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ranges and its consequences
for asymptotics of operator iterates**

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ON INTERPLAY BETWEEN JOINT NUMERICAL RANGES AND ITS CONSEQUENCES FOR ASYMPTOTICS OF OPERATOR ITERATES

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ABSTRACT. We identify subsets of the joint numerical range of an operator tuple in terms of its joint spectrum. This result helps us to transfer weak convergence of operator orbits into certain approximation and interpolation properties for powers in the uniform operator topology. This is a far-reaching generalization of one of the main results in our recent paper [24]. Moreover, it yields an essential (but partial) generalization of Bourin’s “pinching” theorem from [5]. It also allows us to revisit several basic results on joint numerical ranges, provide them with new proofs and find a number of new results.

1. INTRODUCTION

The theory of joint numerical ranges is a developing area of operator theory with several important results obtained in the last years. The geometric structure of joint numerical ranges has got a considerable attention, and many properties of numerical ranges have been transferred or appropriately recasted from the setting of a single operator to the framework of operator tuples, see e.g. [3], [17]-[20], [23], [24] and references therein. At the same time, the relations between spectrum of an operator tuple and its numerical range remained rather obscure until very recent time. We are aware of [32] as the only important contribution to those issues, which moreover was apparently overlooked by the experts. Recently in [24], we have discovered new spectral inclusion results for operator tuples. Specified for tuples formed by powers of a single operator, the results allowed us to identify the unit circle in the spectrum of a bounded linear operator on a Hilbert space in terms of orthogonality and “quasi-orthogonality” relations for the operator orbits, see e.g. [24, Theorem 5.1]. This constituted an essential generalization of the corresponding results by Arveson who dealt with unitary operators only.

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Moreover, in [24], by means of spectral approximations of numerical ranges, we put recent harmonic analysis considerations by Hamdan ([13]) into the operator setting and extended them by, in particular, replacing a single orbit in Hamdan's statement by an infinite-dimensional subspace of orbits. (See below for more on that.)

The present paper brings further insights into the relations between spectrum and numerical range for operator tuples, and uses them to obtain new asymptotic properties of operator orbits under quite general assumptions. More precisely, we extend, complement and sharpen several main results from [24] on numerical ranges $W(T_1, \dots, T_n)$ of tuples $\mathcal{T} = (T_1, \dots, T_n)$ of bounded linear operators on a Hilbert space H , and in this way obtain essential generalizations of results from [24] on asymptotic behavior of compressions of operator powers. As byproduct of our approach, we obtain a partial generalization of a "pinching" theorem by Bourin [5, Theorem 2.1] in a much more demanding setting of operator tuples. For recent applications of [5, Theorem 2.1] see [6].

One of the novelties in our approach, stemming from [24], is that in our studies of geometric properties of operator iterates we rely on the numerical ranges methodology. It is instructive to note that the condition of orthogonality of elements from an orbit of $T \in B(H)$ can be rewritten in terms of the joint numerical range of the tuple $\mathcal{T} = (T, \dots, T^n)$. On the other hand, as we prove below, the joint numerical range $W(\mathcal{T})$ contains the interior of the convex hull of the joint spectrum $\sigma(\mathcal{T})$ (in spite of the fact that the joint numerical range is in general not convex). Using inductive arguments, this fact helps us to construct orbits of T with special geometric properties from the vectors resembling (essential) approximate eigenvectors of T . The constructions are far from being straightforward, and we have to invoke new ideas not present in [24].

More precisely, our considerations rely on the following spectral inclusion result.

Theorem 1.1. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then*

$$(1.1) \quad \text{Int conv } (W_e(\mathcal{T}) \cup \sigma_p(\mathcal{T})) \subset W(\mathcal{T}).$$

Moreover, if the tuple \mathcal{T} is commuting then

$$(1.2) \quad \text{Int conv } \sigma(\mathcal{T}) \subset W(\mathcal{T}).$$

Theorem 1.1 can be considered as a partial generalization of the main result in [32, Theorem 2.2] dealing with numerical ranges of operators on Banach spaces, and also as a generalization of [24, Corollaries 4.2 and 4.4], where $\sigma_p(\mathcal{T})$ was absent in (1.1), and (1.2) was stated with $\sigma(\mathcal{T})$ replaced by $\sigma_e(\mathcal{T})$. Note that while the result in [32] allows to find parts of the spectrum of \mathcal{T} in $\overline{W(\mathcal{T})}$, we may replace $\overline{W(\mathcal{T})}$ by a smaller and more transparent set $W(\mathcal{T})$. The proof of Theorem 1.1 requires new tools, e.g. Zenger's lemma, and it is technically more demanding than the corresponding arguments in [24].

To present our applications of Theorem 1.1 (or rather its predecessor from [24]), let us recall that, motivated by applications in ergodic theory, Hamdan characterized in [13] the size of the spectrum of some unitary operators by a new type of asymptotic assumptions. He proved that if a unitary operator T on H is such that $T^n \rightarrow 0$ in the weak operator topology, then $\sigma(T) = \mathbb{T}$ if and only if for every $\epsilon > 0$ there exists a unit vector $x \in H$ satisfying

$$(1.3) \quad \sup_{n \geq 1} |\langle T^n x, x \rangle| < \epsilon.$$

His result has been extended in [24] to general bounded operators and to the setting allowing to take x 's in (1.3) from an infinite-dimensional subspace. More precisely, we proved in [24, Corollary 6.3 and Remark 6.4] that if T is a bounded linear operator on H such that $T^n \rightarrow 0$ in the weak operator topology, and $\sigma(T) \supset \mathbb{T}$, then for every $\epsilon > 0$ one can find an infinite-dimensional subspace L of H such that the compressions $(T^n)_L$ of T^n to L are asymptotically small in two senses:

$$\lim_{n \rightarrow \infty} \|(T^n)_L\| = 0 \quad \text{and} \quad \sup_{n \geq 1} \|(T^n)_L\| < \epsilon.$$

This result can be complemented as follows.

Theorem 1.2. *Let $T \in B(H)$ be such that $T^n \rightarrow 0$ in the weak operator topology, and let $\sigma(T) \supset \mathbb{T}$. Let \tilde{C} be a strict contraction on a separable Hilbert space, i.e., $\|\tilde{C}\| < 1$. Then for every $\epsilon > 0$ there exists a subspace $L \subset H$ and $C \in B(L)$ unitarily equivalent to \tilde{C} such that*

$$(1.4) \quad \lim_{n \rightarrow \infty} \|(T^n)_L - C^n\| = 0 \quad \text{and} \quad \sup_{n \geq 1} \|(T^n)_L - C^n\| \leq \epsilon.$$

It is natural to ask whether the asymptotic relation (1.4) above can be made an exact equality. Surprisingly, the answer is “yes”, if one restricts oneself to a finite piece of the orbit $(C^n)_{n \geq 1}$. More precisely, the following theorem holds.

Theorem 1.3. *Let $T \in B(H)$ and suppose that the polynomial hull $\hat{\sigma}(T)$ of $\sigma(T)$ contains the unit disc \mathbb{D} . Let $k \in \mathbb{N}$ and let \tilde{C} be a strict contraction on a separable Hilbert space. Then there exists a subspace $L \subset H$ and $C \in B(L)$ unitarily equivalent to \tilde{C} such that $(T^n)_L = C^n$ for $j = 1, \dots, k$.*

Theorem 1.3 is a tuples version of the main result in [5], where the case $k = 1$ was considered.

Despite the main motivation for the paper was to understand how far Hamdan's type results can be pushed by operator-theoretical technique, as a byproduct of our approach we found new arguments for the proofs of recent characterizations of essential and infinite numerical ranges, as well as several new statements concerning numerical ranges which are of independent interest. Recall that the infinite numerical range $W_\infty(\mathcal{T})$ of a tuple $\mathcal{T} = (T_1, \dots, T_n)$ can be defined as

$$W_\infty(\mathcal{T}) := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : PT_jP = \lambda_j P, \quad j = 1, \dots, n\}$$

for some infinite rank projection P . We prove that the essential numerical range $W_e(\mathcal{T})$ of \mathcal{T} can be described in terms of $W_\infty(\mathcal{T})$ as

$$W_e(\mathcal{T}) = \bigcup_{\mathcal{K} \in \mathcal{K}(H)^n} W_\infty(\mathcal{T} - \mathcal{K}).$$

Moreover there exists an n -tuple \mathcal{K} of trace-class operators on H such that $W_e(\mathcal{T}) = \overline{W_\infty(\mathcal{T} - \mathcal{K})}$. We also show that for every tuple $\mathcal{T} \in B(H)^n$ of bounded linear operators on H one has

$$\overline{\text{conv}} W(\mathcal{T}) = \text{conv} (W(\mathcal{T}) \cup W_e(\mathcal{T})),$$

and

$$\text{Int } \overline{W(\mathcal{T})} \subset W(\mathcal{T})$$

if $\text{Int } W_e(\mathcal{T}) \neq \emptyset$.

2. NOTATION

It will be convenient to fix some notations in a separate section. In particular, we let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and $B(H)$ the space of all bounded linear operators on H . For a bounded linear operator T we denote by $\sigma(T)$ its spectrum, and by $N(T)$ its kernel.

If M is a subspace of a Hilbert space H then we denote by P_M the orthogonal projection onto M and by J_M the natural embedding $J_M : M \rightarrow H$, defined by $Jx = x$, $x \in M$. For an operator $T \in B(H)$, we denote by $T_M : M \rightarrow M$ the compression of T to M . Note that $T_M = J_M^* T J_M$.

In the following we consider an n -tuple $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Note that we do not in general assume that the operators T_j commute. For $x, y \in H$ we write shortly $\langle \mathcal{T}x, y \rangle = (\langle T_1x, y \rangle, \dots, \langle T_nx, y \rangle) \in \mathbb{C}^n$ and $\mathcal{T}x = (T_1x, \dots, T_nx) \in H^n$. Similarly for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ we write $\mathcal{T} - \lambda = (T_1 - \lambda_1, \dots, T_n - \lambda_n)$ and $\|\lambda\| = \max\{|\lambda_1|, \dots, |\lambda_n|\}$. If $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ and $R, S \in B(H)$ then

$$(2.1) \quad RTS := (RT_1S, \dots, RT_nS).$$

If $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ and $\mathcal{S} = (S_1, \dots, S_n) \in B(H)^n$ are n -tuples of operators, then we say that \mathcal{T} and \mathcal{S} are unitarily equivalent and write $\mathcal{T} \stackrel{u}{\sim} \mathcal{S}$ if there exists a unitary operator $U : H \rightarrow H$ such that $T_j = US_jU^{-1}$ for every $j = 1, \dots, n$.

For a closed set $K \subset \mathbb{C}^n$ we denote by ∂K the topological boundary of K , by $\text{Int } K$ the interior of K , by $\text{conv } K$ the convex hull of K , and by \hat{K} the polynomial hull of K . Recall that if $K \subset \mathbb{C}$ then \hat{K} is the union of K with all bounded components of the complement $\mathbb{C} \setminus K$.

Finally, we let \mathbb{T} stand for the unit circle $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$, \mathbb{D} for the unit disc $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\mathbb{R}_+ = [0, \infty)$.

3. PRELIMINARIES

We start with recalling certain basic notions and facts from the spectral theory of operator tuples on Hilbert spaces. They can be found e.g. in [22, Chapters 2-3].

Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ be an n -tuple of commuting operators. Recall that its joint (Harte) spectrum $\sigma(\mathcal{T})$ can be defined as the complement of the set of those $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ for which

$$\sum_{j=1}^n L_j(T_j - \lambda_j) = \sum_{j=1}^n (T_j - \lambda_j)R_j = I$$

for some $L_j, R_j, 1 \leq j \leq n$, from the algebra $B(H)$. It is well-known that $\sigma(\mathcal{T})$ is a non-empty compact subset of \mathbb{C}^n . One can define the joint essential spectrum $\sigma_e(\mathcal{T})$ as the (Harte) spectrum of the n -tuple $(T_1 + \mathcal{K}(H), \dots, T_n + \mathcal{K}(H))$ in the Calkin algebra $B(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the ideal of all compact operators on H . Note that if $n = 1$ then $\sigma_e(T_1) = \{\lambda_1 \in \mathbb{C} : T_1 - \lambda_1 \text{ is not Fredholm}\}$, and for $T \in B(H)$ and $\mathcal{T} = (T, T^2, \dots, T^n) \in B(H)^n$, one has $\sigma(\mathcal{T}) = \{(\lambda, \dots, \lambda^n) : \lambda \in \sigma(T)\}$ and $\sigma_e(\mathcal{T}) = \{(\lambda, \dots, \lambda^n) : \lambda \in \sigma_e(T)\}$.

For the rest of this section, we assume that a n -tuple \mathcal{T} is not necessarily commuting. Denote by $\sigma_p(\mathcal{T})$ the point spectrum of \mathcal{T} , i.e., the set of all n -tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that $\bigcap_{j=1}^n N(T_j - \lambda_j) \neq \{0\}$. If $x \in \bigcap_{j=1}^n N(T_j - \lambda_j)$ then we will write $\mathcal{T}x = \lambda x$. Remark, however, that in fact we will not need a somewhat cumbersome spectral theory of non-commuting tuples \mathcal{T} .

As in the case of a single operator, it is often useful to relate $\sigma(\mathcal{T})$ to a larger and more easily computable set $W(\mathcal{T}) \subset \mathbb{C}^n$ called the joint numerical range of \mathcal{T} and defined as

$$W(\mathcal{T}) = \{(\langle T_1 x, x \rangle, \dots, \langle T_n x, x \rangle) : x \in H, \|x\| = 1\}.$$

The set $W(\mathcal{T})$ can be identified with a subset of \mathbb{R}^{2n} if one identifies the n -tuple \mathcal{T} with the $2n$ -tuple $(\operatorname{Re} T_1, \operatorname{Im} T_1, \dots, \operatorname{Re} T_n, \operatorname{Im} T_n)$ of selfadjoint operators. Unfortunately, if $n > 1$, then $W(\mathcal{T})$ is not in general convex, see e.g. [19].

As in the spectral theory, there is also a notion of the joint essential numerical range $W_e(\mathcal{T})$ associated to \mathcal{T} . For $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ we define $W_e(\mathcal{T})$ as the set of all n -tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that there exists an orthonormal sequence $(x_k) \subset H$ with

$$\lim_{k \rightarrow \infty} \langle T_j x_k, x_k \rangle = \lambda_j, \quad j = 1, \dots, n.$$

Alternatively, $W_e(\mathcal{T})$ can be defined as

$$W_e(\mathcal{T}) := \bigcap \overline{W(T_1 + K_1, \dots, T_n + K_n)}$$

where the intersection is taken over all n -tuples K_1, \dots, K_n of compact operators on H . Recall that $W_e(\mathcal{T})$ is a nonempty, compact and, in contrast to

$W(\mathcal{T})$, convex subset of $\overline{W(\mathcal{T})}$, [19]. As a straightforward consequence of the definitions above, if a tuple $\mathcal{T} \in B(H)^n$ is commuting then $\sigma_e(\mathcal{T}) \subset W_e(\mathcal{T})$.

There's also a useful and related notion of the numerical range for tuples of elements of a unital Banach algebra \mathcal{A} . For $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ define

$$(3.1) \quad V(a, \mathcal{A}) := \{(f(a_1), \dots, f(a_n)) : f \in \mathcal{A}^*, f(1) = \|f\| = 1\},$$

and recall that $f \in \mathcal{A}^*$ such that $f(1) = \|f\| = 1$ are called states. With such a definition, one has

$$(3.2) \quad \overline{\text{conv}} W(\mathcal{T}) = V(\mathcal{T}, B(H))$$

and

$$(3.3) \quad W_e(\mathcal{T}) = V(T_1 + \mathcal{K}(H), \dots, T_n + \mathcal{K}(H), B(H)/\mathcal{K}(H)).$$

The proofs of (3.2) and (3.3) can be found in [23, Theorem 1 and Theorem 2], respectively. For a comprehensive account of essential numerical ranges one may consult [19].

The next result due to Zenger is used in a number of operator-theoretical constructions. For its proof one may consult e.g. [4, p. 18-20].

Lemma 3.1 (Zenger Lemma). *Let $u_1, \dots, u_n \in H$ be linearly independent, and let $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$ be such that $\sum_{k=1}^n \alpha_k = 1$. Then there exist $w_1, \dots, w_n \in \mathbb{C}$ and $u \in H$, $\|u\| \leq 1$, satisfying $\|\sum_{j=1}^n w_j u_j\| \leq 1$ and*

$$\langle w_j u_j, u \rangle = \alpha_j, \quad 1 \leq j \leq n.$$

Note that $\langle \sum_{j=1}^n w_j u_j, u \rangle = 1$, and so $u = \sum_{j=1}^n w_j u_j$.

4. SPECTRA AND NUMERICAL RANGES FOR TUPLES

In the following we consider an n -tuple $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Note that we do not assume that the operators T_j commute.

It was proved in [24, Corollary 4.2] that

$$(4.1) \quad \text{Int } W_e(\mathcal{T}) \subset W(\mathcal{T}).$$

Moreover, if $\lambda = (\lambda_1, \dots, \lambda_n) \in \text{Int } W_e(\mathcal{T})$, then there exists an infinite-dimensional subspace L of H such that

$$(4.2) \quad P_L T_j P_L = \lambda_j P_L, \quad j = 1, \dots, n,$$

where P_L is the orthogonal projection on L . So, despite $T_j, 1 \leq j \leq n$, do not commute, in a number of situations of interest they have diagonal compressions to the same subspace. Moreover, (4.1) has important spectral consequences. It was shown in [24, Corollary 4.4] that

$$(4.3) \quad \text{Int conv } \sigma_e(\mathcal{T}) \subset W(\mathcal{T}).$$

On the other hand, (4.1) has certain drawbacks. For instance, if one of the operators $T_j, 1 \leq j \leq n$, is compact then $\text{Int } W_e(T_1, \dots, T_n) = \emptyset$, and (4.1) says nothing. Thus, it is desirable, to obtain extensions of (4.1) shedding also light on $W(\mathcal{T})$ in the case of tuples with ‘‘small’’ essential numerical range. The next theorem serves just that purpose. Extending (4.1), it allows

one to describe "big" subsets of $W(\mathcal{T})$ in spectral terms. The result is also related to [32, Theorem 2.2] where weaker statements have been obtained. As [32, Theorem 2.2], the theorem below depends on Zenger's Lemma, and also uses the following simple statement.

Lemma 4.1. *Let $\mathcal{T} \in B(H)^n$. If $\lambda \in W_e(\mathcal{T})$, then for every $\delta > 0$ and every subspace $M \subset H$ of finite codimension there exists a unit vector $x \in M$ such that $\|\langle \mathcal{T}x, x \rangle - \lambda\| < \delta$.*

Proof. If $\lambda \in W_e(\mathcal{T})$ then there exists an orthonormal sequence $(x_i)_{i \geq 1}$ in H such that $\langle \mathcal{T}x_i, x_i \rangle \rightarrow \lambda$, $i \rightarrow \infty$. Let $M \subset H$ be a subspace of a finite codimension and $\delta > 0$. We have that $\|P_{M^\perp}x_i\| \rightarrow 0$, and so $\|P_Mx_i - x_i\| \rightarrow 0$ as $i \rightarrow \infty$. Set

$$u_i = \frac{P_Mx_i}{\|P_Mx_i\|}, \quad i \geq 1.$$

Then $\lim_{i \rightarrow \infty} \|u_i - x_i\| = 0$ and $\lim_{i \rightarrow \infty} \langle \mathcal{T}u_i, u_i \rangle = \lambda$. Hence there exists i_0 such that $\|\langle \mathcal{T}u_{i_0}, u_{i_0} \rangle - \lambda\| < \delta$. \square

Theorem 4.2. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then*

$$\text{Int conv}(W_e(\mathcal{T}) \cup \sigma_p(\mathcal{T})) \subset W(\mathcal{T}).$$

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \text{Int conv}(W_e(\mathcal{T}) \cup \sigma_p(\mathcal{T}))$. We show that $\lambda \in W(\mathcal{T})$. By considering the n -tuple $(T_1 - \lambda_1, \dots, T_n - \lambda_n)$ instead of (T_1, \dots, T_n) , we can assume without loss of generality that $(\lambda_1, \dots, \lambda_n) = (0, \dots, 0)$.

Let $r > 0$ satisfy

$$\{(\varepsilon_1, \dots, \varepsilon_n) : \max_j |\varepsilon_j| \leq r\} \subset \text{conv}(W_e(\mathcal{T}) \cup \sigma_p(\mathcal{T})).$$

Let $\lambda^{(1)}, \dots, \lambda^{(m)} \in \sigma_p(\mathcal{T}) \setminus W_e(\mathcal{T})$ be a finite set such that

$$\min\{\|\mu - \lambda^{(i)}\| : i = 1, \dots, m\} < \frac{r}{2}$$

for all $\mu \in \sigma_p(\mathcal{T})$.

We show that

$$(4.4) \quad \left\{ \varepsilon \in \mathbb{C}^n : \|\varepsilon\| \leq \frac{r}{2} \right\} \subset \text{conv}(W_e(\mathcal{T}) \cup \{\lambda^{(1)}, \dots, \lambda^{(m)}\}).$$

Let $a_1, \dots, a_n \in \mathbb{C}$, $\sum_{i=1}^n |a_i| = 1$, and $c \in \mathbb{R}$. Assume that

$$\text{Re} \sum_{i=1}^n a_i z_i \geq c$$

for all $z = (z_1, \dots, z_n) \in W_e(\mathcal{T}) \cup \{\lambda^{(1)}, \dots, \lambda^{(m)}\}$. Then

$$\text{Re} \sum_{i=1}^n a_i z_i \geq c - \frac{r}{2}$$

for all $z = (z_1, \dots, z_n) \in W_e(\mathcal{T}) \cup \sigma_p(\mathcal{T})$. Consequently,

$$\operatorname{Re} \sum_{i=1}^n a_i z_i \geq c - \frac{r}{2} \quad \text{for all } z \in \mathbb{C}^n, \quad \|z\| \leq r.$$

Setting $z_i = e^{-i \arg a_i} r$, $1 \leq i \leq n$, in the inequality above we infer that $c + \frac{r}{2} \leq 0$.

If $z \in \mathbb{C}^n$, $\|z\| \leq \frac{r}{2}$, then

$$\operatorname{Re} \sum_{i=1}^n a_i z_i \geq -\frac{r}{2} \geq c.$$

Since the convex hull of $W_e(\mathcal{T}) \cup \{\lambda^{(1)}, \dots, \lambda^{(m)}\}$ is the intersection of all halfspaces containing it, this shows (4.4).

