weak sense) (15), (3) or (10), (3), respectively.

Standard considerations imply that any bifurcation point is simultaneously a critical point, cf. also Lemma 2 in Section 4.

The set of all critical points of the problem (16), (3) is known, and those lying in the positive quadrant form curves C_k introduced below. One of our goals is to show that critical points of (9), (3) are under some assumptions also in a domain where it is impossible for the problem (16), (3).

However, in order to translate our problem to a variational form, we will fix the value $d_1 \in (0, y_1)$, $d_1 \neq y_j := b_{11}/\kappa_j$ for all $j \in \mathbb{N}$, that means only d_2 will be a parameter. In this case we will deal with critical and bifurcation points in the sense of the following Definition 2, cf. [1], [8].

Definition 2. *A parameter $d_2 \in \mathbb{R}$ is a critical point of (9), (3) or (16), (3) with fixed d_1 if there exists a (weak) solution $(u, v) \neq (0, 0)$ of (9), (3) or (16), (3), respectively.*

By a bifurcation point of (15), (3) or (10), (3) with fixed d_1 we mean a point d_2^b for which in any neighborhood of $(d_2^b, 0, 0)$ in $\mathbb{R} \times \mathbb{H}_D \times \mathbb{H}_D$ there exists (d_2, u, v) with $(u, v) \neq 0$ satisfying (in the weak sense) (15), (3) or (10), (3), respectively.

Notation 1. *Let us denote $\{\kappa_k\}_{k\in\mathbb{N}}$, $0 < \kappa_1 < \kappa_2 \le \cdots$, the set of all eigenvalues of the operator $-\triangle$ with boundary conditions (3). We choose an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of the Hilbert space $\mathbb{H}_D$ so that for any $k \in \mathbb{N}$ the function e_k is an eigenfunction corresponding to the eigenvalue κ_k. Let us note that the first eigenvalue κ_1 is simple and the corresponding eigenfunction e_1 does not change sign in Ω.*

It is known that the set of all critical points (d_1, d_2) of the linear problem (16), (3) is $\bigcup_{i=1}^{\infty} \tilde{C}_k$, where

$$\tilde{C}_k = \left\{ d = (d_1, d_2) \in \mathbb{R}^2 \;\middle|\; d_2 := \frac{1}{\kappa_k}\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right)\right\},$$

see [13] or [14] for 1D case or [5] for the general case.

For each $k \in \mathbb{N}$ the set $\tilde{C}_k$ is a hyperbola in $\mathbb{R}^2$. We are interested only in (d_1, d_2) lying in the positive quadrant, i.e. we consider the sets

$$C_k = \left\{ d = (d_1, d_2) \in \mathbb{R}^2_+ \,\middle|\, d_2 := \frac{1}{\kappa_k}\left(\frac{b_{12}b_{21}}{d_1\kappa_k - b_{11}} + b_{22}\right)\right\}.$$

The envelope of the curves C_k will be denoted as C_E. A vertical asymptote y_k of C_k is

$$\left\{ (d_1, d_2) \in \mathbb{R}^2 \,\middle|\, d_1 = y_k := \frac{b_{11}}{\kappa_k} \right\}. \tag{17}$$

The sets C_k are black-colored lines in Fig. 1 on p. 8. We define two sets D_U and D_S as

$$D_U = \left\{ (d = d_1, d_2) \in \mathbb{R}^2_+ \mid \ d \text{ lies to the left from at least one } C_k,\ k \in \mathbb{N}, \text{ i.e. from } C_E \right\},$$

$$D_S = \left\{ d = (d_1, d_2) \in \mathbb{R}^2_+ \mid \ d \text{ lies to the right from all } C_k,\ k \in \mathbb{N}, \text{ i.e. from } C_E \right\}.$$

According to the following remark, the sets D_S and D_U are called the domain of stability and domain of instability, respectively.

Remark 1. *Let us consider an eigenvalue problem*

$$\begin{aligned} d_1\triangle u + b_{11}u + b_{12}v &= \lambda u, \\ d_2\triangle v + b_{21}u + b_{22}v &= \lambda v \end{aligned}$$

with b.c. (3). If $d \in D_S$ then there exists $\varepsilon > 0$ such that Re $\lambda < -\varepsilon < 0$ for all eigenvalues of this problem, and if $d \in D_U$, then there exists at least one real eigenvalue $\lambda > 0$, see [13] or [14] for 1D case or [5] for the general case. Hence, for $d \in D_S$ or $d \in D_U$ the trivial solution of (1), (3) with $\hat{g}_\pm \equiv 0$ is linearly stable or unstable, respectively.

If $\hat{g}_1,\ \hat{g}_2 \not\equiv 0$, then it is not possible to describe analytically analogues of D_S, D_U.

Our main results concerning reaction-diffusion systems are Theorems 1 - 3. The most essential are the last assertions of these theorems, which state that there is a critical point and a bifurcation, respectively, in the domain D_S where neither critical nor bifurcation points can exist in the classical case without unilateral sources and sinks (i.e. with $g_+, g_- \equiv 0$). The existence of bifurcation in D_S is proved only for small s_+, s_- while the existence of critical points is proved for arbitrary nontrivial sources and sinks.

Notation 2. *We will consider fixed $d_1 \in (0, y_1),\ d_1 \neq y_j := b_{11}/\kappa_j$ for all $j \in \mathbb{N}$, and numbers*

$$\lambda_j^S := \frac{b_{12}b_{21}}{d_1\kappa_j^2 - b_{11}\kappa_j} + \frac{b_{22}}{\kappa_j}, \qquad d_2^0 := \max_{j\in\mathbb{N}} \lambda_j^S,$$

$$d_2^m := \sup_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\sum_{j=1}^{\infty} \lambda_j^S \xi_j^2 + \sum_{j=1}^{\infty} \xi_j \int_\Omega \left(\left(\sum_{k=1}^{\infty} \xi_k e_k\right)^- s_- - \left(\sum_{k=1}^{\infty} \xi_k e_k\right)^+ s_+ \right) e_j \ dx}{\sum_{j=1}^{\infty} \xi_j^2}. \tag{18}$$

We define the multiplicity of d_2^0 as the multiplicity of the eigenvalue κ_{j_0} of $-\triangle$, where j_0 is such that the maximum in the definition of d_2^0 is attained for $j = j_0$.

Let us define a convex cone

$$K := \{v \in \mathbb{H}_D \big|\ s_-(x)v^-(x) - s_+(x)v^+(x) = 0 \text{ for a.a. } x \in \Omega\}.$$

The numbers λ_j^S are in fact defined also for parameters $d_1 > y_1$. However, in such a case it follows from (6) that $\lambda_j^S < 0$ for all $j \neq \mathbb{N}$. Moreover, because $\lim_{j\to\infty} \kappa_j = \infty$, we get

$$\sup_{j\in\mathbb{N}} \lambda_j^S = \lim_{j\to\infty} \lambda_j^S = 0,$$

and no maximizer exists. For this case our method cannot give an existence of positive critical and bifurcation points. We are interested only in positive diffusion rates, therefore the parameters $d_1 > y_1$ will not be of our interest.

Later we will see in Sections 3 and 5 that if d_2^m is positive, then it is the largest eigenvalue of a certain compact positively homogeneous operator characterizing the problem (9), (3). The numbers λ_j^S are eigenvalues of a certain linear compact symmetric operator S characterizing the problem (16), (3). Under the assumptions (6) and $d_1 \in (0, y_1)$, $d_1 \neq y_j$ for all $j \in \mathbb{N}$ the number d_2^0 is its largest eigenvalue and simultaneously the largest critical point of (16), (3) with fixed d_1. For more details see Section 3.

Under the assumptions (8) the cone K contains functions which are nonnegative on Ω_- and nonpositive on Ω_+ and zero elsewhere. In particular, K contains nonzero elements.

Theorem 1. *Let (4), (6), (8) be true and let d_1 be fixed, $d_1 \in (0, y_1)$, $d_1 \neq y_j$ for all $j \in \mathbb{N}$. The number d_2^m from (18) can be estimated as*

$$d_2^0 \geq d_2^m \geq \max\left\{ \sup_{j\in\mathbb{N}} \left(\lambda_j^S - \frac{\|s_-\|_{L^\infty} + \|s_+\|_{L^\infty}}{\kappa_j} \right), \sup_{\substack{\{\xi_j\}\in\ell^2\setminus\{0\} \\ \sum \xi_j e_j \in K}} \frac{\sum\limits_{j=1}^{\infty} \lambda_j^S \xi_j^2}{\sum_{i=1}^{\infty} \xi_i^2} \right\}. \tag{19}$$

If d_2^m is positive, then the supremum in (18) is maximum, i.e.

$$d_2^m = \max_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\sum\limits_{j=1}^{\infty} \lambda_j^S \xi_j^2 + \sum\limits_{j=1}^{\infty} \xi_j \int_\Omega \left(\left(\sum\limits_{k=1}^{\infty} \xi_k e_k \right)^- s_- - \left(\sum\limits_{k=1}^{\infty} \xi_k e_k \right)^+ s_+ \right) e_j \; dx}{\sum\limits_{j=1}^{\infty} \xi_j^2}, \tag{20}$$

and it is the largest critical point of the system (9), (3) with fixed d_1. If $d_1 \in (y_2, y_1)$, $\|s_-\|_{L^\infty} > 0$, $\|s_+\|_{L^\infty} > 0$ and $d_2^m > 0$, then $(d_1, d_2^m) \in D_S$.

The proof is postponed to Section 5. Let us note that if the first supremum in (19) is positive, then it is the maximum. This follows from the fact that $\lim_{j\to\infty} \lambda_j^S = 0$, $\lim_{j\to\infty} \kappa_j = \infty$. Similarly, if the supremum over K in (19) is positive, then it is the maximum, see Theorem 3.2 in [1].

Remark 2. *If $d_1 \in (0, y_1)$, $d_1 \neq y_j$ for all $j \in \mathbb{N}$, $\|s_\pm\|_{L^\infty}$ are sufficiently small, then it follows from (6), Notation 2 and (19) that d_2^m from (18) is positive. If, moreover, $d_1 \in (y_2, y_1)$ and $\|s_\pm\|_{L^\infty} > 0$, then $(d_1, d_2^m) \in D_S$ by the last assertion of Theorem 1.*

The following conclusion follows from Remark 3.4 in [1]:

There exists $\varepsilon > 0$ such that if $d_1 \in [y_1 - \varepsilon, y_1)$ then the supremum over K in (19) is positive. Hence, d_2^m in (18) is positive. The only assumption for application of the remark mentioned is an existence of $v \in K$ with $\langle v, e_1 \rangle \neq 0$. However, any v positive on Ω_+ or on Ω_- and zero elsewhere satisfies this condition because

$$\langle v, e_1 \rangle = \int_\Omega \nabla v \nabla e_1 \; dx = \kappa_1 \int_\Omega v e_1 \; dx \neq 0.$$

Hence, if $\|s_\pm\|_{L^\infty} > 0$ and $d_1 \in [y_1 - \varepsilon, y_1)$ then $(d_1, d_2^m) \in D_S$ by the last assertion of Theorem 1.

Let us emphasize that the bounds in (19) can be found explicitly for particular systems.

Let $d_1 \in (0, y_1),\ d_1 \neq y_j$ for all $j \in \mathbb{N}$ be fixed. We denote

$$\mathcal{S} = \overline{\{(d_2, u, v) \in \mathbb{R} \times \mathbb{H}_D \times \mathbb{H}_D \mid (u, v) \neq 0,\ (u, v) \text{ is a solution of (10), (3)}\}}.$$

Theorem 2. *Let (4)–(8), (12), (13) be true, let $d_1 \in (0, y_1),\ d_1 \neq y_j$ for all $j \in \mathbb{N}$, let the multiplicity of d_2^0 be odd. Then for any sufficiently small $\varepsilon > 0$ there exists $\tau_s > 0$ such that if $\|s_-\|_{L^\infty}, \|s_+\|_{L^\infty} \in [0, \tau_s)$ then $d_2^0 - \varepsilon < d_2^m$ and there is a global bifurcation point $d_2^b \in [d_2^0 - \varepsilon, d_2^m]$ of the system (10), (3) with fixed d_1 in the following sense. The connected component $\mathcal{S}_{d_2^b}$ of $\mathcal{S}$ containing the point $(d_2^b, 0)$ satisfies at least one of the following conditions:*

(a) *$\mathcal{S}_{d_2^b}$ is unbounded,*

(b) *there exists $(u, v) \in \mathbb{H}_D \times \mathbb{H}_D,\ (u, v) \neq 0$ such that $(0, u, v) \in \overline{\mathcal{S}_{d_2^b}}$,*

(c) *there exists a critical point $d_2^c \notin [d_2^0 - \varepsilon, d_2^m]$ of (9), (3) with fixed d_1 such that $(d_2^c, 0, 0) \in \overline{\mathcal{S}_{d_2^b}}$.*

If, moreover, $d_1 \in (y_2, y_1)$, $\|s_+\|_{L^\infty}, \|s_-\|_{L^\infty} > 0$ then $(d_1^0, d_2^b) \in D_S$.