Fix now eigenvectors $u_1, \dots, u_m \in H$ such that $\mathcal{T}u_i = \lambda^{(i)}u_i$, $\|u_i\| = 1$.

Let

$$F = \bigvee_{i=1}^m u_i \quad \text{and} \quad M = F^\perp \cap \bigcap_{j=1}^n (T^j F)^\perp \cap \bigcap_{j=1}^n (T_j^* F)^\perp.$$

Clearly $\dim F < \infty$ and $\operatorname{codim} M < \infty$. Note that

$$F \perp M, \quad T_j F \perp M \quad \text{and} \quad T_j M \perp F \quad \text{for all } j = 1, \dots, n.$$

We construct the unit vector $x \in H$ satisfying $\langle T_j x, x \rangle = 0$ for all $1 \leq j \leq n$ inductively as a limit of consecutive approximations. Set $x_0 = v_0 = w_0 = 0$. It will be convenient to separate the following fact.

Claim. Let $r > 0$, $F, M \subset H$ be as above. Let $k \in \mathbb{N} \cup \{0\}$, let $v_k \in M$, $w_k \in F$ and $x_k = v_k + w_k$ satisfy

$$w_k = \sum_{i=1}^m t_i u_i, \quad \langle t_i u_i, w_k \rangle = \alpha_i \geq 0, \quad \|w_k\|^2 = \sum_{i=1}^m \alpha_i,$$

$$\|x_k\|^2 = 1 - 2^{-k} \quad \text{and} \quad \|\langle \mathcal{T} x_k, x_k \rangle\| < \frac{r}{2^{k+2}}.$$

Then there exist v_{k+1}, w_{k+1} and x_{k+1} ,

$$x_{k+1} = v_{k+1} + w_{k+1}, \quad v_{k+1} \in M, \quad w_{k+1} \in F,$$

such that

$$w_{k+1} = \sum_{i=1}^m s_i u_i, \quad \langle s_i u_i, w_{k+1} \rangle = \beta_i \geq 0, \quad \|w_{k+1}\|^2 = \sum_{i=1}^m \beta_i,$$

$$\|x_{k+1}\|^2 = 1 - 2^{-k-1}, \quad \|v_{k+1} - v_k\|^2 \leq \frac{1}{2^{k+1}}$$

and

$$\|\langle \mathcal{T} x_{k+1}, x_{k+1} \rangle\| < \frac{r}{2^{k+3}}.$$

To prove the claim, let $\varepsilon = \langle \mathcal{T}x_k, x_k \rangle$. So $\|\varepsilon\| < \frac{r}{2^{k+2}}$. Thus there exist elements $\lambda^{(m+1)}, \dots, \lambda^{(m')} \in W_e(\mathcal{T})$ and numbers $c_1, \dots, c_{m'} \geq 0$ such that $\sum_{i=1}^{m'} c_i = 1$ and

$$\sum_{i=1}^{m'} c_i \lambda^{(i)} = -\varepsilon 2^{k+1}.$$

By Zenger's lemma, there are complex numbers s_1, \dots, s_n such that $w_{k+1} := \sum_{i=1}^m s_i u_i$ satisfies

$$\langle s_i u_i, w_{k+1} \rangle = \alpha_i + \frac{c_i}{2^{k+1}} \quad \text{and} \quad \|w_{k+1}\|^2 = \sum_{i=1}^m \left(\alpha_i + \frac{c_i}{2^{k+1}} \right).$$

Thus

$$\langle \mathcal{T}w_{k+1}, w_{k+1} \rangle = \sum_{i=1}^m \left(\alpha_i + \frac{c_i}{2^{k+1}} \right) \lambda^{(i)}.$$

The elements $\lambda^{(m+1)}, \dots, \lambda^{(m')}$ belong to $W_e(\mathcal{T})$. Using Lemma 4.1 and the induction argument, we can construct unit vectors $y_{m+1}, \dots, y_{m'}$ in the following way.

Suppose that $m \leq s < m'$ and that the vectors y_{m+1}, \dots, y_s have already been constructed. Set

$$G := \bigvee \{v_k, T_j v_k, y_i, T_j y_i : m+1 \leq i \leq s, 1 \leq j \leq n\},$$

$$L := G^\perp \cap \bigcap_{j=1}^n (T_j^* G)^\perp.$$

Then $\dim G < \infty$ and $\text{codim } L < \infty$. Hence there exists a unit vector $y_{s+1} \in L \cap M$ such that

$$\left\| \langle T y_{s+1}, y_{s+1} \rangle - \lambda^{(s+1)} \right\| < \frac{r}{4}.$$

If the vectors $y_{m+1}, \dots, y_{m'}$ are constructed in this way, set

$$v_{k+1} = v_k + \sum_{i=m+1}^{m'} \left(\frac{c_i}{2^{k+1}} \right)^{1/2} y_i \quad \text{and} \quad x_{k+1} = v_{k+1} + w_{k+1}.$$

Then

$$\|v_{k+1}\|^2 = \|v_k\|^2 + \sum_{i=m+1}^{m'} \frac{c_i}{2^{k+1}},$$

$$\|v_{k+1} - v_k\|^2 = \sum_{i=m+1}^{m'} \frac{c_i}{2^{k+1}} \leq \frac{1}{2^{k+1}}$$

and

$$\begin{aligned} \|x_{k+1}\|^2 &= \|v_{k+1}\|^2 + \|w_{k+1}\|^2 = \|v_k\|^2 + \sum_{i=m+1}^{m'} \frac{c_i}{2^{k+1}} + \sum_{i=1}^m \left(\alpha_i + \frac{c_i}{2^{k+1}} \right) \\ &= \|v_k\|^2 + \frac{1}{2^{k+1}} + \|w_k\|^2 = \|x_k\|^2 + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}. \end{aligned}$$

Finally,

$$\begin{aligned} \|\langle \mathcal{T} x_{k+1}, x_{k+1} \rangle\| &= \left\| \langle \mathcal{T} v_k, v_k \rangle + \sum_{i=m+1}^{m'} \frac{c_i}{2^{k+1}} \langle \mathcal{T} y_i, y_i \rangle + \langle \mathcal{T} w_{k+1}, w_{k+1} \rangle \right\| \\ &\leq \left\| \langle \mathcal{T} v_k, v_k \rangle + \sum_{i=m+1}^{m'} \frac{c_i \lambda^{(i)}}{2^{k+1}} + \sum_{i=1}^m \left(\alpha_i + \frac{c_i}{2^{k+1}} \right) \lambda^{(i)} \right\| \\ &\quad + \sum_{i=m+1}^{m'} \frac{c_i}{2^{k+1}} \|\langle \mathcal{T} y_i, y_i \rangle - \lambda^{(i)}\| \\ &\leq \left\| \langle \mathcal{T} x_k, x_k \rangle + \frac{1}{2^{k+1}} \sum_{i=1}^{m'} c_i \lambda^{(i)} \right\| + \sum_{m+1}^{m'} \frac{c_i}{2^{k+1}} \cdot \frac{r}{4} \\ &= \frac{r}{2^{k+3}}. \end{aligned}$$

This finishes the proof of the claim.

Now construct the vectors v_k, w_k and $x_k = v_k + w_k, k \in \mathbb{N}$, inductively as described in the Claim. Clearly $(v_k)_{k \geq 1}$ is a Cauchy sequence, and we let $v \in M$ be its limit. The sequence $(w_k)_{k \geq 1}$ is a bounded sequence in a finite-dimensional space F . By passing to a subsequence if necessary, we may assume that $(w_k)_{k \geq 1}$ is convergent, $w_k \rightarrow w \in F, k \rightarrow \infty$. The vector

$$x = v + w = \lim_{k \rightarrow \infty} (v_k + w_k) = \lim_{k \rightarrow \infty} x_k$$

satisfies $\|x\| = 1$ and $\langle \mathcal{T} x, x \rangle = 0$. This finishes the proof. \square

Theorem 4.2 yields the following generalization of [24, Corollary 4.4], replacing $\sigma_e(\mathcal{T})$ by $\sigma(\mathcal{T})$ there, cf. (4.3). The generalization complements [32, Corollary 2.3] where, for a commuting tuple $\mathcal{T} \in B(H)^n$, it was shown that

$$(4.5) \quad \text{conv } \sigma(\mathcal{T}) \subset \overline{W(\mathcal{T})}.$$

(Note that as it will be clear below we will be interested in spectral inclusions for the numerical range $W(\mathcal{T})$, rather than for its closure.)

Corollary 4.3. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ be a commuting n -tuple. Then*

$$\text{Int conv } \sigma(\mathcal{T}) \subset W(\mathcal{T}).$$

Proof. Since $\sigma_e(\mathcal{T}) \subset W_e(\mathcal{T})$, by [22, Theorem 19.18], we have

$$\operatorname{conv} \sigma(T) \subset \operatorname{conv} (\sigma_e(\mathcal{T}) \cup \sigma_p(\mathcal{T})) \subset \operatorname{conv} (W_e(\mathcal{T}) \cup \sigma_p(\mathcal{T})),$$

The statement follows then from Theorem 4.2. \square

The statements like Theorem 4.2 specified for power tuples (T, T^2, \dots, T^n) allow one to find appropriate tuples of powers of complex numbers in their joint numerical ranges $W(T, T^2, \dots, T^n)$, thus revealing certain geometric properties of the orbits of T . For instance, the fact that $(0, \dots, 0) \in W(T, T^2, \dots, T^n)$ yields an element $x \in H$ such that $x \perp T^k x$ for all k between 1 and n . The latter property was introduced and characterized in spectral terms for unitary T by Arveson, [2]. For its far-reaching realizations see [24].

The next theorem was proved in [24, Corollary 4.2 and Corollary 4.10]. It will be instrumental in Section 6 below dealing with asymptotic properties of compressions of powers.

Theorem 4.4. *Let $T \in B(H)$ and let λ belong to the interior of polynomial hull of $\sigma(T)$. Then*

$$(\lambda, \lambda^2, \dots, \lambda^n) \in \operatorname{Int} W_e(T, T^2, \dots, T^n) \subset W(T, T^2, \dots, T^n).$$

for all $n \in \mathbb{N}$.

Note that the assumption on λ as in the theorem above is quite natural and apparently close to optimal as the following statements show.

Proposition 4.5. *Let $T \in B(H)$, $n \in \mathbb{N}$. Suppose that*

- (i) $(0, \dots, 0) \notin \operatorname{Int} \operatorname{conv} W(T, T^2, \dots, T^n)$;
- (ii) *there exists a unit vector $x \in H$ such that $x \perp Tx, T^2x, \dots, T^{2n}x$.*

Then $\sigma_p(T) \neq \emptyset$.

Proof. Suppose $x \in H$ satisfy (ii). Then by (i) there exist complex numbers c_1, \dots, c_n such that

$$\operatorname{Re} \sum_{j=1}^n \langle c_j T^j u, u \rangle \geq 0$$

for all $u \in H$.