The proof is postponed to Section 5. It will be seen from it that "sufficiently small ε" means $\varepsilon \in (0, \min(d_2^0, (d_2^0 - d_2^2)/2))$, where d_2^2 is the second largest critical point of the system (16), (3) with fixed d_1. Especially if $d_1 \in (y_2, y_1)$, then $d_2^2 < 0$ and therefore $(d_2^0 - d_2^2)/2 > d_2^0/2$. Thus ε can be taken from the interval $(0, d_2^0/2)$. See also Fig. 1 and Observation 3.

If $s_\pm \equiv 0$, then it is known that the global bifurcation is exactly at the point $d_2^m = d_2^0$.

If $n_1(u, v) \equiv n_1(v)$, then we do not have to suppose the condition (13), see also Remark 6 in Section 3.

Remark 3. *In the proof of the last assertion of Theorem 1 we will use the fact that the first eigenvalue κ_1 of the Laplacian is simple and the eigenfunction e_1 does not change its sign in Ω. Under the assumption $\|s_+\|_{L^\infty}, \|s_-\|_{L^\infty} > 0$ it means that $e_1 \notin K$. Under a more general assumption*

$$e \notin K \ \text{ for all eigenfunctions } e \text{ corresponding to the eigenvalue } \kappa_{j_0},\ j_0 \text{ such that } d_2^0 = \lambda_{j_0}^S, \tag{21}$$

the proof in Section 5 can be modified to get $d_2^m < d_2^0$. However, in the case $k > 1$ it does not imply that $(d_1, d_2^m) \in D_S$ because the point (d_1, d_2^0) can lie above the hyperbolas C_j with $j \leq k$, see Observation 3 in Section 3. Therefore the case $k > 1$ is not included in the last statements of Theorems 1, 2.

The situation for small $\|s_\pm\|_{L^\infty}$ is sketched in Fig. 1. Black lines are the curves C_k. Grey lines are lower bounds to the largest critical points of the system (9), (3) given by the expressions in (19), which continuously depend on d_1. Grey filling marks an area in D_S containing critical points and bifurcations of the problem (9), (3) and (10), (3), respectively, see Remark 2.

Remark 4. *Applying Theorem 1.1 from [17], we can arrive at the following conclusions. Let (4)–(8), (12), (13) for both $n_1,\ n_2$ be fulfilled, let $d_1 \in (0, y_1), d_1 \neq y_j$ for all $j \in \mathbb{N}$, let $d_2 \in \mathbb{R}$ be an arbitrary critical point (not necessarily the largest one) with the multiplicity one of the system (16), (3) with fixed d_1 and let (u_0, v_0) denote the associated nontrivial solution. Let $s_\pm$ be fixed. Then there are $\tau_0 > 0$ and Lipschitz continuous mappings $d_{2+}, d_{2-} : [0, \tau_0) \to \mathbb{R}$, $u_+, v_+, u_-, v_- : [0, \tau_0) \to \mathbb{H}_D$ such that for any $\tau \in (0, \tau_0)$ the numbers $d_{2+}(\tau)$, $d_{2-}(\tau)$ are in a certain neighborhood of d_2 the only critical points of*

$$d_1 \triangle u + b_{11} u + b_{12} v = 0 \tag{22}$$

$$d_2 \triangle v + b_{21} u + b_{22} v + \tau(s_-(x) v^- - s_+(x) v^+) = 0 \tag{23}$$

with b.c. (3) and with fixed d_1, the functions $(u_+(\tau), v_+(\tau))$, $(u_-(\tau), v_-(\tau))$ are the associated normalized nontrivial solutions, $v_+(0) = v_0$, $v_-(0) = -v_0$, $d_{2+}(0) = d_{2-}(0) = d_2$. Moreover, $d_{2\pm}(\tau)$ are simultaneously bifurcation points of the system

$$\begin{aligned} &d_1 \triangle u + b_{11} u + b_{12} v + n_1(u, v) = 0 \\ &d_2 \triangle v + b_{21} u + b_{22} v + n_2(u, v) + \tau(\hat{g}_-(x, v^-) - \hat{g}_+(x, v^+)) = 0 \end{aligned} \tag{24}$$

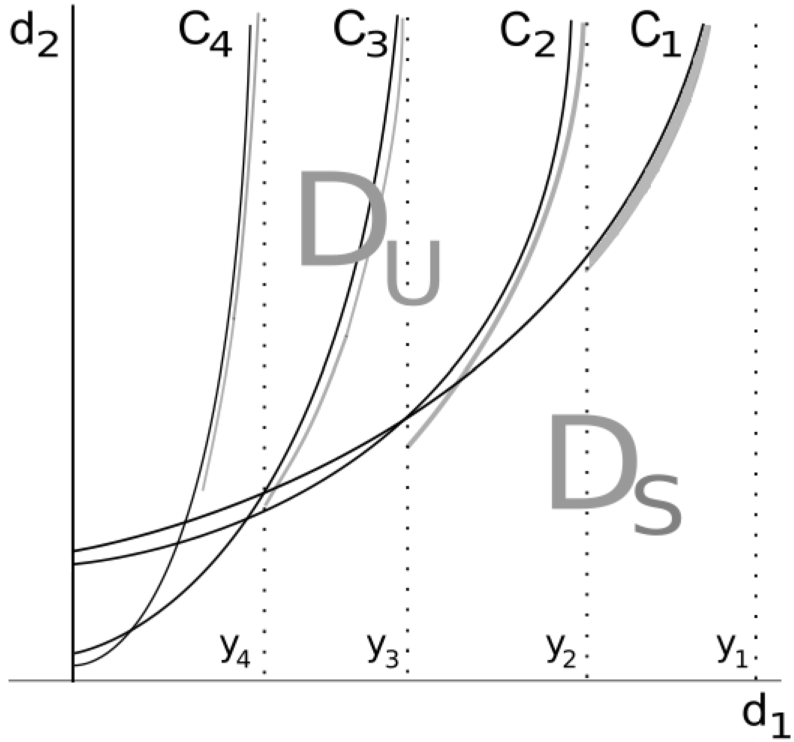

Figure 1: Sketch of hyperbolas.

with (3) and with fixed d_1. *Furthermore, if* $v_0 \notin K \cup (-K)$ *then* $d_{2\pm}(\tau) < d_2$ *for all* $\tau \in (0, \tau_0)$. *In particular, if* $d_1 \in (y_2, y_1)$ *and* $\|s_\pm\|_{L^\infty} > 0$, *then* $(d_1, d_{2\pm}(\tau)) \in D_S$ *for all* $\tau \in (0, \tau_0)$. *The parameter* τ_0 *has here a similar role as the parameter* τ_s *in Theorem 2, i.e. it is related to the strength of the source. Unlike our Theorems 1, 2, no estimate of the type (19) can be derived from [17], multiplicity of the critical point of system (16), (3) with fixed* d_1 *is assumed to be one and it is not known whether the bifurcation branches are global. Cf. also Remark 8 in Section 4 and [17], Section 4, Theorem 4.3. However, the statement mentioned above does not follow directly from Section 4 in [17], where applications to our system are given, but the authors assume* $\Omega \in C^{1,1}$, *and solutions are considered in different spaces.*

Remark 5. *In the paper [4], existence of bifurcation points in* D_S *was proved for the case that unilateral terms* $s_-^{(1)}u^- - s_+^{(1)}u^+$ *and* $s_-^{(2)}v^- - s_+^{(2)}v^+$ *are added into the first and second equation, respectively. A curve* σ *intersecting transversally the border* C_E *between the domain of stability and instability in some point* $[d_1^0, d_2^0]$ *was considered and the existence of a bifurcation point lying on this curve in* D_S *was shown. An essential assumption was that if* $[d_1^0, d_2^0] \in C_k$ *then* $s_\pm^{(j)}e \equiv 0$ *for some eigenfunction corresponding to* κ_k. *This assumption is not fulfilled in the most interesting situation* $d_1 \in (y_2, y_1)$, $\|s_\pm\|_{L^\infty} > 0$ *in our Theorems 1, 2, because then* $[d_1, d_2^0] \in C_1$ *and* e_1 *does not change its sign in* Ω. *Cf. also Remark 3. The result [4] is based on a topological method related (in a non-direct way) to the well-known Dancer's global bifurcation theorem, but it gives only a local bifurcation without any information about existence of a connected branch of bifurcating solutions.*

2.2. Systems with unilateral sources and sinks on the boundary

Let us consider the problem

$$\begin{aligned} & d_1\triangle u + b_{11}u + b_{12}v + n_1(u,v) = 0, \\ & d_2\triangle v + b_{21}u + b_{22}v + n_2(u,v) = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} & u = v = 0 \text{ on } \Gamma_D, \\ & \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma_N, \quad \frac{\partial v}{\partial \vec{n}} = s_-(x)v^- - s_+(x)v^+ \text{ on } \Gamma_N, \end{aligned} \tag{26}$$

where $s_\pm \in L^\infty(\Gamma_N)$, $s_\pm(x) \geq 0$ for a.a. $x \in \Gamma_N$, and $s_\pm(x) > 0$ on $\Gamma_\pm$, where $\Gamma_\pm$ are open subsets of Γ_N but not both coinciding with Γ_N, cf. (8). We do not consider here general $\hat{g}_+, \hat{g}_-$ as in (10), because it would mean some additional technical complications. The corresponding homogenization of (25), (26) is (16), (26). We define a cone

$$K_N := \{v \in \mathbb{H}_D \big| \; s_-v^- - s_+v^+ = 0 \text{ on } \Gamma_N \text{ in the sense of traces}\}.$$

Notation 3. *Let us define*

$$\tilde{d}_2^m := \sup_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\sum_{j=1}^{\infty} \lambda_j^S \xi_j^2 + \sum_{j=1}^{\infty} \xi_j \int_{\Gamma_N} \left(\left(\sum_{k=1}^{\infty} \xi_k e_k \right)^- s_- - \left(\sum_{k=1}^{\infty} \xi_k e_k \right)^+ s_+ \right) e_j \; dS}{\sum_{j=1}^{\infty} \xi_j^2}. \tag{27}$$

We will use λ_j^S and d_2^0 from Notation 2.

The following Theorem 3 is an analogue of Theorems 1 and 2.

Theorem 3. *Let $d_1 \in (0, y_1)$, $d_1 \neq y_j$ for all $j \in \mathbb{N}$, let (4)-(6), (13) and the first line of (12) be true. The estimate (19) with d_2^m, K and $\|s_\pm\|_{L^\infty}$ replaced by $\tilde{d}_2^m$, K_N and $C_T\|s_\pm\|_{L^\infty(\Gamma_N)}$, respectively, is valid, where C_T is a constant from the embedding $\mathbb{H}_D \hookrightarrow L^2(\partial\Omega)$.*

If $\tilde{d}_2^m$ is positive, then the supremum in (27) is maximum, i.e.

$$\tilde{d}_2^m = \max_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\sum_{j=1}^{\infty} \lambda_j^S \xi_j^2 + \sum_{j=1}^{\infty} \xi_j \int_{\Gamma_N} \left(\left(\sum_{k=1}^{\infty} \xi_k e_k \right)^- s_- - \left(\sum_{k=1}^{\infty} \xi_k e_k \right)^+ s_+ \right) e_j \; dS}{\sum_{j=1}^{\infty} \xi_j^2}, \tag{28}$$

and it is the largest critical point of the system (16), (26) with fixed d_1.