Let $S = \sum_{j=1}^n c_j T^j$. Let $\alpha \in \mathbb{C}$ and $y = \alpha x + Sx$. Then

$$0 \leq \operatorname{Re} \langle Sy, y \rangle = \operatorname{Re} \langle \alpha Sx + S^2x, Sx \rangle = \operatorname{Re} \left(\alpha \|Sx\|^2 + \langle S^2x, Sx \rangle \right).$$

Since this is true for all $\alpha \in \mathbb{C}$, we have $Sx = 0$. So $0 \in \sigma_p(S) = \sigma_p(\sum_{j=1}^n c_j T^j) = \{\sum_{j=1}^n c_j \lambda^j : \lambda \in \sigma_p(T)\}$ by the spectral mapping theorem for the point spectrum. Hence $\sigma_p(T) \neq \emptyset$. \square

Corollary 4.6. *Let $T \in B(H)$, $\sigma_p(T) = \emptyset$. Suppose that for all $n \in \mathbb{N}$ one has $(0, \dots, 0) \in W(T, \dots, T^n)$. Then*

$$(0, \dots, 0) \in \operatorname{Int} \operatorname{conv} W(T, \dots, T^n)$$

for all $n \in \mathbb{N}$.

5. JOINT NUMERICAL RANGES REVISITED

In this section we use the numerical range inclusion results proved above to provide alternative and, we believe, sometimes simpler proofs of the theorems describing essential and so-called infinite numerical ranges for tuples in terms of their compressions and higher rank numerical ranges. The results were (essentially) obtained in [19] and [20], see also [17], [27], and [30] for their single operator analogues, and [23] for complementary results.

Moreover, our techniques allow us to prove several new results on joint numerical ranges of independent interest.

Let for the rest of this section H will stand for an infinite-dimensional separable Hilbert space. We start with general considerations on joint numerical ranges. Recall that the joint numerical range $W(\mathcal{T})$ is, in general, neither convex nor closed. Thus, it makes sense to describe the closed convex hull of $W(\mathcal{T})$ in terms of $W(\mathcal{T})$ and the related set $W_e(\mathcal{T})$. The following result is an extension of a similar theorem due to Lancaster for single operators [16]. Its proof is based on an idea of Williams from [31] leading to an alternative proof of Lancaster's theorem.

Theorem 5.1. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then*

$$\overline{\text{conv}} W(\mathcal{T}) = \text{conv} (W(\mathcal{T}) \cup W_e(\mathcal{T})).$$

Proof. Since $W_e(\mathcal{T}) \subset \overline{W(\mathcal{T})}$, we have the inclusion “ \supset ”.

Conversely, let $(\lambda_1, \dots, \lambda_n) \in \overline{\text{conv}} W(\mathcal{T})$. Recall that by (3.2) one has $\overline{\text{conv}} W(\mathcal{T}) = V(\mathcal{T}, B(H))$. So there exists a state $f \in B(H)^*$ such that $f(T_j) = \lambda_j$ for all $j = 1, \dots, n$. By Dixmier's theorem [8], one has a decomposition $f = \alpha f_0 + (1 - \alpha)f_1$, where $0 \leq \alpha \leq 1$ and f_0, f_1 are states on $B(H)$ such that f_0 annihilates the ideal of compact operators $\mathcal{K}(H)$ on H , and $f_1(A) := \text{trace}(AS)$ for a fixed trace class operator $S \geq 0$ and all $A \in B(H)$. Hence there exist an orthonormal system $(e_k)_{k \geq 1} \subset H$ and positive numbers β_k with $\sum_{k \geq 1} \beta_k = 1$ such that $S = \sum_{k \geq 1} \beta_k e_k \otimes e_k$. Thus

$$f_1(A) = \sum_{k \geq 1} \beta_k \langle Ae_k, e_k \rangle$$

for all $A \in B(H)$. Recall that a convex set in \mathbb{C}^n is invariant with respect to taking infinite convex combinations of its elements (note that the set may be not closed), see e.g. [7] or [28]. Thus, since clearly $\{(Te_k, e_k) : k \geq 1\} \subset \text{conv} W(\mathcal{T})$, we have

$$f_1(\mathcal{T}) \in \text{conv} W(\mathcal{T}).$$

By (3.3), $V(T_1 + \mathcal{K}(H), \dots, T_n + \mathcal{K}(H), B(H)/\mathcal{K}(H)) = W_e(\mathcal{T})$. Hence $f_0(\mathcal{T}) \in W_e(\mathcal{T})$, and thus

$$(\lambda_1, \dots, \lambda_n) = (f(T_1), \dots, f(T_n)) \in \text{conv} (W(\mathcal{T}) \cup W_e(\mathcal{T})).$$

□

Despite topological properties of joint numerical ranges are much more complicated than the properties of numerical ranges for single operators, they can be described in terms of other numerical ranges that are somewhat simpler to deal with. Let us recall now the definition of higher rank numerical ranges.

Definition 5.2. Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Let $k \in \mathbb{N} \cup \{\infty\}$. Define the k -th rank numerical range of \mathcal{T} as the set of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that there exists a subspace $L \subset H$ with $\dim L = k$ satisfying

$$P_L T_j P_L = \lambda_j P_L, \quad j = 1, \dots, n.$$

The set $W_\infty(\mathcal{T})$ is called the infinite numerical range of \mathcal{T} .

Clearly $W_1(\mathcal{T})$ is the usual joint numerical range and

$$W_1(\mathcal{T}) \supset W_2(\mathcal{T}) \supset \dots \supset W_\infty(\mathcal{T}).$$

It is easy to see that $W_\infty(\mathcal{T})$ can be empty even for $n = 1$ — consider an injective compact operator T_1 . Note that property (4.2) implies that for any $\mathcal{T} \in B(H)^n$,

$$(5.1) \quad \text{Int}(W_e(\mathcal{T})) \subset W_\infty(\mathcal{T}),$$

so $W_\infty(\mathcal{T})$ is large if $W_e(\mathcal{T})$ is large. On the other hand, in infinite-dimensional spaces the k -th rank numerical range is always nonempty for each $k \in \mathbb{N}$, as the following proposition (implicit in [20]) shows.

Proposition 5.3. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then $W_k(\mathcal{T}) \neq \emptyset$ for all $k \in \mathbb{N}$.*

Proof. We prove the statement by induction on n . By e.g. [18, Theorem 1], one infers that $W_k(T_1) \neq \emptyset$ for any operator $T_1 \in B(K)$ with $\dim K \geq 3k - 2$. In particular, $W_k(T_1) \neq \emptyset$ for all $k \in \mathbb{N}$.

Suppose the statement is true for some $n - 1 \geq 1$. Let $k \in \mathbb{N}$. By the induction assumption, $W_{4k}(T_1, \dots, T_{n-1}) \neq \emptyset$. So there exists $(\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{C}^{n-1}$ and a subspace $L \subset H$ with $\dim L = k$ such that

$$P_L T_j P_L = \lambda_j P_L, \quad j = 1, \dots, n - 1.$$

By the same result from [18], $W_k(P_L T_n P_L) \neq \emptyset$. So there exists $\lambda_n \in \mathbb{C}$ and a subspace $L' \subset L$ with $\dim L' = k$ and $P_{L'} T_n P_{L'} = \lambda_n P_{L'}$. Hence $P_{L'} T_i P_{L'} = \lambda_i P_{L'}$, $1 \leq i \leq n$, so that $(\lambda_1, \dots, \lambda_n) \in W_k(T_1, \dots, T_n)$ and $W_k(T_1, \dots, T_n) \neq \emptyset$. \square

For $k < \infty$ the higher rank numerical ranges $W_k(\mathcal{T})$ are, in general, not convex. However, they are always star-shaped, as we prove below. See [20, Proposition 4.1] for an analogous statement.

Theorem 5.4. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then for every $k \in \mathbb{N}$ the set $W_k(\mathcal{T})$ is star-shaped with star centers taken from $W_m(\mathcal{T})$ for any $m > k(2n + 1)$.*

Proof. Let $k \in \mathbb{N}$. Fix $m > k(2n + 1)$. By Proposition 5.3, it follows that $W_m(\mathcal{T}) \neq \emptyset$, so we can choose $\lambda \in W_m(\mathcal{T}) \subset W_k(\mathcal{T})$. We show that $W_k(\mathcal{T})$ is star-shaped with the center λ .

Let $\mu \in W_k(\mathcal{T})$ and $t \in [0, 1]$. There exists a subspace $L \subset H$ with $\dim L = k$ and $P_L \mathcal{T} P_L = \mu P_L$. Let x_1, \dots, x_k be an orthonormal set in L .

Let $M \subset H$ satisfy $\dim M = m$ and $P_M \mathcal{T} P_M = \lambda P_M$. We construct an orthonormal set $y_1, \dots, y_k \in M$ in the following way: Let y_1 be any unit vector in $M \cap \{L, T_j L, T_j^* L : 1 \leq j \leq n\}^\perp$. Choose inductively unit vectors $y_s \in M, 2 \leq s \leq k$, such that

$$\begin{aligned} y_s &\perp \{L, T_j L, T_j^* L : 1 \leq j \leq n, \} \\ y_s &\perp \{y_i, T_j y_i, T_j^* y_i : 1 \leq i \leq s-1, 1 \leq j \leq n\}. \end{aligned}$$

Let

$$u_s := \sqrt{t}x_s + \sqrt{1-t}y_s, \quad s = 1, \dots, k \quad \text{and} \quad L' := \bigvee_{s=1}^k u_s.$$

Clearly $\dim L' = k$ and the vectors u_1, \dots, u_k form an orthonormal basis in L' . If $y \in L', \|y\| = 1$, then $y = \sum_{s=1}^k \alpha_s u_s$ for some $\{\alpha_s : 1 \leq s \leq k\} \subset \mathbb{C}$ with $\sum_{s=1}^k |\alpha_s|^2 = 1$. We have

$$\begin{aligned} \langle \mathcal{T}y, y \rangle &= \left\langle \mathcal{T} \sum_{s=1}^k \alpha_s \sqrt{t}x_s, \sum_{s=1}^k \alpha_s \sqrt{t}x_s \right\rangle + 2\operatorname{Re} \left\langle \mathcal{T} \sum_{s=1}^k \alpha_s \sqrt{t}x_s, \sum_{s=1}^k \alpha_s \sqrt{1-t}y_s \right\rangle \\ &\quad + \left\langle \mathcal{T} \sum_{s=1}^k \alpha_s \sqrt{1-t}y_s, \sum_{s=1}^k \alpha_s \sqrt{1-t}y_s \right\rangle \\ &= \mu \left\| \sum_{s=1}^k \alpha_s \sqrt{t}x_s \right\|^2 + \lambda \left\| \sum_{s=1}^k \alpha_s \sqrt{1-t}y_s \right\|^2 \\ &= t\mu + (1-t)\lambda. \end{aligned}$$

Hence $t\mu + (1-t)\lambda \in W_k(\mathcal{T})$, so the set $W_k(\mathcal{T})$ is star-shaped with the center at $\lambda \in W_m(\mathcal{T})$, as required. \square

Remark 5.5. It is easy to see that the closure $\overline{W_k(\mathcal{T})}$ is also star-shaped.

The infinite and essential numerical range have ‘‘infinite-dimensional’’ nature. However it is possible to describe them in terms of ‘‘finite-dimensional’’ higher rank numerical ranges. Moreover, we characterize the infinite and essential numerical ranges of tuples by means of compressions of tuples to infinite-dimensional subspaces.

The equivalence (i) \Leftrightarrow (ii) in the proposition below was first proved in [20, Theorem 4.1].

Proposition 5.6. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. The following statements are equivalent:*

- (i) $\lambda \in W_\infty(\mathcal{T})$;

- (ii) $\lambda \in \bigcap_{k=1}^{\infty} W_k(\mathcal{T})$;
- (iii) for every subspace $M \subset H$ of finite codimension there exists a unit vector $x \in M$ such that $\langle \mathcal{T}x, x \rangle = \lambda$.