Let the multiplicity of d_2^0 be odd. Then for any sufficiently small $\varepsilon > 0$ there exists $\tau_s > 0$ such that if $s_-, s_+ \in L^\infty(\Gamma_N)$, $\|s_-\|_{L^\infty(\Gamma_N)}, \|s_+\|_{L^\infty(\Gamma_N)} \in [0, \tau_s)$ then $d_2^0 - \varepsilon < \tilde{d}_2^m$ and there is a global bifurcation point $\tilde{d}_2^b \in [d_2^0 - \varepsilon, \tilde{d}_2^m]$ of the system (25), (26) in the sense of Theorem 2. If $d_1 \in (y_2, y_1)$, $\|s_-\|_{L^\infty(\Gamma_N)}$, $\|s_+\|_{L^\infty(\Gamma_N)} \in (0, \tau_0)$ then $(d_1, \tilde{d}_2^b) \in D_S$.

An analogue of Remark 2 applies here as well. The proof of Theorem 3 is postponed to Section 5.

3. Abstract formulation and reduction to one equation

In the whole section we will assume (4)–(8), (12), (13). Of course, for the problem (9), (3) the assumptions (5), (12), (13) are trivially fulfilled.

3.1. Basic definitions and abstract formulation

Using Riesz Representation Theorem and compact embedding $\mathbb{H}_D \hookrightarrow\hookrightarrow L^2(\Omega)$ we can define a linear, compact, symmetric and positive operator $A : \mathbb{H}_D \to \mathbb{H}_D$ by

$$\langle Av, \varphi\rangle = \int_\Omega v\varphi \,\mathrm{dx} \quad \text{for all } v, \varphi \in \mathbb{H}_D,$$

and nonlinear operators $\beta^+, \beta^- : \mathbb{H}_D \to \mathbb{H}_D$ by

$$\begin{aligned} \langle \beta^-(v), \varphi\rangle &= -\int_\Omega s_-(x) v^- \varphi \,\mathrm{dx} \;\text{ for all } v, \varphi \in \mathbb{H}_D, \\ \langle \beta^+(v), \varphi\rangle &= \int_\Omega s_+(x) v^+ \varphi \,\mathrm{dx} \;\text{ for all } v, \varphi \in \mathbb{H}_D. \end{aligned} \tag{29}$$

Clearly, $\beta^\pm$ are positively homogeneous, i.e. $\beta^\pm(tv) = t\beta^\pm(v)$ for all $t > 0,\ v \in \mathbb{H}_D$.

Let us recall that if $v \in \mathbb{H}_D$, then $v^- \in \mathbb{H}_D$ and $\|v^-\| \le \|v\|$, see e.g. [19], p. 47, Corollary 2.1.8.

Observation 1. *One can easily show that the eigenvalues of A are $\lambda_k := \kappa_k^{-1}$, $k = 1, 2, ...,$ and the corresponding eigenfunctions of $-\triangle$ and A coincide.*

Using Riesz Representation Theorem, the growth conditions (12), well-known properties of Nemyckii operator and compact embedding $\mathbb{H}_D \hookrightarrow\hookrightarrow L^p(\Omega)$ with p satisfying (14), we can define continuous compact nonlinear operators $N_1, N_2 : \mathbb{H}_D \times \mathbb{H}_D \to \mathbb{H}_D$ by

$$\langle N_i(u, v), \varphi\rangle = \int_\Omega n_i(u, v)\varphi \,\mathrm{dx} \quad \text{for all } u, v, \varphi \in \mathbb{H}_D,\ i = 1, 2, \tag{30}$$

and $\hat{G}_-, \hat{G}_+ : \mathbb{H}_D \to \mathbb{H}_D$ by

$$\langle \hat{G}_-(v), \varphi\rangle = -\int_\Omega \hat{g}_-(x, v^-)\varphi \,\mathrm{dx}, \quad \langle \hat{G}_+(v), \varphi\rangle = \int_\Omega \hat{g}_+(x, v^+)\varphi \,\mathrm{dx} \quad \text{for all } v, \varphi \in \mathbb{H}_D.$$

The operators N_1, N_2 satisfy

$$N_1(0, 0) = N_2(0, 0) = 0, \quad \lim_{u,v\to 0} \frac{N_1(u, v)}{\|u\| + \|v\|} = \lim_{u,v\to 0} \frac{N_2(u, v)}{\|u\| + \|v\|} = 0. \tag{31}$$

The first condition in (31) follows directly from (5), the proof of the second condition in (31) can be found in [10].

If $N_1(u, v) \equiv N_1(v)$, then the condition (13) is superfluous. This will be seen from the proof of Theorem 4 describing a reduction of our system to a single equation, where (13) guarantees $N_1 \in C^1(\mathbb{H}_D \times \mathbb{H}_D, \mathbb{H}_D)$, see Theorem 7 in Appendix.

The relation between the operators $\hat{G}_-$ and β^-, $\hat{G}_+$ and β^+ is explained by the following lemma.

Lemma 1. *If $v_n \rightharpoonup v$ in $\mathbb{H}_D$ then $\beta^-(v_n) \to \beta^-(v),\ \beta^+(v_n) \to \beta^+(v)$. If $v_n \to 0$, $v_n/\|v_n\| \rightharpoonup w$ in $\mathbb{H}_D$, then*

$$\frac{\hat{G}_-(v_n)}{\|v_n\|} \to \beta^-(w) \quad \text{and} \quad \frac{\hat{G}_+(v_n)}{\|v_n\|} \to \beta^+(w).$$

Proof. Let $v_n \rightharpoonup v$. The compact embedding $\mathbb{H}_D \hookrightarrow\hookrightarrow L^2(\Omega)$ gives

$$\left\|\beta^-(v_n) - \beta^-(v)\right\| = \sup_{\varphi\in\mathbb{H}_D, \|\varphi\|=1} -\int_\Omega s_-(x)\left(v_n{}^- - v^-\right)\varphi \,\mathrm{dx} \le C\|s_-\|_{L^\infty}\|v_n - v\|_{L^2} \to 0. \tag{32}$$

Let us define the operator $G : \mathbb{H}_D \to \mathbb{H}_D$ by

$$\langle G(v), \varphi\rangle = -\int_\Omega (g_-(x, v) - s_-(x)v)\varphi \,\mathrm{dx} \;\text{ for all } v, \varphi \in \mathbb{H}_D.$$

The assumption (7) implies

$$\lim_{\xi\to 0}\frac{g_-(x,\xi)-s_-(x)\xi}{\xi}=0 \quad \text{for a.a. } x\in\Omega,$$

and this together with (12) give

$$\lim_{v\to 0}\frac{G(v)}{\|v\|}=0,$$

see Proposition 3.2 from [6]. If $v_n\to 0$, then also $v_n^-\to 0$, and the choice $v:=v_n^-$ yields

$$\lim_{n\to\infty}\frac{\|\hat{G}_-(v_n)-\beta^-(v_n)\|}{\|v_n\|}=\lim_{n\to\infty}\frac{\|G(v_n^-)\|}{\|v_n\|}\le\lim_{n\to\infty}\frac{\|G(v_n^-)\|}{\|v_n^-\|}=0. \tag{33}$$

If $v_n/\|v_n\|\rightharpoonup w$, then this together with the positive homogeneity of β^- and (32) with v_n replaced by $v_n/\|v_n\|$ and v replaced by w give

$$\frac{\hat{G}_-(v_n)}{\|v_n\|}\to\beta^-(w).$$

The proof for β^+, $\hat{G}_+$ is analogous. □

Observation 2. *A couple $(u,v)\in\mathbb{H}_D\times\mathbb{H}_D$ is a weak solution of the problem (10), (3) or (9), (3) if and only if it is a solution of the system of operator equations*

$$\begin{aligned} &d_1u-b_{11}Au-b_{12}Av-N_1(u,v)=0,\\ &d_2v-b_{21}Au-b_{22}Av-N_2(u,v)+\hat{G}_-(v)+\hat{G}_+(v)=0, \end{aligned} \tag{34}$$

or

$$\begin{aligned} &d_1u-b_{11}Au-b_{12}Av=0,\\ &d_2v-b_{21}Au-b_{22}Av+\beta^-(v)+\beta^+(v)=0, \end{aligned}$$

respectively.

The abstract formulation of (16), (26) and (25), (26) can be done similarly, see the end of Section 5.

3.2. Reduction to one equation

Theorem 4. *Let d_1 be fixed, $d_1\neq y_j$ for all $j\in\mathbb{N}$. Then there exist neighborhoods U,V of 0 in $\mathbb{H}_D$ such that $(u,v)\in U\times V$ satisfies (34) if and only if*

$$\begin{aligned} &d_2v-Sv-N(v)+\beta^+(v)+\beta^-(v)=0,\\ &u=F(v), \end{aligned} \tag{35}$$

where $F:V\to U$ is a bijective and continuously differentiable map, $S:=b_{12}b_{21}A(d_1I-b_{11}A)^{-1}A+b_{22}A$ is a linear, compact and symmetric operator and $N:\mathbb{H}_D\to\mathbb{H}_D$ is a continuous and compact nonlinear operator satisfying

$$\lim_{v\to 0}\frac{N(v)}{\|v\|}=0. \tag{36}$$

We postpone the proof to the end of this section. Let us note that Theorem 4 is crucial for the proofs of Theorems 1 and 2.

Remark 6. *It will be seen from the proof that if $n_1(u,v)\equiv n_1(v)$, then $U=V=\mathbb{H}_D$,*

$$F(v)=b_{12}(d_1I-b_{11}A)^{-1}Av+(d_1I-b_{11}A)^{-1}N_1(v),$$

and the condition (13) is not needed.

According to Theorem 4, it will be sufficient to study only the first equation in (35).

Corollary 1. *It follows from Theorem 4 and Observation 2 that d_2 is a critical point of (9), (3) with fixed d_1 if and only if d_2 is an eigenvalue of the operator $S - \beta^+ - \beta^-$ (see Definition 3 in Section 4). A point d_2 is a bifurcation point of the system (10), (3) with fixed d_1 if and only if d_2 is a bifurcation point of the first equation in (35).*

In particular, d_2 is a critical point of (16), (3) with fixed d_1 or a bifurcation point of the system (15), (3) with fixed d_1 if and only if d_2 is simultaneously an eigenvalue of the operator S or a bifurcation point of the first equation in (35) with $\beta^{\pm} \equiv 0$, respectively.

First, let us apply Theorem 4 to the system (34) with $\hat{G}_- \equiv 0$, $\hat{G}_+ \equiv 0$, i.e. to the system

$$\begin{aligned} d_1 u - b_{11} A u - b_{12} A v - N_1(u,v) = 0, \\ d_2 v - b_{21} A u - b_{22} A v - N_2(u,v) = 0, \end{aligned} \tag{37}$$

and to the corresponding linearization

$$\begin{aligned} d_1 u - b_{11} A u - b_{12} A v = 0, \\ d_2 v - b_{21} A u - b_{22} A v = 0. \end{aligned} \tag{38}$$

Theorem 4 reduces an analysis of (37) and (38) to a study of equations

$$d_2 v = Sv + N(v) \quad \text{and} \quad d_2 v = Sv, \tag{39}$$

respectively. The operator S is compact and symmetric, thus it has a sequence of eigenvalues, all of them are real and have the only accumulation point zero. Using Observation 1 and Notation 1 we can easily find that

$$Se_k = b_{12} b_{21} A (d_1 I - b_{11} A)^{-1} A e_k + b_{22} A e_k = \frac{1}{\kappa_k} \left(\frac{b_{12} b_{21}}{d_1 \kappa_k - b_{11}} + b_{22} \right) e_k. \tag{40}$$

Hence,

$$\lambda_k^S = \frac{1}{\kappa_k} \left(\frac{b_{12} b_{21}}{d_1 \kappa_k - b_{11}} + b_{22} \right), \ k \in \mathbb{N} \tag{41}$$

introduced already in Notation 2 are eigenvalues of S with the corresponding eigenfunctions e_k. Since the set $\{e_k\}_{k \in \mathbb{N}}$ is an orthonormal base of $\mathbb{H}_D$, there are no other eigenvalues of S.