Proof. The implication (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii): Let $M \subset H$ be a subspace of a finite codimension. Let $k \in \mathbb{N}$, $k > \text{codim } M$. By (ii) there exists a subspace $F \subset H$ with $\dim F = k$ and $P_F \mathcal{T} P_F = \lambda P_F$. Then $F \cap M \neq \{0\}$, and any unit vector in $F \cap M$ satisfies (iii).

(iii) \Rightarrow (i): Using (iii), find a unit vector $x_1 \in H$ such that $\langle \mathcal{T}x_1, x_1 \rangle = \lambda$. Construct inductively a sequence $(x_i)_{i \geq 1} \subset H$ of unit vectors such that

$$x_{i+1} \perp \{x_m, T_j x_m, T_j^* x_m : 1 \leq m \leq i, 1 \leq j \leq n\}$$

and

$$\langle \mathcal{T}x_i, x_i \rangle = \lambda$$

for all $i \in \mathbb{N}$. Let $L = \bigvee_{i=1}^{\infty} x_i$. Clearly L is an infinite-dimensional subspace with an orthonormal basis $(x_i)_{i \geq 1}$. Let $y \in L$, $\|y\| = 1$, so that $y = \sum_{i=1}^{\infty} \alpha_i x_i$ where $\sum_{i=1}^{\infty} |\alpha_i|^2 = 1$. Then

$$\langle \mathcal{T}y, y \rangle = \sum_{i=1}^{\infty} |\alpha_i|^2 \langle \mathcal{T}x_i, x_i \rangle = \lambda \sum_{i=1}^{\infty} |\alpha_i|^2 = \lambda,$$

so that $P_L \mathcal{T} P_L = \lambda P_L$. □

Incidentally, in the general setting of operator tuples, Proposition 5.6 gives a partial answer to the old question of Fillmore, Stampfli and Pearcy [11, p. 190, Remark (4)] on the description for $T \in B(H)$ of the set $\lambda \in \mathbb{C}$ such that $P(T - \lambda)P = 0$ for an infinite-rank projection P . It can also be considered as a sharper version of [3, Theorem 3.1.1] where an approximate version of the proposition has been proved.

The equivalence (i) \Leftrightarrow (ii) in the next result has been first obtained in [20, Corollary 4.5].

Proposition 5.7. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. The following statements are equivalent:*

- (i) $\lambda \in W_e(\mathcal{T})$;
- (ii) $\lambda \in \bigcap_{k=1}^{\infty} \overline{W_k(\mathcal{T})}$;
- (iii) for every $\delta > 0$ and every subspace $M \subset H$ of finite codimension there exists a unit vector $x \in M$ such that $\|\langle \mathcal{T}x, x \rangle - \lambda\| < \delta$.

Proof. (iii) \Rightarrow (i) is clear.

(ii) \Rightarrow (iii): Let $M \subset H$ be a subspace of a finite codimension and $\delta > 0$. Let $k \in \mathbb{N}$, $k > \text{codim } M$. By (ii), there exists a $\mu \in \overline{W_k(\mathcal{T})}$ such that $\|\lambda - \mu\| < \delta$. Let F be a subspace of H with $\dim F = k$ and $P_F \mathcal{T} P_F = \mu P_F$. Then $F \cap M \neq \{0\}$. Let $x \in F \cap M$ be any unit vector. Then

$$\|\langle \mathcal{T}x, x \rangle - \lambda\| = \|\mu - \lambda\| < \delta.$$

(i) \Rightarrow (iii): This is proved in Lemma 4.1.

(iii) \Rightarrow (ii): Let $\lambda \in \mathbb{C}^n$ satisfy (iii) for some $\delta > 0$. Let $k \in \mathbb{N}$ be fixed. Choose inductively an orthonormal sequence $(x_i)_{i \geq 1} \subset H$ such that

$$x_{i+1} \perp \{x_m, T_j x_m, T_j^* x_m : 1 \leq m \leq i, 1 \leq j \leq n\}$$

and

$$\|\langle T x_{i+1}, x_{i+1} \rangle - \lambda\| < \delta.$$

Let $L = \bigvee_{i=1}^{\infty} x_i$. Then $\dim L = \infty$ and $\|\langle T y, y \rangle - \lambda\| < \delta$ for all $y \in L$, $\|y\| = 1$. Hence

$$\|P_L T_j P_L - \lambda_j P_L\| < 2\delta, \quad j = 1, \dots, n.$$

Note that $W_k(P_L T P_L) \neq \emptyset$, and let $\mu \in W_k(P_L T P_L)$. Then $\|\mu - \lambda\| < 2\delta$. Since $\delta > 0$ was arbitrary, $\lambda \in \overline{W_k(\mathcal{T})}$. \square

One important application of item (iii) in the propositions above is the proof of convexity for $W_{\infty}(\mathcal{T})$ and $W_e(\mathcal{T})$. By different arguments, the convexity of $W_e(\mathcal{T})$ was first proved in [19, Theorem 3.1], while the fact that $W_{\infty}(\mathcal{T})$ is convex was discovered in [20, Theorem 4.2], see also [27].

Theorem 5.8. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then the sets $W_{\infty}(\mathcal{T})$ and $W_e(\mathcal{T})$ are convex.*

Proof. Let $\lambda, \mu \in W_{\infty}(\mathcal{T})$ and $t \in [0, 1]$. Let $M \subset H$, $\text{codim } M < \infty$. By Proposition 5.6 there exists $x \in M$, $\|x\| = 1$ and $\langle T x, x \rangle = \lambda$.

Similarly there exists a unit vector $y \in M \cap \{x, T_j x, T_j^* x : 1 \leq j \leq n\}^{\perp}$ such that $\langle T y, y \rangle = \mu$.

Let $u = \sqrt{t}x + \sqrt{1-t}y$. Clearly $u \in M$, $\|u\| = 1$ and

$$\langle T u, u \rangle = t \langle T x, x \rangle + (1-t) \langle T y, y \rangle = t\lambda + (1-t)\mu.$$

By Proposition 5.6 again, $t\lambda + (1-t)\mu \in W_{\infty}(\mathcal{T})$.

The convexity of $W_e(\mathcal{T})$ can be proved similarly using Proposition 5.7 instead of Proposition 5.6. \square

By (4.2),

$$(5.2) \quad \text{Int } W_e(\mathcal{T}) \subset W_{\infty}(\mathcal{T}) \subset W_e(\mathcal{T})$$

for any $\mathcal{T} \in B(H)^n$ and $W_e(\mathcal{T})$ is stable under compact perturbations. The behaviour of $W_{\infty}(\mathcal{T})$ under compact perturbations is given in Theorem 5.10 below. To prove it, we need a result of independent interest.

Proposition 5.9. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$, and let $\Lambda \subset W_e(\mathcal{T})$ be a countable set. Then there exists an n -tuple $\mathcal{K} = (K_1, \dots, K_n)$ of trace-class normal operators on H such that*

$$\Lambda \subset W_{\infty}(\mathcal{T} - \mathcal{K}).$$

Proof. Let $\Lambda = \{\lambda_1, \lambda_2, \dots\}$. Let $f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ be a bijection. For $s \in \mathbb{N}$ write $f(s) = (f_1(s), f_2(s))$.

We construct inductively an orthonormal sequence $(e_s)_{s \geq 1} \subset H$ in the following way:

Fix $s \geq 1$ and suppose that the vectors $e_1, \dots, e_{s-1} \in H$ have already been constructed. Since $\lambda_{f_1(s)} \in W_e(\mathcal{T})$, there exists an orthonormal sequence $(x_k) \subset H$ such that $\lim_{k \rightarrow \infty} \langle \mathcal{T}x_k, x_k \rangle = \lambda_{f_1(s)}$.

Let

$$F_s = \bigvee \{e_i, T_j e_i, T_j^* e_i : 1 \leq i \leq s-1, 1 \leq j \leq n\}.$$

Since $\dim F_s < \infty$, there exists $m \in \mathbb{N}$ such that

$$\|P_{F_s} x_m\| \leq 2^{-s} \quad \text{and} \quad \|\langle \mathcal{T}x_m, x_m \rangle - \lambda_{f_1(s)}\| \leq 2^{-s}.$$

Set $e_s = \frac{(I - P_{F_s})x_m}{\|(I - P_{F_s})x_m\|}$. Then $\|e_s\| = 1$ and

$$e_s \perp e_i, T_j e_i, T_j^* e_i, \quad 1 \leq i \leq s-1, 1 \leq j \leq n.$$

We also have

$$\begin{aligned} \|x_m - e_s\| &\leq \|x_m - (I - P_{F_s})x_m\| + \left\| (I - P_{F_s})x_m - \frac{(I - P_{F_s})x_m}{\|(I - P_{F_s})x_m\|} \right\| \\ &\leq \|P_{F_s} x_m\| + \left(1 - \frac{1}{1 - 2^{-s}}\right) \\ &\leq 3 \cdot 2^{-s}. \end{aligned}$$

Moreover, if $\varepsilon_s = (\varepsilon_{s,1}, \dots, \varepsilon_{s,n}) \in \mathbb{C}^n$ is given by $\varepsilon_s = \langle \mathcal{T}e_s, e_s \rangle - \lambda_{f_1(s)}$ then

$$\begin{aligned} |\varepsilon_{s,j}| &= |\langle T_j e_s, e_s \rangle - \lambda_{f_1(s),j}| \\ &\leq |\langle T_j(e_s - x_m), e_s \rangle| + |\langle T_j x_m, e_s - x_m \rangle| + |\langle T_j x_m, x_m \rangle - \lambda_{f_1(s),j}| \\ &\leq 3 \cdot 2^{-s} \|T_j\| + 3 \cdot 2^{-s} \|T_j\| + 2^{-s}, \quad 1 \leq j \leq n. \end{aligned}$$

Now, for every $j = 1, \dots, n$ define $K_j \in B(H)$ by

$$K_j = \sum_{s=1}^{\infty} \varepsilon_{s,j} e_s \otimes e_s.$$

By construction, K_j is a trace-class normal operator and $\langle (T_j - K_j)e_s, e_s \rangle = \lambda_{f_1(s),j}$ for all $s \in \mathbb{N}$.

For $k \in \mathbb{N}$ let $L_k = \bigvee \{e_s : f_1(s) = k\}$. Clearly $\dim L_k = \infty$. If $f_1(s) = k$ then $\langle T_j e_s, e_s \rangle = \lambda_{k,j}$. Moreover, if $f_1(s) = k = f_1(s')$ and $s \neq s'$ then $\langle T_j e_s, e_{s'} \rangle = 0 = \langle T_j e_{s'}, e_s \rangle$ for all $j = 1, \dots, n$. It is easy to see that this means that

$$P_{L_k}(\mathcal{T} - \mathcal{K})P_{L_k} = \lambda_k P_{L_k}.$$

Hence $\lambda_k \in W_{\infty}(\mathcal{T} - \mathcal{K})$. \square

Using Proposition 5.9, we can now express $W_e(\mathcal{T})$ in terms of the infinite numerical ranges of compact perturbations of $W_{\infty}(\mathcal{T})$. The result below seems to be new even for single operators.