Observation 3. *Let $d_1 \in (0, y_1)$, $d_1 \neq y_j$ for all $j \in \mathbb{N}$. In our notation the sequence λ_k^S is not monotone and the largest eigenvalue of S denoted as $\lambda_{\max}^S$ is in general not λ_1^S but λ_j^S, where*

$$\lambda_j^S = \max_{\substack{w \in \mathbb{H}_D \\ \|w\| \neq 0}} \frac{\langle Sw, w \rangle}{\|w\|^2} = \max_{k \in \mathbb{N}} \frac{1}{\kappa_k} \left(\frac{b_{12} b_{21}}{d_1 \kappa_k - b_{11}} + b_{22} \right). \tag{42}$$

Let us write for a moment $S(d_1)$ and $\lambda_k^S(d_1)$ to emphasize the dependence of S and λ_k^S on d_1. It follows from the definition of C_k that the graph of the function $\lambda_k^S(d_1)$ is the curve C_k, see Fig. 1. If $d_1 < y_k = b_{11}/\kappa_k$ for some $k \in \mathbb{N}$, i.e. d_1 is to the left from k-th asymptote (17), then $d_1 < b_{11}/\kappa_j$ for any $j \leq k$ and consequently $\lambda_j^S > 0$ as can be seen from formulae (41) and (6). If $d_1 \in (y_2, y_1)$, then λ_1^S is simple because κ_1 is simple, and it is the only positive eigenvalue of S. Hence, $\lambda_{\max}^S = \lambda_1^S$.

Proof of Theorem 4. We will show that the assumptions of Implicit Function Theorem are fulfilled for the map $\Phi : \mathbb{H}_D \times \mathbb{H}_D \to \mathbb{H}_D$ defined by

$$\Phi(u,v) := d_1 u - b_{11} A u - b_{12} A v - N_1(u,v).$$

Evidently $\Phi(0,0) = 0$. Since $N_1 \in C^1(\mathbb{H}_D \times \mathbb{H}_D, \mathbb{H}_D)$ under the assumptions (12), (13), see Theorem 7 in Appendix, and the operator A is linear and bounded, we have $\Phi \in C^1(\mathbb{H}_D \times \mathbb{H}_D, \mathbb{H}_D)$.

The conditions (31) and $d_1 \neq y_k$ for all $k \in \mathbb{N}$, that means $d_1 \notin \sigma(b_{11}A)$, guarantee that the partial derivative

$$\Phi'_u(0,0) = d_1 I - b_{11}A - (N_1)'_u(0,0) = d_1 I - b_{11}A$$

is a linear isomorphism of the space $\mathbb{H}_D$. Implicit Function Theorem gives that there exist neighborhoods U, V of the origin in $\mathbb{H}_D$ and a bijection $F : V \to U$ such that

$$\Phi(u,v) = 0 \text{ for } (u,v) \in U \times V \text{ if and only if } u = F(v).$$

Moreover, $F \in C^1(V,U)$. We calculate the partial derivative

$$\Phi'_v(0,0) = -b_{12}A - (N_1)'_v(0,0) = -b_{12}A$$

to find that

$$F'(0) = -\Phi'_u(0,0)^{-1}\Phi'_v(0,0) = b_{12}\left[d_1 I - b_{11}A\right]^{-1} A.$$

The first equation in (34) gives $u = F(v)$ and introducing the linear compact symmetric operator $S := b_{21}AF'(0) + b_{22}A$, the second equation in (34) can be written as

$$d_2 v - Sv - N_2(F(v),v) - b_{21}A(F(v) - F'(0)(v)) + \hat{G}_-(v) + \hat{G}_+(v) = 0.$$

It follows that system (34) is on $U \times V$ equivalent to (35) with

$$N(v) := N_2(F(v),v) + b_{21}A(F(v) - F'(0)(v)) - \hat{G}_-(v) + \beta^-(v) - \hat{G}_+(v) + \beta^+(v).$$

The operators $A, N_1, N_2, \hat{G}_\pm, \beta^\pm$ are continuous and compact, therefore N is continuous and compact. It remains to show that $N(v)$ satisfies (36). Mean Value Theorem gives an existence of $C > 0$ such that

$$\frac{\|F(v)\| + \|v\|}{\|v\|} \leq C\|v\| \quad \text{for all } v \in V,$$

which together with (31) implies

$$\lim_{v\to 0} \frac{N_2(F(v),v)}{\|v\|} = \lim_{v\to 0} \frac{N_2(F(v),v)}{\|F(v)\| + \|v\|} \frac{\|F(v)\| + \|v\|}{\|v\|} \to 0. \tag{43}$$

We can use Mean Value Theorem again to get

$$\|F(v) - F'(0)v\| = \sup_{t\in[0,1]} \|F'(tv)(v) - F'(0)(v)\|.$$

Due to the continuity of F' we have

$$\lim_{v\to 0} \sup_{t\in[0,1]} \frac{\|F'(tv)(v) - F'(0)(v)\|}{\|v\|} \leq \lim_{v\to 0} \sup_{t\in[0,1]} \left\| F'(tv)\left(\frac{v}{\|v\|}\right) - F'(0)\left(\frac{v}{\|v\|}\right) \right\| = 0. \tag{44}$$

Using (43), (44) and (33) (which holds for any $v_n \to 0$) and its analogue for $\hat{G}_+, \beta^+$ we get (36). $\square$

4. Eigenvalues and bifurcations for equations with positively homogeneous operators

In this section we will always consider a general real Hilbert space $\mathbb{H}$ with a scalar product $\langle\cdot,\cdot\rangle$, the corresponding norm $\|\cdot\|$ and a nonlinear compact operator $P : \mathbb{H} \to \mathbb{H}$ which is positively homogeneous. Moreover, we will assume that P fulfills the conditions

$$v_n \rightharpoonup v \text{ in } \mathbb{H} \;\Rightarrow\; P(v_n) \to P(v) \text{ in } \mathbb{H}, \tag{45}$$

$$\sup_{v\in\mathbb{H}, \|v\|=1} \langle P(v), v\rangle > 0. \tag{46}$$

It follows easily that the number

$$|P| := \sup_{v\in\mathbb{H},\|v\|=1} \|P(v)\|$$

is finite and

$$\|P(v)\| \le |P|\|v\| \ \text{ for all } v \in \mathbb{H}.$$

We will work mainly with operators $P := S - B$ satisfying (46), where $S : \mathbb{H} \to \mathbb{H}$ is always a linear symmetric compact operator, B is a positively homogeneous operator satisfying (45) (with P replaced by B) and

$$\langle B(v), v\rangle \ge 0 \quad \text{for all } v \in \mathbb{H}. \tag{47}$$

Furthermore, $N : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ will denote always a compact nonlinear operator satisfying

$$\lim_{v\to 0} \frac{N(\lambda, v)}{\|v\|} = 0 \ \text{ uniformly for } \lambda \text{ from compact subsets of } \mathbb{R}. \tag{48}$$

We will consider a nonlinear problem

$$\lambda v - Sv + B(v) - N(\lambda, v) = 0. \tag{49}$$

Definition 3. *We call $\lambda \in \mathbb{R}$ an eigenvalue of a positively homogeneous operator $P : \mathbb{H} \to \mathbb{H}$ if there exists a nontrivial $v \in \mathbb{H}$ (an eigenvector corresponding to λ) such that $P(v) = \lambda v$. We will denote $\lambda^P_{\max}$ the largest eigenvalue of the operator P, if it exists.*

Of course, eigenvectors of a positively homogeneous operator corresponding to an eigenvalue λ do not form a subspace in general.

The goal of this section is to give a variational characterization of the largest eigenvalue of a positively homogeneous operator and in some cases also to give an existence of a global bifurcation points for the equation (49).

Theorem 5. *There exists $v_0 \in \mathbb{H}$ such that $\|v_0\| = 1$ and*

$$\max_{v\in\mathbb{H},\|v\|\neq 0} \frac{\langle P(v), v\rangle}{\|v\|^2} = \max_{v\in\mathbb{H},\|v\|=1} \langle P(v), v\rangle = \langle P(v_0), v_0\rangle. \tag{50}$$

If there is v_0 with $\|v_0\| = 1$ satisfying (50) and in addition the condition

$$\lim_{t\to 0} \frac{1}{t}\left[\langle P(v_0 + th) - P(v_0), v_0\rangle\right] = \langle P(v_0), h\rangle \quad \textit{for all} \ \ h \in \mathbb{H}, \tag{51}$$

then the largest eigenvalue of P is $\lambda^P_{\max} = \langle P(v_0), v_0\rangle$ and v_0 is the corresponding eigenvector.

If (51) is fulfilled for all $v_0 \in \mathbb{H}, \|v_0\| = 1$, then $v_0 \in \mathbb{H}$ with $\|v_0\| = 1$ is an eigenvector of P corresponding to $\lambda^P_{\max}$ if and only if v_0 fulfills (50).

Condition (51) means that the functional $\Phi(v) := \frac{1}{2}\langle P(v), v\rangle$ has at v_0 the Frechet derivative $\Phi'(v_0) = P(v_0)$. Hence, if (51) is fulfilled for all $v_0 \in \mathbb{H}$, $\|v_0\| = 1$, then P is a potential operator with the potential Φ.

We will prove in Section 5 that particular operators β^+ and β^- introduced in Section 3, which are neither linear, nor differentiable, satisfy (51) for all $v_0 \in \mathbb{H}, \|v_0\| = 1$. The following simple example demonstrates that sometimes only some maximizers of (50) satisfy (51).

Example 1. *Let $\mathbb{H} = \mathbb{R}^3$ and let P be a linear operator defined by a matrix*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -a \\ 0 & a & a \end{pmatrix},$$

$a > 0$ being a parameter. The eigenvalues are $\lambda_{\max} = 1$, $\lambda_2 = a(1+i)$, $\lambda_3 = a(1-i)$. If $a = 1$, then $\langle v, Pv\rangle = \|v\|^2$ for all $v \in \mathbb{R}^3$ and the maximum in (50) is 1. However, only the maximizers $v := \pm(1,0,0)$ satisfy the condition (51) and they are the only eigenvectors corresponding to the eigenvalue $\lambda^P_{\max}$. If $a = 2$, then the maximum in (50) is 2, no maximizer satisfies (51) and the largest eigenvalue $\lambda^P_{\max}$ is not characterized by the formula (50).

Proof of Theorem 5. The first equality in (50) follows from positive homogenity if this maximum exist. There exists a sequence $\{v_n\} \subset \mathbb{H}$, $\|v_n\| = 1$ for all $n \in \mathbb{N}$ such that

$$\sup_{v\in\mathbb{H},\|v\|=1} \langle P(v), v\rangle = \lim_{n\to\infty} \langle P(v_n), v_n\rangle.$$

We can assume without loss of generality that $v_n \rightharpoonup v_0$ with some $v_0 \in \mathbb{H}$ and it follows from (45) that

$$P(v_n) \to P(v_0).$$

Moreover, $v_0 \neq 0$ due to (46). We will prove that $\|v_0\| = 1$. Clearly $\|v_0\| \leq 1$. If there were $\|v_0\| < 1$ then we would have

$$\sup_{v\in\mathbb{H},\|v\|=1} \langle P(v), v\rangle = \langle P(v_0), v_0\rangle < \frac{\langle P(v_0), v_0\rangle}{\|v_0\|^2},$$

which is a contradiction. Thus, $\|v_0\| = 1$, the maximum in (50) exists and is attained at v_0.

Now let v_0 satisfy also (51). Then for arbitrary $t \in \mathbb{R}$ and $h \in \mathbb{H}$ with $\|v_0 + th\| \neq 0$ we have

$$\frac{\langle P(v_0+th), v_0+th\rangle}{\|v_0+th\|^2} \leq \langle P(v_0), v_0\rangle.$$

Since $\|v_0\| = 1$, this can be rewritten as

$$\langle P(v_0+th), v_0+th\rangle \leq \langle P(v_0), v_0\rangle(1 + 2t\langle v_0, h\rangle + t^2\|h\|^2).$$

Rearranging the inequality and setting $\lambda^P_{\max} = \langle P(v_0), v_0\rangle$ give

$$\langle P(v_0+th), v_0\rangle - \langle P(v_0), v_0\rangle + t\langle P(v_0+th), h\rangle \leq \lambda^P_{\max}(2t\langle v_0, h\rangle + t^2\|h\|^2).$$

For $t \neq 0$ we can divide the last inequality by $2t$. Then the limiting processs $t \to 0+, t \to 0-$ together with (51) and the continuity of P give successively

$$\langle P(v_0), h\rangle \leq \lambda^P_{\max}\langle v_0, h\rangle,$$
$$\langle P(v_0), h\rangle \geq \lambda^P_{\max}\langle v_0, h\rangle.$$

Since h was arbitrary, we get

$$P(v_0) = \lambda^P_{\max} v_0.$$

Hence, $\lambda^P_{\max}$ is an eigenvalue and v_0 is a corresponding eigenvector of the operator P. Let $\lambda^P \in \mathbb{R}$ be an arbitrary eigenvalue of P and v_1 a corresponding eigenvector. We multiply the equation $\lambda^P v_1 = P(v_1)$ by $v_1/\|v_1\|^2$ and use (50) to get

$$\lambda^P = \frac{\langle P(v_1), v_1\rangle}{\|v_1\|^2} \leq \max_{v\in\mathbb{H},v\neq 0} \frac{\langle P(v), v\rangle}{\|v\|^2} = \lambda^P_{\max}.$$

Hence, $\lambda^P_{\max}$ is the largest eigenvalue of the operator P.

Let (51) be fulfilled for all $v_0 \in \mathbb{H}, \|v_0\| = 1$. If $v_0 \in \mathbb{H}, \|v_0\| = 1$ is an eigenvector to $\lambda^P_{\max}$, then

$$\lambda^P_{\max} = \lambda^P_{\max}\|v_0\|^2 = \langle P(v_0), v_0\rangle \leq \max_{v\in\mathbb{H},\|v\|=1} \langle P(v), v\rangle = \lambda^P_{\max},$$

i.e. v_0 satisfies (50). Conversely, if v_0 fulfills (50), then according to the first statement of Theorem 5 it must be an eigenvector of P corresponding to $\lambda^P_{\max}$. □

Remark 7. *Theorem 5 is a generalization of the well-known fact that the largest eigenvalue of a linear symmetric compact operator S satisfying (46) is*

$$\lambda^S_{\max} = \max_{v\in\mathbb{H},\|v\|=1} \langle Sv, v\rangle \tag{52}$$

and the eigenvectors of S corresponding to $\lambda^S_{\max}$ are exactly all maximizers. Any such operator has a potential $\Phi(v) = \frac{1}{2}\langle Sv, v\rangle$, that means it satisfies (51) for all v_0.

Let us also recall that the second largest eigenvalue λ^S_2 of the operator S satisfies

$$\lambda^S_2 = \max_{v\in\tilde{\mathbb{H}},\|v\|=1} \langle Sv, v\rangle, \tag{53}$$

where $\tilde{\mathbb{H}}$ is the orthogonal complement to the eigenspace of $\lambda^S_{\max}$.

Even in the case of a symmetric linear operator, the assumption (46) is crucial. If S is a negative operator, then $\sup_{\|v\|=1}\langle Sv, v\rangle = 0$, no maximizer exists and 0 is not an eigenvalue of S.

In particular, if (51) is fulfilled for v_0 satisfying (50) with $P := B$, then it is fulfilled also for $P := S - B$ and Theorem 5 gives that the largest eigenvalue $\lambda^{S-B}_{\max}$ of $S - B$ is

$$\lambda^{S-B}_{\max} = \max_{v\in\mathbb{H},\|v\|=1} \left(\langle Sv, v\rangle - \langle B(v), v\rangle\right).$$

Observation 4. *Since we assume that B is a positively homogeneous operator satisfying (47), we see that*

$$\begin{aligned}\lambda^{S-B}_{\max} &= \max_{v\in\mathbb{H},\|v\|=1} \left(\langle Sv, v\rangle - \langle B(v), v\rangle\right) \geq \max_{v\in\mathbb{H},\|v\|=1}\langle Sv, v\rangle - \max_{v\in\mathbb{H},\|v\|=1}\langle B(v), v\rangle = \\ &= \lambda^S_{\max} - \max_{v\in\mathbb{H},\|v\|=1}\langle B(v), v\rangle \geq \lambda^S_{\max} - |B|.\end{aligned}$$

Remark 8. *The following assertions can be obtained from Theorem 1.1 in [17]. If the operators S and B satisfying our assumptions are given and λ^S is an arbitrary simple eigenvalue of S with an eigenvector v_0, $\|v_0\| = 1$, then there are $\tau_0 > 0$ and Lipschitz continuous maps $\lambda_+ : (-\tau_0, \tau_0) \to \mathbb{R}$, $\lambda_- : (-\tau_0, \tau_0) \to \mathbb{R}$, $v_+ : (-\tau_0, \tau_0) \to \mathbb{H}$, $v_- : (-\tau_0, \tau_0) \to \mathbb{H}$ such that $\lambda_+(\tau)$, $\lambda_-(\tau)$ are the only eigenvalues of the operator $S - \tau B$ in a certain neighborhood of λ^S, $v_+(\tau)$, $v_-(\tau)$ are the corresponding normalized eigenvectors, $v_+(0) = v_0$, $v_-(0) = -v_0$ and $\lambda_\pm(0) = \lambda^S$. Moreover, $\lambda_\pm(\tau)$ are simultaneously bifurcation points of the equation*

$$\lambda v - Sv + \tau B(v) - N(\lambda, v) = 0,$$

but this bifurcation is only local. In fact, problems studied in [17] are more general but the simplicity of λ^S is essential. The information that the eigenvalues $\lambda_\pm(\tau)$ are isolated is not self-evident. Let us emphasize that a variational inequality, which can be approximated by an equation with a large positively homogeneous penalty term, can have an interval of eigenvalues, see [9].

In our Theorem 5 we need no assumption concerning multiplicity and no smallness of the positively homogeneous perturbation B, but we deal only with the largest eigenvalue. In Theorem 6 below we get a global bifurcation point for small positively homogeneous perturbations B under the assumption that $\lambda^S_{\max}$ is of odd multiplicity, but unfortunately we do not show that the bifurcation is in λ^{S-B}_{max}, only in its small left neighborhood. However, this is a plausible information from the point of view of an application to reaction-diffusion systems from Section 2.

For any positively homogeneous operator B, let us denote

$$\mathcal{S}(B) = \overline{\{(\lambda, v) \in (0,\infty) \times \mathbb{H} \mid v \neq 0,\ (\lambda, v) \text{ satisfies } (49)\}}.$$

Let us recall that λ^S_2 denotes the second largest eigenvalue of S, see (53).

Theorem 6. *Let the multiplicity of* $\lambda^S_{\max}$ *be odd. Then for any* $\varepsilon \in (0, \min\{(\lambda^S_{\max} - \lambda^S_2)/2, \lambda^S_{\max}\})$ *there exists* $\tau_0 > 0$ *such that the following assertion is true. If* B *satisfies besides our basic assumptions also* $|B| \le \tau_0$ *and* $P := S - B$ *fulfills (51) with* v_0 *satisfying (50), then* $\lambda^S_{\max} - \varepsilon < \lambda^{S-B}_{\max}$ *and there is a global bifurcation point* $\lambda_b \in [\lambda^S_{\max} - \varepsilon, \lambda^{S-B}_{\max}]$ *of the problem (49) in the following sense. The connected component* $\mathcal{S}_{\lambda_b}$ *of* $\mathcal{S}(B)$ *containing the point* $(\lambda_b, 0) \in \mathbb{R} \times \mathbb{H}$ *satisfies at least one of the following conditions:*

(a) $\mathcal{S}_{\lambda_b}$ *is unbounded,*
(b) *there exists* $v \in \mathbb{H}, v \ne 0$ *such that* $(0, v) \in \overline{\mathcal{S}_{\lambda_b}}$*,*
(c) *there exists an eigenvalue* $\lambda_c \notin [\lambda^S_{\max} - \varepsilon, \lambda^{S-B}_{\max}]$ *of the operator* $S - B$ *such that* $(\lambda_c, 0) \in \overline{\mathcal{S}_{\lambda_b}}$*.*

For the proof of this Theorem we will need two auxiliary lemmas.

Lemma 2. *Let* $\{\lambda_n\}$ *be a sequence of real numbers such that* $\lambda_n \to \lambda \ne 0$*, let* $\{v_n\}$ *be a sequence in* $\mathbb{H}$ *satisfying* $v_n \to 0$*,* $v_n/\|v_n\| \rightharpoonup w$ *and*

$$\lambda_n v_n - S v_n + B(v_n) - N(\lambda_n, v_n) = 0. \tag{54}$$

Then

$$\frac{v_n}{\|v_n\|} \to w, \ \|w\| = 1 \quad \text{and} \quad \lambda w - S w + B(w) = 0.$$

Proof. Dividing (54) by $\|v_n\|$ gives

$$\lambda_n \frac{v_n}{\|v_n\|} = S\left(\frac{v_n}{\|v_n\|}\right) - B\left(\frac{v_n}{\|v_n\|}\right) + \frac{N(\lambda_n, v_n)}{\|v_n\|}. \tag{55}$$

The operator S is compact and linear, the operator $P := B$ satisfies (45) and the nonlinear operator N satisfies (48), therefore the r.h.s. of the equation (55) converges strongly. Since $\lambda_n \to \lambda \ne 0$, it implies that $v_n/\|v_n\|$ converges strongly and the only possible limit is the vector w, $\|w\| = 1$. Providing the limit in the equation (55) and using (48) yields

$$\lambda w = S w - B(w).$$

□

Notation 4. *For* $\lambda \ne 0$ *we denote by* $deg(I - (1/\lambda)P, B_r, 0)$ *the Leray-Schauder degree of the map* $I - (1/\lambda)P$ *with respect to the ball* B_r *with the radius* $r > 0$*, centred at the origin and with respect to the point 0. This degree is independent of* r *due to the positive homogenity. We use the symbol* $\sigma(S)$ *for the spectrum of the operator* S*. The set* $\sigma(S)$ *consist of* 0 *and all eigenvalues of the operator* S*.*

Lemma 3. *For any* $\varepsilon > 0$ *there exists* $\tau_0 > 0$ *such that*

$$deg\left(I - \frac{1}{\lambda}(S + B), B_r, 0\right) = deg\left(I - \frac{1}{\lambda}S, B_r, 0\right)$$

for any $\lambda \in \mathbb{R}$ *satisfying* $dist(\lambda, \sigma(S)) > \varepsilon$*, any* B *satisfying in addition to our standard assumptions* $|B| \le \tau_0$*, and all* $r > 0$*.*

Proof. Due to a homotopy invariance of the degree it suffices to prove that for any $\varepsilon > 0$ there is $\tau_0 > 0$ such that

$$v - \frac{1}{\lambda}(S v + t B(v)) \ne 0 \ \text{ for all } \lambda, B \text{ from the assumptions, } \ t \in [0,1], \ \|v\| = 1.$$

Let us suppose that it is not true. Then there exist $\varepsilon > 0$, $t_n \in [0,1]$, λ_n with $\text{dist}(\lambda_n, \sigma(S)) > \varepsilon$, B_n and v_n with $\|v_n\| = 1$ for all $n \in \mathbb{N}$, satisfying

$$\lambda_n \to \lambda, \quad v_n \rightharpoonup v, \quad \|B_n(v_n)\| \to 0, \tag{56}$$

and

$$\lambda_n v_n - S v_n + t_n B_n(v_n) = 0. \tag{57}$$

Since $0 \in \sigma(S)$ and $\operatorname{dist}(\lambda_n, \sigma(S)) > \varepsilon$ we have $|\lambda_n| > \varepsilon$ for all $n \in \mathbb{N}$. The compactness of S together with (56), (57) gives $v_n \to v$, $\|v\| = 1$. Providing the limit in the equation (57) leads to

$$\lambda v - S v = 0,$$

but simultaneously $\operatorname{dist}(\lambda, \sigma(S)) > \varepsilon$ which is a contradiction. □