Theorem 5.10. *Let $\mathcal{T} \in B(H)^n$. Then:*

- (i) $W_e(\mathcal{T}) = \bigcup_{\mathcal{K} \in \mathcal{K}(H)^n} W_\infty(\mathcal{T} - \mathcal{K});$
- (ii) *there exists an n -tuple \mathcal{K} of compact operators such that $W_e(\mathcal{T}) = \overline{W_\infty(\mathcal{T} - \mathcal{K})}$.*

Proof. To show (i), observe that $W_e(\mathcal{T}) = W_e(\mathcal{T} - \mathcal{K})$ for any n -tuple of compact operators \mathcal{K} . Hence, in view of (5.2), we have the inclusion “ \supset ”. The other inclusion is clear by Proposition 5.9.

To prove (ii), it suffices to apply Proposition 5.9 with Λ being any dense countable set in $W_e(\mathcal{T})$ and to use once again that $W_e(\mathcal{T})$ is invariant under compact perturbations. Since

$$W_e(\mathcal{T}) = \overline{\Lambda} \subset \overline{W_\infty(\mathcal{T} - \mathcal{K})} \subset W_e(\mathcal{T}),$$

the assertion follows. \square

Theorem 5.10 is a counterpart of [23, Corollary 13] where it was proved that for any n -tuple $\mathcal{T} \in B(H)^n$ there exists a tuple of compact operators \mathcal{K} such that $W_e(\mathcal{T}) = \overline{W(\mathcal{T} - \mathcal{K})}$. This result should also be compared to the property $W_e(\mathcal{T}) = \bigcap \{W_\infty(\mathcal{T} - \mathcal{K}) : \mathcal{K} \in \mathcal{K}(H)^n\}$ established in [19]. Note finally the theorem provides a one more proof of convexity of $W_e(\mathcal{T})$ once the convexity of $W_\infty(\mathcal{T})$ is established.

The notion of the infinite numerical range allows to prove an inclusion result for numerical ranges which complements Theorem 4.2 and generalizes a similar result (4.2). (Note however that its proof uses (4.2) essentially.)

Let us first remark that if $V \subset \mathbb{C}$ is a convex set, then $\text{Int}(\overline{V}) \subset V$. Indeed, let $\lambda \in \text{Int} \overline{V}$. We show that $\lambda \in V$. Without loss of generality we can assume that $\lambda = 0$. Since $0 \in \text{Int} \overline{V}$, there exists $r > 0$ such that $\{a \in \mathbb{C} : |a| \leq r\} \subset \overline{V}$. In particular, $a_0 := r$, $a_1 := r\eta$ and $a_2 := r\eta^2$ are elements of \overline{V} , where $\eta = e^{2\pi i/3}$. Now if the elements b_0, b_1 , and b_2 belong to V , and are sufficiently close to a_0, a_1 and a_2 , respectively, then $0 \in \text{conv}\{b_0, b_1, b_2\}$. Since V is convex, we have $0 \in V$.

Thus by convexity of the numerical range $W(T)$ for any $T \in B(H)$, we infer that

$$(5.3) \quad \text{Int} \overline{W(T)} \subset W(T).$$

While for tuples $\mathcal{T} \in B(H)^n$ the set $W(\mathcal{T})$ is in general not convex, the property (5.3) nevertheless holds for tuples of operators if $\text{Int} W_e(\mathcal{T}) \neq \emptyset$.

Theorem 5.11. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$, $\text{Int} (W_e(\mathcal{T})) \neq \emptyset$. Then $\text{Int} \overline{W(\mathcal{T})} \subset W(\mathcal{T})$.*

Proof. Let $\text{Int} (W_e(\mathcal{T})) \neq \emptyset$. Without loss of generality we may assume that $(0, \dots, 0) \in \text{Int} W_e(\mathcal{T})$. Let $r > 0$ satisfy

$$\{(z = (z_1, \dots, z_n) : \|z\| \leq r\} \subset \text{Int} (W_e(\mathcal{T})).$$

Let $\lambda \in \text{Int} \overline{W(\mathcal{T})}$, so there exists $s > 0$ such that $(1 + s)\lambda \in \text{Int} (\overline{W(\mathcal{T})})$. Let $0 < \delta < sr$. Then there is a $\mu \in W(\mathcal{T})$ such that $\|\mu - (1 + s)\lambda\| < \delta$.

Set $\eta = \lambda + \frac{\lambda - \mu}{s}$. Then

$$\|\eta\| \leq \left\| \lambda + \frac{\lambda - (1+s)\lambda}{s} \right\| + \frac{\delta}{s} < r.$$

So, in view of (5.1), $\eta \in \text{Int}(W_e(\mathcal{T})) \subset W_\infty(\mathcal{T})$. Furthermore,

$$\frac{1}{1+s}\mu + \frac{s}{1+s}\eta = \frac{1}{1+s}\mu + \frac{s}{1+s}\lambda + \frac{\lambda - \mu}{1+s} = \lambda.$$

By Theorem 5.4, $\lambda \in W(\mathcal{T})$. \square

Since the interior of the essential numerical range played an important role above, it is natural to realize when it is non-empty. The following simple proposition clarifies the situation in algebraic terms.

Let \mathcal{S} stand for the real linear subspace of $B(H)$ formed by the sums of selfadjoint compact operators on H and real scalar multiples of the identity.

Proposition 5.12. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. The following statements are equivalent:*

- (i) $\text{Int } W_e(\mathcal{T}) \neq \emptyset$;
- (ii) $\text{Int } W_\infty(\mathcal{T}) \neq \emptyset$;
- (iii) *the operators $\text{Re } T_1, \text{Im } T_1, \dots, \text{Re } T_n, \text{Im } T_n$ are linearly independent in the real vector space of all selfadjoint operators modulo \mathcal{S} .*

More precisely, if c, t_1, \dots, t_{2n} are real numbers such that $\sum_{j=1}^n (t_{2j-1} \text{Re } T_j + t_{2j} \text{Im } T_j) - cI$ is compact, then $t_1 = \dots = t_{2n} = 0$.

Proof. The equivalence (i) \Leftrightarrow (ii) was proved above. Thus, it is enough to prove the equivalence (i) \Leftrightarrow (iii).

Recall that $W_e(\mathcal{T})$ is a nonempty convex set. We consider $W_e(\mathcal{T})$ to be a subset of \mathbb{R}^{2n} . So $\text{Int } W_e(\mathcal{T}) = \emptyset$ if and only if $W_e(\mathcal{T})$ is contained in a proper hyperplane in \mathbb{R}^{2n} . This is equivalent to the existence of a non-trivial $(2n)$ -tuple $(t_1, \dots, t_{2n}) \in \mathbb{R}^{2n}$ and $c \in \mathbb{R}$ such that

$$\sum_{j=1}^{2n} t_j z_j - c = 0$$

for all $(z_1, \dots, z_{2n}) \in W_e(\mathcal{T}) \subset \mathbb{R}^{2n}$. This means that

$$W_e\left(\sum_{j=1}^n (t_{2j-1} \text{Re } T_j + t_{2j} \text{Im } T_j) - cI\right) \subset \{0\},$$

i.e.,

$$\sum_{j=1}^n (t_{2j-1} \text{Re } T_j + t_{2j} \text{Im } T_j) - cI$$

is a compact operator. \square

6. ASYMPTOTICS OF COMPRESSIONS FOR OPERATOR ITERATES

In this section the numerical ranges ideology will be used to study asymptotical properties of powers of bounded operators. We will show that if the powers of $T \in B(H)$ vanish in the *weak* operator topology and the spectrum of T is large enough, then for any strict contraction C it is possible to find a subspace $L \subset H$ such that compressions of $(T^n)_L$ to L match asymptotically the powers of C in the *uniform* operator topology. Moreover, even if the assumption $T^n \rightarrow 0$ in the weak operator topology has been dropped then for each $k \in \mathbb{N}$ we are able to construct a subspace $L \subset H$ such that $(T^n)_L = C_u^n$, $1 \leq n \leq k$, where $C_u \in B(L)$ is a contraction unitarily equivalent to C .

In the rest of this section we fix a separable infinite-dimensional Hilbert space H . The next statement, of interest in itself, is an extension of [25, Proposition 1.1] from the case of a single operator to the case of operator tuples. It will allow us to identify a convex combination of operator tuples with a compression of their direct sum.

Note that in the next lemma, $\mathcal{A}_j \in B(H)^n$ for each j , so $\mathcal{A}_j = (A_{j1}, \dots, A_{jn})$. The direct sum $\bigoplus_{j=1}^r \mathcal{A}_j$ is then the n -tuple $(\bigoplus_{j=1}^r A_{j1}, \dots, \bigoplus_{j=1}^r A_{jn}) \in B(\bigoplus_{j=1}^r H)^n$. For $M \subset \bigoplus_{j=1}^r H$, the compression $(\bigoplus_{j=1}^r \mathcal{A}_j)_M$ is the n -tuple

$$J_M^* \left(\bigoplus_{j=1}^r \mathcal{A}_j \right) J_M = \left(J_M^* \left(\bigoplus_{j=1}^r A_{j1} \right) J_M, \dots, J_M^* \left(\bigoplus_{j=1}^r A_{jn} \right) J_M \right) \in B(M)^n.$$

Lemma 6.1. *Let $n, r \in \mathbb{N}$, and let $\mathcal{A}_j \in B(H)^n$, $j = 1, \dots, r$. Let $\alpha_1, \dots, \alpha_r \geq 0$, $\sum_{j=1}^r \alpha_j = 1$. Then there exists a subspace $M \subset \bigoplus_{j=1}^r H$ such that*

$$\left(\bigoplus_{j=1}^r \mathcal{A}_j \right)_M \stackrel{u}{\sim} \sum_{j=1}^r \alpha_j \mathcal{A}_j.$$

Proof. For $r = 1$ the statement is trivial.

We prove the statement first for $r = 2$.

Consider the operator $U : H \oplus H \rightarrow H \oplus H$ defined by

$$U = \begin{pmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} \\ \sqrt{\alpha_2} & -\sqrt{\alpha_1} \end{pmatrix}.$$

It is easy to verify that $U^* = U = U^{-1}$. Moreover,

$$U \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} U = \begin{pmatrix} \alpha_1 \mathcal{A}_1 + \alpha_2 \mathcal{A}_2 & * \\ * & * \end{pmatrix}.$$

Let $M = U(H \oplus \{0\})$. Let $J_M : M \rightarrow H \oplus H$ be the natural embedding. Then

$$J_M J_M^* = P_M = U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U.$$

Let $J_1 : H \rightarrow H \oplus H$ be defined by $J_1 h = h \oplus 0$, $h \in H$. Then $J_M^* U J_1 : H \rightarrow M$ is a unitary operator.