Proof of Theorem 6. The assumptions (46) for $P := S - B$ and (47) imply that $\sup_{\|v\|=1} \langle Sv, v\rangle > 0$, therefore $\lambda^S_{\max}$ exists and is positive. Due to Leray-Schauder Index Formula and the assumed odd multiplicity of $\lambda^S_{\max}$ we have

$$\deg\left(I - \frac{1}{\lambda} S, B_r, 0\right) = 1 \quad \text{for all } \lambda > \lambda^S_{\max},\ r > 0$$
$$\deg\left(I - \frac{1}{\lambda} S, B_r, 0\right) = -1 \quad \text{for all } \lambda \in (\lambda^S_2, \lambda^S_{\max}),\ r > 0,$$

where λ^S_2 is the second largest eigenvalue of S. Lemma 3 gives that for any ε from the assumptions there exists $\tau_0 > 0$ such that for B satisfying $|B| \le \tau_0$ we have

$$\deg\left(I - \frac{1}{\lambda}(S+B), B_r, 0\right) = \deg\left(I - \frac{1}{\lambda} S, B_r, 0\right) = 1 \quad \text{for all } \lambda > \lambda^S_{\max} + \varepsilon,\ r > 0$$
$$\deg\left(I - \frac{1}{\lambda}(S+B), B_r, 0\right) = \deg\left(I - \frac{1}{\lambda} S, B_r, 0\right) = -1 \quad \text{for all } \lambda \in (\lambda^S_2 + \varepsilon, \lambda^S_{\max} - \varepsilon),\ r > 0.$$

It follows from known results [16] and [18], see Theorem 8 in Appendix for details, that there exists $\lambda_b \in [\lambda^S_{\max} - \varepsilon, \lambda^S_{\max} + \varepsilon]$ such that the connected component $\mathcal{S}_{\lambda_b}$ of $\mathcal{S}(\mathcal{B})$ containing the point $(\lambda_b, 0)$ satisfies at least one of the alternatives (a)–(c) with the interval $[\lambda^S_{\max} - \varepsilon, \lambda^{S-B}_{\max}]$ replaced by $[\lambda^S_{\max} - \varepsilon, \lambda^S_{\max} + \varepsilon]$. Due to Theorem 5 the largest eigenvalue $\lambda^{S-B}_{\max}$ of $S - B$ exists. Observation 4 gives that $\lambda^S_{\max} - \varepsilon < \lambda^{S-B}_{\max}$ if τ_0 is small enough. According to Lemma 2 the number λ_b must be in the interval $[\lambda^S_{\max} - \varepsilon, \lambda^{S-B}_{\max}]$ and at least one of the conditions (a)–(c) must be fulfilled. □

5. Proofs of Theorems 1, 2 and 3

In this section we will always consider the operators $A, \beta^\pm, S, N$ introduced in Section 3.

Lemma 4. *The operators β^+ and β^- are positively homogeneous and satisfy (47) with $B = \beta^\pm$ and (51) with $P = \beta^\pm$ for any $v_0 \in \mathbb{H}_D, \|v_0\| = 1$. The operator $P := S - \beta^- - \beta^+$ satisfies (51) for any $v_0 \in \mathbb{H}_D, \|v_0\| = 1$.*

Proof. The condition (47) for $\beta^\pm$ follows directly from the definition and (8). Let $t \in \mathbb{R}$ and $h, v_0 \in \mathbb{H}_D,\ \|v_0\| = 1$ be arbitrary. We will show that (51) is true for the operator $P := \beta^-$. We introduce sets $\Omega_0^+, \Omega_0^-, \Omega_{th}^+, \Omega_{th}^-$ such that $\Omega_0^+ \cup \Omega_0^- \cup \Omega_{th}^+ \cup \Omega_{th}^- = \Omega$ and

$$(v_0 + th)(x) < 0 \quad \text{for a.a. } x \in \Omega_{th}^-, \qquad v_0 + th \ge 0 \quad \text{for a.a. } x \in \Omega_{th}^+,$$
$$v_0(x) < 0 \quad \text{for a.a. } x \in \Omega_0^-, \qquad v_0(x) \ge 0 \quad \text{for a.a. } x \in \Omega_0^+.$$

Then

$$\begin{aligned}
&\frac{1}{t}\left[\langle \beta^-(v_0 + th) - \beta^-(v_0), v_0\rangle\right] - \langle \beta^-(v_0), h\rangle = \\
&\qquad = \frac{1}{t}\left[-\int_\Omega (v_0 + th)^- v_0 - v_0^- v_0 \text{ dx}\right] + \int_\Omega v_0^- h \text{ dx} = \\
&\qquad = \frac{1}{t}\left[\int_{\Omega_{th}^-} (v_0 + th) v_0 \text{ dx} - \int_{\Omega_0^-} v_0^2 \text{ dx}\right] - \int_{\Omega_0^-} v_0 h \text{ dx} = \\
&\qquad = \frac{1}{t}\left[\int_{\Omega_{th}^-} v_0^2 \text{ dx} - \int_{\Omega_0^-} v_0^2 \text{ dx}\right] + \int_{\Omega_{th}^-} v_0 h \text{ dx} - \int_{\Omega_0^-} v_0 h \text{ dx}.
\end{aligned} \tag{58}$$

Dominated Convergence Theorem gives

$$\lim_{t\to 0}\left(\int_{\Omega_{th}^-} v_0 h \text{ dx}\right) = \lim_{t\to 0}\int_\Omega v_0 h\chi_{\Omega_{th}^-} \text{ dx} = \int_\Omega v_0 h \chi_{\Omega_0^-} \text{ dx} = \int_{\Omega_0^-} v_0 h \text{ dx}, \tag{59}$$

where χ denotes the characteristic function of a set in a subscript. Now we introduce sets $\Omega_{th1}, \Omega_{th2}, \Omega_{th3}$ such that

$$\Omega_{th}^- = \Omega_{th1} \cup \Omega_{th2}, \quad \Omega_0^- = \Omega_{th1} \cup \Omega_{th3},$$

and

$$\begin{aligned} &v_0(x) < -th(x) \text{ and } v_0(x) < 0 \text{ for a.a. } x \in \Omega_{th1},\\ &v_0(x) < -th(x) \text{ and } v_0(x) \geq 0 \text{ for a.a. } x \in \Omega_{th2},\\ &v_0(x) \geq -th(x) \text{ and } v_0(x) < 0 \text{ for a.a. } x \in \Omega_{th3}. \end{aligned}$$

Then

$$\begin{aligned} \int_{\Omega_{th}^-} v_0^2 \text{ dx} - \int_{\Omega_0^-} v_0^2 \text{ dx} &= \int_{\Omega_{th1}} v_0^2 \text{ dx} + \int_{\Omega_{th2}} v_0^2 \text{ dx} - \int_{\Omega_{th1}} v_0^2 \text{ dx} - \int_{\Omega_{th3}} v_0^2 \text{ dx} = \\ &= \int_{\Omega_{th2}} v_0^2 \text{ dx} - \int_{\Omega_{th3}} v_0^2 \text{ dx}. \end{aligned} \tag{60}$$

Since $|v_0(x)| < |th(x)|$ for a.a. $x \in \Omega_{th2} \cup \Omega_{th3}$, we get

$$\lim_{t\to 0}\frac{1}{t}\left|\int_{\Omega_{th2}} v_0^2 - \int_{\Omega_{th3}} v_0^2\right| \leq \lim_{t\to 0}\frac{1}{t}\left|\int_{\Omega_{th2}} (th)^2 - \int_{\Omega_{th3}} (th)^2\right| = 0. \tag{61}$$

It follows from (58) – (61) that

$$\lim_{t\to 0}\left|\frac{1}{t}\left[\langle \beta^-(v_0+th) - \beta^-(v_0), v_0\rangle\right] - \langle \beta^-(v_0), h\rangle\right| = 0,$$

which proves that β^- satisfies (51). The proof for β^+ is similar. The operator S is linear, compact and symmetric, therefore it also satisfies (51) for any $v_0 \in \mathbb{H}_D, \|v_0\| = 1$. Clearly, the operator $P = S - \beta^+ - \beta^-$ satisfies (51) for any $v_0 \in \mathbb{H}_D, \|v_0\| = 1$ as well.

□

Proof of Theorem 1. Since $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal base in $\mathbb{H}_D$, see Notation 1, for any $v \in \mathbb{H}_D$ there exists $\{\xi_i\} \in \ell^2$ such that

$$v = \sum_{i=1}^{\infty} \xi_i e_i.$$

As $Se_k = \lambda_k^S e_k$ by (40), (41) we get

$$\begin{aligned} &\sup_{v\in\mathbb{H}_D, \|v\|=1} \langle Sv - \beta^-(v) - \langle \beta^+(v), v\rangle = \\ &= \sup_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\left\langle S\left(\sum\limits_{k=1}^{\infty}\xi_k e_k\right), \sum\limits_{j=1}^{\infty}\xi_j e_j\right\rangle - \left\langle \beta^-\left(\sum\limits_{k=1}^{\infty}\xi_k e_k\right) + \beta^+\left(\sum\limits_{k=1}^{\infty}\xi_k e_k\right), \sum\limits_{j=1}^{\infty}\xi_j e_j\right\rangle}{\left\|\sum\limits_{j=1}^{\infty}\xi_j e_j\right\|^2} = \\ &= \sup_{\{\xi_j\}\in\ell^2\setminus\{0\}} \frac{\sum\limits_{j=1}^{\infty}\lambda_j^S\xi_j^2 + \sum\limits_{j=1}^{\infty}\xi_j\int_\Omega\left(\left(\sum\limits_{k=1}^{\infty}\xi_k e_k\right)^- s_- - \left(\sum\limits_{k=1}^{\infty}\xi_k e_k\right)^+ s_+\right) e_j \text{ dx}}{\sum\limits_{j=1}^{\infty}\xi_j^2} = d_2^m \end{aligned} \tag{62}$$

by Notation 2. Now we will prove the estimate (19). Due to (8) we have

$$
\begin{aligned}
-\langle\beta^-(v),v\rangle &= \int_\Omega s_-(x)v^-v \geq -\|s_-\|_{L^\infty}\int_\Omega v^2 = -\|s_-\|_{L^\infty}\langle Av,v\rangle,\\
-\langle\beta^+(v),v\rangle &= -\int_\Omega s_+(x)(v^+)^2 \geq -\|s_+\|_{L^\infty}\int_\Omega v^2 = -\|s_+\|_{L^\infty}\langle Av,v\rangle
\end{aligned}
\tag{63}
$$

for all $v\in\mathbb{H}_D$. The eigenvalues of the operator $S-\|s_-\|_{L^\infty}A-\|s_+\|_{L^\infty}A$ are

$$
\lambda_k^S - \frac{\|s_+\|_{L^\infty}+\|s_-\|_{L^\infty}}{\kappa_k},
$$

cf. Observation 1 and (41). By use of (52) with S replaced by $S-\|s_-\|_{L^\infty}A-\|s_+\|_{L^\infty}A$ we get

$$
\sup_{v\in\mathbb{H}_D,\|v\|=1}\langle Sv-\|s_-\|_{L^\infty}Av-\|s_+\|_{L^\infty}Av,v\rangle = \sup_{j\in\mathbb{N}}\left(\lambda_j^S-\frac{\|s_-\|_{L^\infty}+\|s_+\|_{L^\infty}}{\kappa_j}\right). \tag{64}
$$

If the last supremum is positive, then it is maximum. If it is equal to zero, then no maximizer exists, cf. Remark 7. Notation 2, Observation 3, the first statement of Lemma 4 and the formulae (62)– (64) give

$$
\begin{aligned}
d_2^0 = &\max_{j\in\mathbb{N}}\lambda_j^S = \lambda_{\max}^S = \max_{v\in\mathbb{H}_D,\|w\|=1}\langle Sv,v\rangle \geq \sup_{v\in\mathbb{H}_D,\|v\|=1}\langle Sv-\beta^-(v)-\beta^+(v),v\rangle =\\
=&d_2^m \geq \sup_{v\in\mathbb{H}_D,\|v\|=1}\langle Sv-(\|s_-\|_{L^\infty}+\|s_+\|_{L^\infty})Av,v\rangle =\\
=&\sup_{j\in\mathbb{N}}\left(\lambda_j^S-\frac{\|s_-\|_{L^\infty}+\|s_+\|_{L^\infty}}{\kappa_j}\right).
\end{aligned}
\tag{65}
$$

Hence, the upper estimate of d_2^m and a part of the lower estimate in (19) is proved. Due to the definition of K and (62) we see that

$$
\begin{aligned}
d_2^m &= \sup_{v\in\mathbb{H}_D,\|v\|=1}\langle Sv-\beta^-(v)-\beta^+(v),v\rangle \geq \sup_{v\in K,\|w\|=1}\langle Sv-\beta^-(v)-\beta^+(v),v\rangle =\\
&= \sup_{v\in K,\|w\|=1}\langle Sv,v\rangle = \sup_{\substack{\xi_j\in\ell^2\setminus\{0\}\\ \sum\xi_je_j\in K}}\frac{\sum_{j=1}^{\infty}\lambda_j^S\xi_j^2}{\sum_{i=1}^{\infty}\xi_i^2},
\end{aligned}
$$

which finishes the proof of (19).