We have

$$\begin{aligned}
\begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}_M &= J_M^* \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} J_M \\
&\stackrel{u}{\sim} (J_1^* U J_M) J_M^* \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} J_M (J_M^* U J_1) \\
&= J_1^* U P_M \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} P_M U J_1 \\
&= J_1^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix} U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} J_1 \\
&= J_1^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \mathcal{A}_1 + \alpha_2 \mathcal{A}_2 & * \\ * & * \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} J_1 \\
&= J_1^* \begin{pmatrix} \alpha_1 \mathcal{A}_1 + \alpha_2 \mathcal{A}_2 & 0 \\ 0 & 0 \end{pmatrix} J_1 \\
&= \alpha_1 \mathcal{A}_1 + \alpha_2 \mathcal{A}_2.
\end{aligned}$$

For $r > 2$ the statement can be proved by induction. Let $r \geq 3$ be fixed and suppose the statement is true for $r-1$. We may assume that $\sum_{j=1}^{r-1} \alpha_j \neq 0$. By the induction hypothesis, there exists a subspace $L \subset \bigoplus_{j=1}^{r-1} H$ such that

$$J_L^* \left(\bigoplus_{j=1}^{r-1} \mathcal{A}_j \right) J_L \stackrel{u}{\sim} \left(\sum_{j=1}^{r-1} \alpha_j \right)^{-1} \sum_{j=1}^{r-1} \alpha_j \mathcal{A}_j.$$

Consider the Hilbert space $L \oplus H$. By the statement for $r=2$ there exists a subspace $M \subset L \oplus H \subset \bigoplus_{j=1}^r H$ such that

$$J_M^* \left(\bigoplus_{j=1}^r \mathcal{A}_j \right) J_M \stackrel{u}{\sim} \left(\sum_{j=1}^{r-1} \alpha_j \right)^{-1} \cdot \frac{\sum_{j=1}^{r-1} \alpha_j \mathcal{A}_j}{\sum_{j=1}^{r-1} \alpha_j} + \alpha_r \mathcal{A}_r = \sum_{j=1}^r \alpha_j \mathcal{A}_j,$$

and the statement is thus true for r . This completes the proof. \square

Remark 6.2. Let $\mathcal{A}_j \in B(H)^n$ and $\alpha_1, \dots, \alpha_r$ be as above. Let $\tilde{\mathcal{A}}_j \in B(H)^n$ be n -tuples unitarily equivalent to \mathcal{A}_j , i.e., $\tilde{\mathcal{A}}_j = U_j^{-1} \mathcal{A}_j U_j$ for some unitary operators $U_j \in B(H)$ ($j = 1, \dots, r$).

Since $\bigoplus_{j=1}^r \mathcal{A}_j \stackrel{u}{\sim} \bigoplus_{j=1}^r \tilde{\mathcal{A}}_j$, the previous lemma implies also that there exists a subspace $\tilde{L} \subset H$ such that

$$\left(\bigoplus_{j=1}^r \mathcal{A}_j \right)_{\tilde{L}} \stackrel{u}{\sim} \sum_{j=1}^r \alpha_j \tilde{\mathcal{A}}_j.$$

Let $S \subset \mathbb{C}^n$. Denote by $\mathcal{M}(S)$ the set of all n -tuples of operators $\mathcal{A} = (A_1, \dots, A_n) \in B(H)^n$ for which there exist an orthonormal basis $(x_i)_{i \geq 1}$ in H and elements $\lambda_i \in S, i \geq 1$, such that $\mathcal{A} x_i = \lambda_i x_i, i \geq 1$.

Using Proposition 6.1 we will further identify a compression of an operator tuple \mathcal{T} with a tuple of diagonal operators \mathcal{A} whose diagonals belong to the infinite numerical range of \mathcal{T} .

Proposition 6.3. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Let $\mathcal{A} \in \text{conv } \mathcal{M}(W_\infty(\mathcal{T}))$. Then there exists a subspace $L \subset H$ such that the compression \mathcal{T}_L is unitarily equivalent to \mathcal{A} .*

Proof. We have

$$\mathcal{A} = \sum_{j=1}^r \alpha_j \mathcal{A}_j$$

for some $\mathcal{A}_j \in \mathcal{M}(W_\infty(\mathcal{T}))$ and $\alpha_j \geq 0$ such that $\sum_{j=1}^r \alpha_j = 1$. So $\mathcal{A}_j = \text{diag}(\lambda_{j,1}, \lambda_{j,2}, \dots)$ with $\lambda_{j,i} \in W_\infty(\mathcal{T})$.

Consider the set $\{1, \dots, r\} \times \mathbb{N}$ with the lexicographic order: $(j, i) \prec (j', i')$ if either $i < i'$ or $i = i'$ and $j < j'$. We construct inductively an orthonormal sequence $(x_{j,i}) \subset H$ in the following way: Let $j \in \{1, \dots, r\}$, $i \in \mathbb{N}$ and suppose that we have already constructed vectors $x_{j',i'}$ for all $(j', i') \prec (j, i)$. Since $\lambda_{j,i} \in W_\infty(\mathcal{T})$, we can find a unit vector $x_{j,i} \in H$ such that

$$\begin{aligned} x_{j,i} &\perp x_{j',i'}, & (j', i') &\prec (j, i), \\ x_{j,i} &\perp T_s x_{j',i'}, & (j', i') &\prec (j, i), \quad s = 1, \dots, k, \\ x_{j,i} &\perp T_s^* x_{j',i'}, & (j', i') &\prec (j, i), \quad s = 1, \dots, k, \end{aligned}$$

and

$$\langle \mathcal{T} x_{j,i}, x_{j,i} \rangle = \lambda_{j,i}.$$

Suppose we have constructed the vectors $x_{j,i}$ in this way. For $j = 1, \dots, r$ let $H_j = \bigvee_{i \in \mathbb{N}} x_{j,i}$. Let $\tilde{H} = \bigoplus_{j=1}^r H_j$. For $j = 1, \dots, r$ we have

$$J_{H_j}^* \mathcal{T} J_{H_j} \stackrel{u}{\sim} \mathcal{A}_j$$

and

$$J_{\tilde{H}}^* \mathcal{T} J_{\tilde{H}} \stackrel{u}{\sim} \bigoplus_{j=1}^r \mathcal{A}_j.$$

By Lemma 6.1, there exists a subspace $L \subset \tilde{H} \subset H$ such that

$$J_L^* \mathcal{T} J_L \stackrel{u}{\sim} \sum_{j=1}^r \alpha_j \mathcal{A}_j.$$

□

Now we are able to prove a partial generalization of [5, Theorem 2.1] by putting the result into the setting of operator tuples and slightly weakening the assumptions on operator T .

Theorem 6.4. *Let $T \in B(H)$ be such that the polynomial hull $\hat{\sigma}(T)$ contains \mathbb{D} . Let $n \in \mathbb{N}$ and $\mathcal{T} = (T, T^2, \dots, T^n)$. Let $A \in B(H)$, $\|A\| < 1$ and let $\mathcal{A} = (A, A^2, \dots, A^n)$. Then there exists a subspace $L \subset H$ such that*

$$(\mathcal{T})_L \overset{u}{\sim} \mathcal{A}.$$

Proof. Let $\|A\| = c < 1$. Since A has the (power) dilation cU where U is a unitary operator, we can assume that $A = cU$.

We have $(\lambda, \lambda^2, \dots, \lambda^n) \in W_\infty(\mathcal{T})$ for all $\lambda \in \mathbb{D}$. Since the function

$$\lambda \mapsto \text{dist} \{(\lambda, \dots, \lambda^n), \mathbb{C}^n \setminus W_\infty(\mathcal{T})\}$$

is continuous, we can find $c' < 1$ such that $c < c' < 1$ and $\frac{c'}{c}(\lambda, \lambda^2, \dots, \lambda^n) \in W_\infty(\mathcal{T})$ for all λ with $|\lambda| = c$. Set

$$\eta = \left(1 - \frac{c}{c'}\right)(2k)^{-1}.$$

Let δ be such that $(\varepsilon_1, \dots, \varepsilon_n) \in W_\varepsilon(\mathcal{T})$ for all $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{C}$ with $\max_j |\varepsilon_j| < \delta$. By the Weyl-von Neumann diagonalization theorem (see e.g. [14, Section X.2.1, p. 525] for a version for selfadjoint operators), we can decompose A as $A = D + K_1$, where D is a diagonal operator with entries of modulus c and K_1 is a compact operator satisfying $\|K_1\| < \delta\eta k^{-1}$. Set $K_j := A^j - D^j$, $1 \leq j \leq n$. We have

$$K_j = A^j - D = \sum_{i=0}^{j-1} A^i (A - D) D^{j-i-1}.$$

So for every j the operator K_j is compact and

$$\|K_j\| \leq n c^{j-1} \|A - D\| \leq n \|K_1\| \leq \delta\eta.$$

Write $K_j = \text{Re } K_j + i \text{Im } K_j$, $1 \leq j \leq n$. The operators $\text{Re } K_j$ and $i \text{Im } K_j$ are diagonal operators with entries of modulus at most $\delta\eta$. We have

$$\begin{aligned} (A, A^2, \dots, A^n) &= \frac{c}{c'} \cdot \frac{c'}{c} (D, D^2, \dots, D^n) + \eta \cdot \eta^{-1} (\text{Re } K_1, 0, \dots, 0) \\ &\quad + \eta \cdot \eta^{-1} (i \text{Im } K_1, 0, \dots, 0) + \eta \cdot \eta^{-1} (0, \text{Re } K_2, 0, \dots, 0) + \dots \\ &\quad \dots + \eta \cdot \eta^{-1} (0, \dots, 0, i \text{Im } K_n), \end{aligned}$$

where $\frac{c}{c'} + 2k\eta = 1$ and the n -tuples

$$\frac{c'}{c} (D, D^2, \dots, D^n), \eta^{-1} (\text{Re } K_1, 0, \dots, 0), \dots, \eta^{-1} (0, \dots, 0, i \text{Im } K_n)$$

belong to $\mathcal{M}(W_\infty(\mathcal{T}))$. By Proposition 6.3, there exists a subspace $L \subset H$ such that

$$(\mathcal{T})_L \overset{u}{\sim} \mathcal{A}.$$

□

We proceed with an asymptotic version of Theorem 6.4. To this aim several auxiliary lemmas will be needed. First we prove a slight generalization of [24, Lemma 6.1] where we replace the element $(0, \dots, 0)$ by $(\lambda, \lambda^2, \dots, \lambda^n)$, $|\lambda| < 1$, and choose a unit vector $x \in H$ more carefully. The next statement can be considered as a variant of the main result, Theorem 6.7 below, for a single vector.

Lemma 6.5. *Let $T \in B(H)$ be such that $T^n \rightarrow 0$ in the weak operator topology, and let $\lambda \in \mathbb{C}$, $|\lambda| < 1$. Suppose that*

$$(6.1) \quad (\lambda, \lambda^2, \dots, \lambda^n) \in W_\infty(T, \dots, T^n).$$

for all $n \in \mathbb{N}$. Let $A \subset H$ be a finite set, $\varepsilon > 0$ and $M \subset H$ be a subspace of a finite codimension. Then there exists a unit vector $x \in M \cap A^\perp$ such that

$$\sup_{n \geq 1} |\langle (T^n - \lambda^n)x, x \rangle| \leq \varepsilon, \quad \sup_{n \geq 1} |\langle T^n x, a \rangle| \leq \varepsilon \quad \text{and} \quad \sup_{n \geq 1} |\langle T^{*n} x, a \rangle| \leq \varepsilon$$

for all $a \in A$.

Proof. Clearly T is power bounded by the uniform boundedness principle. Let $K = \sup\{\|T^n\| : n = 0, 1, \dots\}$. It is apparent that also $T^{*n} \rightarrow 0$ in the weak operator topology. Without loss of generality we may assume that $\max\{\|a\| : a \in A\} \leq 1$.

Choose $s \in \mathbb{N}$ such that $s > 25K^2\varepsilon^{-2}$, and find $n_0 \in \mathbb{N}$ such that $|\lambda|^{n_0} < \frac{\varepsilon}{5}$.