We will verify that the assumptions of Theorem 5 are fulfilled for the positively homogeneous operator $P:=S-\beta^--\beta^+$. The equality (62) together with assumed positiveness of d_2^m yield

$$
\sup_{v\in\mathbb{H}_D,\|v\|=1}\langle Sv-\beta^-(v)-\beta^+(v),v\rangle > 0,
$$

and therefore (46) is true. The first assertion of Lemma 1 together with linearity and compactness of S imply (45). Lemma 4 guarantees that (51) is fulfilled for any $v_0\in\mathbb{H}_D$. Theorem 5 gives the existence of v_0, $\|v_0\|=1$ such that

$$
\lambda_{\max}^{S-\beta^--\beta^+} = \max_{v\in\mathbb{H}_D,\|v\|=1}\langle Sv-\beta^-(v)-\beta^+(v),v\rangle = \langle Sv_0-\beta^-(v_0)-\beta^+(v_0),v_0\rangle \tag{66}
$$

is the largest eigenvalue of the operator $S-\beta^+-\beta^-$. Now it follows from (62) and (66) that the supremum in (18) is maximum, i.e. (20) is proved. Due to Corollary 1, the point $d_2^m=\lambda_{\max}^{S-\beta^--\beta^+}$ is the largest critical point of the system (9), (3) with fixed d_1.

If $d_1 \in (y_2, y_1)$ then $\lambda^S_{\max} = \lambda^S_1$, $\lambda^S_{\max}$ is simple and the corresponding eigenfunction of S is e_1 by Observation 3 and (40). Since e_1 has a constant sign in Ω, see Notation 1, we get under the assumption $\|s_+\|_{L^\infty} > 0$ and $\|s_-\|_{L^\infty} > 0$ that

$$\langle \beta^+(e_1) + \beta^-(e_1), e_1\rangle = \int_\Omega s_+(x)(e_1^+)^2 + s_-(x)(e_1^-)^2 \text{ dx} > 0.$$

Let v_0 be from (66). If $v_0 \neq e_1$ then $\langle Sv_0, v_0\rangle < \lambda^S_{\max}$ by Remark 7 and

$$\langle Sv_0, v_0\rangle - \langle \beta^-(v_0) + \beta^+(v_0), v_0\rangle < \lambda^S_{\max}.$$

If $v_0 = e_1$ then $\langle Sv_0, v_0\rangle = \lambda^S_{\max}$, $\langle \beta^-(v_0) + \beta^+(v_0), v_0\rangle > 0$ and therefore

$$\langle Sv_0, v_0\rangle - \langle \beta^-(v_0) + \beta^+(v_0), v_0\rangle < \lambda^S_{\max}.$$

Summarizing, we get

$$d_2^m = \max_{v\in\mathbb{H}, \|v\|=1} \langle Sv - \beta^-(v) - \beta^+(v), v\rangle < \lambda^S_{\max} = d_2^0,$$

which together with the assumption $d_2^m > 0$ implies $(d_1, d_2^m) \in D_S$. □

Proof of Theorem 2. We have

$$\langle \beta^-(v), \varphi\rangle = -\int_\Omega s_-(x)v^-\varphi \text{ dx} \leq \|s_-\|_{L^\infty}\|v^-\|_{L^2}\|\varphi\|_{L^2} \leq C\|s_-\|_{L^\infty}\|v\|\|\varphi\|,$$

where C is a constant from Poincaré inequality. This implies

$$\|\beta^-(v)\| = \sup_{\varphi\in\mathbb{H}_D, \|\varphi\|=1} \langle \beta^-(v), \varphi\rangle \leq C\|s_-\|_{L^\infty}\|v\|.$$

Similarly for β^+ and therefore

$$\|\left(\beta^- + \beta^+\right)(v)\| \leq C\left(\|s_-\|_{L^\infty} + \|s_+\|_{L^\infty}\right)\|v\|.$$

We assume that d_2^0 has an odd multiplicity. By Observation 3, the eigenvalue $\lambda^S_{\max} = d_2^0$ has an odd multiplicity and therefore it follows from Theorem 6 and Observation 4 with $B := \beta^- + \beta^+$ that for any $\varepsilon \in (0, \min\{\lambda^S_{\max}, (\lambda^S_{\max} - \lambda^S_2)/2)$ there exists $\tau_0 > 0$ such that if $\|s_-\|_{L^\infty}, \|s_+\|_{L^\infty} < \tau_s := \tau_0/(2C)$, then $\lambda^S_{\max} - \varepsilon < \lambda^{S-\beta^--\beta^+}_{\max}$ and there is a global bifurcation point $\lambda_b \in [\lambda^S_{\max} - \varepsilon, \lambda^{S-\beta^--\beta^+}_{\max}]$ of the equation

$$\lambda v - Sv - N(v) + \beta^+(v) + \beta^-(v) = 0$$

in the sense of Theorem 6. The formulae (62) and (66) imply that $\lambda^{S-\beta^--\beta^+}_{\max} = d_2^m$. Due to Theorem 4 and Corollary 1, $d_2^b = \lambda_b \in [d_2^0 - \varepsilon, d_2^m] = [\lambda^S_{\max} - \varepsilon, \lambda^{S-\beta^--\beta^+}_{\max}]$ is simultaneously a global bifurcation point of the system (10), (3) with fixed d_1 in the sense of Theorem 2. □

To give an abstract formulation of the problem from Section 2.2, we define the operators $\beta^\pm_N : \mathbb{H}_D \to \mathbb{H}_D$ by

$$\langle \beta^-_N(v), \varphi\rangle = -\int_{\Gamma_N} s_-(x)v^-\varphi \text{ dS} \quad \text{for all } v, \varphi \in \mathbb{H}_D,$$
$$\langle \beta^+_N(v), \varphi\rangle = \int_{\Gamma_N} s_+(x)v^+\varphi \text{ dS} \quad \text{for all } v, \varphi \in \mathbb{H}_D.$$

The weak form of (25), (26) is

$$\begin{aligned} &d_1 u - b_{11}Au - b_{12}Av - N_1(u,v) = 0, \\ &d_2 v - b_{21}Au - b_{22}Av - N_2(u,v) + \beta^-_N(v) + \beta^+_N(v) = 0, \end{aligned} \tag{67}$$

and the weak form of (16), (26) is

$$
\begin{aligned}
&d_1 u - b_{11} A u - b_{12} A v = 0, \\
&d_2 v - b_{21} A u - b_{22} A v + \beta_N^-(v) + \beta_N^+(v) = 0.
\end{aligned} \tag{68}
$$

An analogue of Theorem 4 can be proved to get that

$$
\begin{aligned}
d_2 v - Sv - N(v) + \beta_N^-(v) + \beta_N^+(v) = 0, \\
u = F(v)
\end{aligned} \tag{69}
$$

is equivalent on a neighborhood of the origin with (67). Here F, S are the same as in Theorem 4, and N is a small nonlinear compact perturbation, i.e. $N(v)/\|v\| \to 0$ as $v \to 0$ in $\mathbb{H}_D$.

An analogue of Corollary 1 applies here as well. To be more precise, any eigenvalue of $S - \beta_N^- - \beta_N^+$ is simultaneously a critical point of (16), (26) with fixed d_1 and vice versa. A bifurcation of (69) occurs if and only if a bifurcation of (25), (26) with fixed d_1 occurs.

Proof of Theorem 3. An analogue of Lemma 4 can be proved. The operators $P := \beta_N^{\pm}$ fulfill (45) due to the compact embedding $\mathbb{H}_D \hookrightarrow\hookrightarrow L^2(\partial\Omega)$. For the proof of (51) with $P := \beta_N^-$ we introduce sets $\Gamma_{th}^+, \Gamma_{th}^-, \Gamma_0^+, \Gamma_0^-$ such that $\Gamma_N = \Gamma_{th}^+ \cup \Gamma_{th}^- = \Gamma_0^+ \cup \Gamma_0^-$,

$$
\begin{aligned}
&(v_0 + th)(x) < 0 \;\text{ for a.a. } x \in \Gamma_{th}^-, &&(v_0 + th)(x) \geq 0 \;\text{ for a.a. } x \in \Gamma_{th}^+, \\
&v_0(x) < 0 \;\text{ for a.a. } x \in \Gamma_0^-, &&v_0(x) \geq 0 \;\text{ for a.a. } x \in \Gamma_0^+,
\end{aligned}
$$

and $\Gamma_{th1}, \Gamma_{th2}, \Gamma_{th3}$ such that $\Gamma_{th}^- = \Gamma_{th1} \cup \Gamma_{th2}$, $\Gamma_0^- = \Gamma_{th1} \cup \Gamma_{th3}$,

$$
\begin{aligned}
&v_0(x) < -th(x) \text{ and } v_0(x) < 0 \text{ for a.a. } x \in \Gamma_{th1}, \\
&v_0(x) < -th(x) \text{ and } v_0(x) \geq 0 \text{ for a.a. } x \in \Gamma_{th2}, \\
&v_0(x) \geq -th(x) \text{ and } v_0(x) < 0 \text{ for a.a. } x \in \Gamma_{th3}.
\end{aligned}
$$

Similarly for the operator β_N^+. Then we can follow the proof of Lemma 4. The proof of the first part of Theorem 3 is now almost the same as the proof of Theorem 1. To prove the second part of Theorem 3, we will use Theorem 6 in the same way as in the proof of Theorem 2. ☐

6. Appendix

For completeness, we will explain here in more details a result concerning a smoothness of the map N_1 used in Section 3 and a general result concerning a global bifurcation from an interval used in Section 4.

6.1. Smoothness of the map N_1 from (30)

Theorem 7. *Under the assumptions (5), (12), (13) the operator N_1 defined in (30) satisfies $N_1 \in C^1(\mathbb{H}_D \times \mathbb{H}_D, \mathbb{H}_D)$ and its Fréchet derivative is given by*

$$
\langle N_1'(u,v)(h_1,h_2), \varphi \rangle = \int_\Omega n_1'(u,v)(h_1,h_2) \cdot \varphi \; dx = \int_\Omega (\partial_u n_1(u,v) h_1 + \partial_v n_1(u,v) h_2) \varphi \; dx, \quad \text{for all } u, v, \varphi, h_1, h_2 \in \mathbb{H}_D. \tag{70}
$$

Proof. Under the assumptions (12), (13), Nemyckii operators $(u,v) \to \partial_u n_1(u,v)$, $(u,v) \to \partial_v n_1(u,v)$ map $L^p \times L^p$ into $L^{\frac{p}{p-2}}$. Hence, using the embedding $H \hookrightarrow L^p$, for any $u, v, h_1, h_2 \in \mathbb{H}_D$ we can define $N_1'(u,v)(h_1,h_2) \in \mathbb{H}_D$ by (70). We will show that $N_1'(u,v)(h_1,h_2)$ is a directional derivative of N_1 at the point (u,v) and in the direction (h_1,h_2). Let $B_1 \subset \mathbb{H}_D$ be the unit ball centered at the origin. Using Hölder inequality we get

$$\begin{aligned}\lim_{t\to 0}&\left\|\frac{N_1((u,v)+t(h_1,h_2))-N_1(u,v)}{t}-N_1'(u,v)(h_1,h_2)\right\|=\\ &=\lim_{t\to 0}\sup_{\varphi\in B_1}\int_\Omega\left(\frac{n_1((u,v)+t(h_1,h_2))-n_1(u,v)}{t}-n_1'(u,v)(h_1,h_2)\right)\varphi\ \mathrm{dx}\le\\ &\le C\lim_{t\to 0}\left(\int_\Omega\left|\frac{n_1((u,v)+t(h_1,h_2))-n_1(u,v)}{t}-n_1'(u,v)(h_1,h_2)\right|^{p'}\mathrm{dx}\right)^{\frac{1}{p'}},\end{aligned}$$

where $p' = p/(p-1)$, p is from (12). We want to apply Dominated Convergence Theorem to exchange limit and integral, hence, we have to find an integrable majorant. We use Mean Value Theorem to get

$$\left|\frac{n_1((u,v)+t(h_1,h_2))-n_1(u,v)}{t}-n_1'(u,v)(h_1,h_2)\right|=|(n_1'((u,v)+t\theta(h_1,h_2))-n_1'(u,v))(h_1,h_2)|$$
$$\text{for a.a. } x\in\Omega,$$

where $\theta(x)\in[0,1]$ for a.a. $x\in\Omega$. From now we will use one universal symbol C for various constants. We use the triangle inequality and condition (13) to get the existence of $C>0$ such that

$$\begin{aligned}|n_1'((u,v)+\theta t(h_1,h_2))(h_1,h_2)|&\le\left|\frac{\partial n_1}{\partial u}(u+\theta th_1,v+\theta th_2)\right||h_1|+\left|\frac{\partial n_1}{\partial v}(u+\theta th_1,v+\theta th_2)\right||h_2|\le\\ &\le C\left(1+|u+\theta th_1|^{p-2}+|v+\theta th_1|^{p-2}\right)(|h_1|+|h_2|).\end{aligned}\tag{71}$$

The Young inequality with $(p-1)/(p-2)$ and $(p-1)$ implies

$$|u+\theta th_1|^{p-2}|h_1|\le C(|u|^{p-2}|h_1|+(\theta t)^{p-2}|h_1|^{p-1})\le C(|u|^{p-1}+(1+(\theta t)^{p-2})|h_1|^{p-1}).$$

Analogous estimates can be done for the other terms in (71). Using all these estimates together with the embedding $\mathbb{H}_D\hookrightarrow L^p$ we get for sufficiently small t that

$$\begin{aligned}&|(n_1'((u,v)+t\theta(h_1,h_2))-n_1'(u,v))(h_1,h_2)|\le\\ &\qquad\le C\left(|u|^{p-1}+|v|^{p-1}+|h_1|^{p-1}+|h_2|^{p-1}\right)\in L^{p'}\ \text{ for any } u,v,h_1,h_2\in\mathbb{H}_D.\end{aligned}$$

Summarizing, we obtain

$$\begin{aligned}&\left|\frac{n_1((u,v)+t(h_1,h_2))-n_1(u,v)}{t}-n_1'(u,v)(h_1,h_2)\right|^{p'}\le\\ &\qquad\le C\left(|u|^{p-1}+|v|^{p-1}+|h_1|^{p-1}+|h_2|^{p-1}\right)^{p'}\in L^1,\end{aligned}$$

and Dominated Convergence Theorem gives

$$\lim_{t\to 0}\left\|\frac{N_1((u,v)+t(h_1,h_2))-N_1(u,v)}{t}-N_1'(u,v)(h_1,h_2)\right\|=0.$$

Hence, $N_1'(u,v)(h_1,h_2)$ is a directional derivative of $N_1(u,v)$ in an arbitrary direction (h_1,h_2).

Let $(u,v)\in\mathbb{H}_D\times\mathbb{H}_D$ be arbitrary fixed. It is clear that the operator $N_1'(u,v):(h_1,h_2)\mapsto N_1'(u,v)(h_1,h_2)$ from (70) is linear. Using the generalized Hölder inequality and (13) we get

$$\begin{aligned}\|N_1'(u,v)(h_1,h_2)\|&=\sup_{\varphi\in B_1}\int_\Omega n_1'(u,v)(h_1,h_2)\varphi\ \mathrm{dx}\le\\ &\le C(1+\|u\|_{L^p}+\|v\|_{L^p})(\|h_1\|_{L^p}+\|h_2\|_{L^p})\le C(\|h_1\|+\|h_2\|).\end{aligned}$$

Hence, the linear operator $N_1'(u,v)$ is bounded and therefore it is a Gâteaux derivative.

Let $(u_0,v_0)\in\mathbb{H}_D\times\mathbb{H}_D$ be arbitrary. Then

$$\lim_{(u,v)\to(u_0,v_0)}\|N_1'(u,v)-N_1'(u_0,v_0)\|_{\mathcal{L}(\mathbb{H}_D\times\mathbb{H}_D,\mathbb{H}_D\times\mathbb{H}_D)}=$$
$$=\lim_{(u,v)\to(u_0,v_0)}\sup_{\varphi\in B_1}\sup_{(h_1,h_2)\in B_1\times B_1}\int_\Omega\left(n_1'(u,v)-n_1'(u_0,v_0)\right)(h_1,h_2)\cdot\varphi\,\mathrm{dx}.$$

The growth conditions (13) and the generalized Hölder inequality leads to

$$\int_\Omega\left(n_1'(u,v)-n_1'(u_0,v_0)\right)(h_1,h_2)\cdot\varphi\le$$
$$\le\left\|\frac{\partial n_1}{\partial u}(u,v)-\frac{\partial n_1}{\partial u}(u_0,v_0)\right\|_{L^{\frac{p}{p-2}}}\|h_1\|_{L^p}\|\varphi\|_{L^p}+\left\|\frac{\partial n_1}{\partial v}(u,v)-\frac{\partial n_1}{\partial v}(u_0,v_0)\right\|_{L^{\frac{p}{p-2}}}\|h_2\|_{L^p}\|\varphi\|_{L^p}.$$

The Nemyckii operators $(u,v)\to\partial_u n_1(u,v)$, $(u,v)\to\partial_v n_1(u,v)$ are under the conditions (13) continuous from $L^p\times L^p$ into $L^{\frac{p}{p-2}}$. Hence,

$$\lim_{(u,v)\to(u_0,v_0)}\left\|\frac{\partial n_1}{\partial u}(u,v)-\frac{\partial n_1}{\partial u}(u_0,v_0)\right\|_{L^{\frac{p}{p-2}}}=0,\quad\lim_{(u,v)\to(u_0,v_0)}\left\|\frac{\partial n_1}{\partial v}(u,v)-\frac{\partial n_1}{\partial v}(u_0,v_0)\right\|_{L^{\frac{p}{p-2}}}=0,$$

and

$$\lim_{(u,v)\to(u_0,v_0)}\|N_1'(u,v)-N_1'(u_0,v_0)\|_{\mathcal{L}(\mathbb{H}_D\times\mathbb{H}_D,\mathbb{H}_D\times\mathbb{H}_D)}=0,$$

i.e. the map $(u,v)\to N'(u,v)$ from $\mathbb{H}_D\times\mathbb{H}_D$ into $\mathcal{L}(\mathbb{H}_D\times\mathbb{H}_D)$ is continuous and therefore it is a Fréchet derivative, see e.g. Proposition 3.2.15 in [3]. □

6.2. A global bifurcation for positively homogeneous problems

We give below a Rabinowitz type global bifurcation result used in the proof of Theorem 6. Since we need only some of the assumptions considered in Section 4, we formulate them explicitly. We will denote by $\mathcal{S}(B)$ the set introduced before Theorem 6 but now for more general operators considered in the following theorem.

Theorem 8. *Let X be Banach space, $S:X\to X$ a linear compact operator, $B:X\to X$ a positively homogeneous continuous compact operator and $N:\mathbb{R}\times X\to X$ a nonlinear compact operator satisfying (48). Let us assume that positive $\lambda_1<\lambda_2$ are not eigenvalues of the operator $S-B$ and*

$$deg\left(I-\frac{1}{\lambda_1}(S-B),B_r,0\right)\neq deg\left(I-\frac{1}{\lambda_2}(S+B),B_r,0\right)\ \text{for all } r>0.\tag{72}$$

Then there exists $\lambda_b\in[\lambda_1,\lambda_2]$ such that the connected component $\mathcal{S}_{\lambda_b}$ of the set $\mathcal{S}(B)$ containing the point $(\lambda_b,0)$ satisfies at least one of the following conditions:

(a) *$\mathcal{S}_{\lambda_b}$ is unbounded,*
(b) *there exists $v\in X, v\neq 0$ such that $(0,v)\in\overline{\mathcal{S}_{\lambda_b}}$,*
(c) *there exists an eigenvalue $\lambda_c\notin[\lambda_1,\lambda_2]$ of the operator $S-B$ such that $(\lambda_c,0)\in\overline{\mathcal{S}_{\lambda_b}}$.*

Proof. Our assertion can be obtained as a particular case of very abstract Theorem 7 in [18], where we set

$$\Lambda=(0,\infty),\ \Omega=X,\ \Omega_0=B_r,\ r>0\text{ small enough},$$
$$F=I,\ \phi(\lambda,v)=\lambda^{-1}(Sv-B(v)+N(\lambda,v)),\ x_0=0$$

and $\mathcal{B} = \mathcal{B}_0$ can be the system of all bounded subsets of $(0, \infty) \times X$, see also remarks below Proposition 8 in [18]. The assumptions (7), (8) in that theorem and (a),(b) on the top of the p. 217 can be written in our particular situation as the following conditions:

$$\text{zero is an isolated solution of (49) for any } \lambda \text{ in a neighbourhood of } \lambda_1 \text{ and } \lambda_2, \tag{73}$$

$$\deg(I - \frac{1}{\lambda_1}(S + B - N), B_r, 0) \neq \deg(I - \frac{1}{\lambda_2}(S - B + N), B_r, 0) \text{ for } r > 0 \text{ small enough}, \tag{74}$$

$$\text{the set of all } (\lambda, v) \text{ satisfying (49) is closed in } (0, \infty) \times X, \tag{75}$$

$$\text{any closed and bounded set of } (\lambda, v) \text{ satisfying (49) is compact.} \tag{76}$$

Let us verify these conditions. If (73) were not true then $\lambda^{(n)}, v_n$ satisfying (49) would exist such that $\lambda^{(n)} \to \lambda_j$, $j = 1$ or $j = 2$, $v_n \to 0$. Dividing (49) by $\|v_n\|$ and using the compactness of S and B and the condition (48) we would get a subsequence of v_n satisfying $v_{n_k}/\|v_{n_k}\| \to w$ with some $w \in X$ and $\lambda_j w = Sw - B(w)$. Therefore λ_j would be an eigenvalue of the operator $S - B$, which is a contradiction with the assumptions. The condition (74) for sufficienly small $r > 0$ follows easily from (72) by the homotopy invariance of the degree by using the homotopy $H(t, v) = v - \frac{1}{\lambda_j}(Sv + B - tN(\lambda_j, v))$, $t \in [0, 1]$ and the assumption (48). The condition (75) is clearly fulfilled due to continuity of our maps. The condition (76) follows from the compactness of the operator $S - B + N$.

Now, the assertion of Theorem 7 in [18] translated to our particular situation gives the assertion of our Theorem 8. Let us only recall that we have chosen $\mathcal{B}_0$ as the system of all bounded subsets of $(0, \infty) \times X$ and therefore our case (a) coincides with the condition (i) in Theorem from [18] stating that $\mathcal{S}_b$ is not contained in a set from $\mathcal{B}_0$.

□

Proof of Theorem 8 can be done also directly in a similar way as that of Theorem 1.3 in [16]. The difference is that the case $B \equiv 0$ is considered and no assumption about the degree is necessary in [16] because bifurcation from a characteristic value μ of S of odd multiplicity is discussed. Our condition (72) follows then by using Leray-Schauder formula even for arbitrary λ_1, λ_2 sufficiently close to $\frac{1}{\mu}$ and $\lambda_1 < \frac{1}{\mu}$, $\lambda_2 > \frac{1}{\mu}$. We have to realize that if (72) is true then also

$$\deg\left(I - \left(\frac{1}{\lambda_1} - \delta\right)(S + B), B_r, 0\right) \neq \deg\left(I - \left(\frac{1}{\lambda_1} - \delta\right)(S + B), B_r, 0\right)$$

for all $\delta > 0$ small enough and all $r > 0$. Then we can modify the proof from [16] replacing the component $\mathcal{C}_\mu$ of the set $\mathcal{S}$ discussed in [16] by the component containing $\mathcal{S}_{\lambda_1,\lambda_2}$ to get that $\mathcal{S}_{\lambda_1,\lambda_2}$ satisfies at least one of the conditions (a), (b), (c).

7. Acknowledgement

Research of M. Kučera has been supported by the Project 13-00863S of the Grant Agency of the Czech Republic and by RVO:67985840. J. Navrátil has been supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS16/239/OHK4/3T/14.

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