We construct unit vectors $u_1, u_2, \dots, u_s \in M$ and positive integers $n_0 < n_1 < \dots < n_s$ in the following way: Let $1 \leq r \leq s-1$ and suppose that the unit vectors $u_1, \dots, u_r \in M$ and numbers $n_0 < n_1 < \dots < n_r$ have already been constructed.

By Proposition 5.7, there exists a unit vector $u_{r+1} \in M$ such that

$$u_{r+1} \perp \{T^n u_k, T^{*n} u_k, T^n a, T^{*n} a : 0 \leq n \leq n_r, 1 \leq k \leq r, a \in A\}$$

and

$$\langle T^n u_{r+1}, u_{r+1} \rangle = \lambda^n, \quad 1 \leq n \leq n_r.$$

Find $n_{r+1} > n_r$ such that

$$\begin{aligned} |\langle T^n u_{r+1}, u_k \rangle| &< \frac{\varepsilon}{5s}, \\ |\langle T^{*n} u_{r+1}, u_k \rangle| &< \frac{\varepsilon}{5s}, \\ |\langle T^n u_{r+1}, a \rangle| &< \frac{\varepsilon}{5s}, \end{aligned}$$

and

$$|\langle T^{*n} u_{r+1}, a \rangle| < \frac{\varepsilon}{5s}$$

for all $n \geq n_{r+1}$, $1 \leq k \leq r+1$ and $a \in A$.

Let u_1, \dots, u_s and n_0, \dots, n_s be constructed in this way. Set

$$x = \frac{1}{\sqrt{s}} \sum_{k=1}^s u_k.$$

Clearly $x \in M$. Moreover, $\|x\| = 1$ since the vectors u_k are orthonormal.

For $n > n_s$ we have

$$|\langle (T^n - \lambda^n)x, x \rangle| \leq s^{-1} \sum_{k,k'=1}^s |\langle T^n u_k, u_{k'} \rangle| + |\lambda|^n \leq s^{-1} s^2 \frac{\varepsilon}{5s} + \frac{\varepsilon}{5} < \varepsilon.$$

Let $0 \leq r \leq s-1$ and $n_r < n \leq n_{r+1}$. Then

$$\begin{aligned} |\langle (T^n - \lambda^n)x, x \rangle| &\leq |\langle T^n x, x \rangle| + \frac{\varepsilon}{5} \\ &\leq s^{-1} \sum_{k,k'=1}^r |\langle T^n u_k, u_{k'} \rangle| + s^{-1} \sum_{k=1}^{r+1} |\langle T^n u_{r+1}, u_k \rangle| \\ &\quad + s^{-1} \sum_{k=1}^r |\langle T^n u_k, u_{r+1} \rangle| + s^{-1} \sum_{k=r+2}^s |\langle T^n u_k, u_k \rangle| \\ &\quad + s^{-1} \sum_{\substack{1 \leq k,k' \leq s, k \neq k' \\ \max\{k,k'\} \geq r+2}} |\langle T^n u_k, u_{k'} \rangle| + \frac{\varepsilon}{5}, \end{aligned}$$

where the last sum is equal to 0 by the construction. So

$$\begin{aligned} |\langle (T^n - \lambda^n)x, x \rangle| &\leq s^{-1} r^2 \frac{\varepsilon}{5s} + s^{-1} \|T^n u_{r+1}\| \cdot \left\| \sum_{k=1}^{r+1} u_k \right\| \\ &\quad + s^{-1} \|T^{*n} u_{r+1}\| \cdot \left\| \sum_{k=1}^r u_k \right\| + s^{-1} (s-r-1) |\lambda|^n + \frac{\varepsilon}{5} \\ &\leq \frac{\varepsilon}{5} + s^{-1} K \sqrt{r+1} + s^{-1} K \sqrt{r} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} \\ &\leq \varepsilon. \end{aligned}$$

Let $1 \leq n \leq n_0$. Then

$$\begin{aligned} &\langle (T^n - \lambda^n)x, x \rangle \\ &= s^{-1} \sum_{k=1}^s \langle (T^n - \lambda^n)u_k, u_k \rangle + s^{-1} \sum_{1 \leq k,k' \leq s, k \neq k'} \langle (T^n - \lambda^n)u_k, u_{k'} \rangle = 0. \end{aligned}$$

Hence

$$\sup_{n \geq 1} |\langle (T^n - \lambda^n)x, x \rangle| \leq \varepsilon.$$

Let $a \in A$. For $n \geq n_s$ we have

$$|\langle T^n x, a \rangle| \leq \frac{1}{\sqrt{s}} \sum_{k=1}^s |\langle T^n u_k, a \rangle| \leq \frac{1}{\sqrt{s}} \cdot s \cdot \frac{\varepsilon}{5s} < \varepsilon.$$

Let $0 \leq r \leq s-1$ and $n_r \leq n < n_{r+1}$. Then

$$\begin{aligned} |\langle T^n x, a \rangle| &\leq \frac{1}{\sqrt{s}} \sum_{k=1}^r |\langle T^n u_k, a \rangle| + \frac{1}{\sqrt{s}} |\langle T^n u_{r+1}, a \rangle| + \frac{1}{\sqrt{s}} \sum_{k=r+2}^s |\langle T^n u_k, a \rangle| \\ &\leq \frac{1}{\sqrt{s}} \cdot r \cdot \frac{\varepsilon}{5s} + \frac{1}{\sqrt{s}} \cdot K + 0 < \varepsilon. \end{aligned}$$

Finally, for $1 \leq n \leq n_0$ we have

$$\langle T^n x, a \rangle = s^{-1} \sum_{k=1}^s \langle T^n u_k, a \rangle = 0.$$

So

$$\sup_{n \geq 1} |\langle T^n x, a \rangle| \leq \varepsilon$$

for all $a \in A$.

The property $\sup_{n \geq 1} |\langle T^{*n} x, a \rangle| \leq \varepsilon$ for all $a \in A$ can be proved similarly. \square

The previous lemma enables us to prove the main result in a particular situation when a compression of powers a bounded operator is approximated by powers of any strictly contractive diagonal operator.

Lemma 6.6. *Let $T \in B(H)$ be such that $T^n \rightarrow 0$ in the weak operator topology and $\sigma(T) \supset \mathbb{T}$. Let $(\lambda_k)_{k \geq 1} \subset \mathbb{D}$, $\sup_{k \geq 1} |\lambda_k| < 1$ and $\varepsilon > 0$. Then there exists an orthonormal sequence $(e_k)_{k \geq 1}$ in H such that*

$$\begin{aligned} \sup_{n \geq 1} \|(T^n)_L - D^n\| &\leq \varepsilon \\ \text{and } \lim_{n \rightarrow \infty} \|(T^n)_L - D^n\| &= 0, \end{aligned}$$

where $L = \bigvee_{k=1}^{\infty} e_k$ and $D \in B(L)$ is the diagonal operator defined by $De_k = \lambda_k e_k$, $k \in \mathbb{N}$.

Proof. Let $r = \sup_k |\lambda_k| < 1$. By Theorem 4.4,

$$(\lambda_k, \lambda_k^2, \dots, \lambda_k^n) \in W_{\infty}(T, T^2, \dots, T^n)$$

for all $n, k \in \mathbb{N}$.

We use the previous lemma inductively to construct the sequence $(e_k)_{k \geq 1}$.

Let $n_0 \in \mathbb{N}$ satisfy $r^{n_0} < \frac{\varepsilon}{16}$.

We find vectors e_1, e_2, \dots and positive integers $n_0 < n_1 < \dots$ inductively.

Suppose that $s \geq 0$ and orthonormal vectors $e_1, \dots, e_s \in H$ and numbers $n_0 < n_1 < \dots < n_s$ have already been constructed. Find a unit vector $e_{s+1} \in H$ such that

$$\begin{aligned} e_{s+1} &\perp \{T^n e_k, T^{*n} e_k : 0 \leq n \leq n_s, 1 \leq k \leq s\}, \\ \sup_{n \geq 1} |\langle (T^n - \lambda_{s+1}^n) e_{s+1}, e_{s+1} \rangle| &< \frac{\varepsilon}{2^{s+4}(s+1)}, \\ \sup_{n \geq 1} |\langle T^n e_{s+1}, e_k \rangle| &< \frac{\varepsilon}{2^{s+4}(s+1)}, \quad 1 \leq k \leq s, \end{aligned}$$

and

$$\sup_{n \geq 1} |\langle T^{*n} e_{s+1}, e_k \rangle| < \frac{\varepsilon}{2^{s+4}(s+1)}, \quad 1 \leq k \leq s.$$

Find $n_{s+1} > n_s$ such that

$$r^{n_{s+1}} < \frac{\varepsilon}{2^{s+5}}$$

and

$$|\langle T^n e_k, e_{k'} \rangle| \leq \frac{\varepsilon}{2^{s+4}(s+1)}, \quad 1 \leq k, k' \leq s+1, n \geq n_{s+1}.$$

Let $L = \bigvee_{k=1}^{\infty} e_k$ and let the diagonal operator $D : L \rightarrow L$ be defined by $De_k = \lambda_k e_k$, $k \in \mathbb{N}$.

Let $x \in L$, $\|x\| = 1$. Then $x = \sum_{k \geq 1} \alpha_k e_k$ where $\sum_{k \geq 1} |\alpha_k|^2 = 1$. Note that $\sum_{k=1}^s |\alpha_s| \leq \sqrt{s}$ for all $s \in \mathbb{N}$.

For $1 \leq n \leq n_0$ we have

$$\begin{aligned} |\langle (T^n - D^n)x, x \rangle| &\leq \sum_{k, k'=1}^{\infty} |\alpha_k \bar{\alpha}_{k'}| \cdot |\langle (T^n - D^n)e_k, e_{k'} \rangle| \\ &= \sum_{k=1}^{\infty} |\alpha_k|^2 \cdot |\langle (T^n - \lambda_k^n)e_k, e_k \rangle| \leq \frac{\varepsilon}{2}. \end{aligned}$$

Thus $\|(T^n)_L - D^n\| \leq \varepsilon$.

Let $s \geq 0$ and $n_s < n \leq n_{s+1}$. We have

$$\begin{aligned} |\langle (T^n - D^n)x, x \rangle| &\leq |\langle T^n x, x \rangle| + r^n \\ &\leq \left| \sum_{k, k'=1}^{\infty} \alpha_k \bar{\alpha}_{k'} \langle T^n e_k, e_{k'} \rangle \right| + \frac{\varepsilon}{2^{s+4}} \\ &\leq \sum_{k, k'=1}^s |\alpha_k \bar{\alpha}_{k'}| \cdot |\langle T^n e_k, e_{k'} \rangle| + \sum_{k=1}^s |\alpha_{s+1} \bar{\alpha}_k| \cdot |\langle T^n e_{s+1}, e_k \rangle| \\ &\quad + \sum_{k=1}^s |\alpha_k \bar{\alpha}_{s+1}| \cdot |\langle T^n e_k, e_{s+1} \rangle| + \sum_{k=s+1}^{\infty} |\alpha_k|^2 \cdot |\langle T^n e_k, e_k \rangle| \\ &\quad + \sum_{k \neq k', \max\{k, k'\} \geq s+2} |\alpha_k \bar{\alpha}_{k'}| \cdot |\langle T^n e_k, e_{k'} \rangle| + \frac{\varepsilon}{2^{s+4}} \\ &\leq \frac{s\varepsilon}{2^{s+3}} + \frac{\varepsilon\sqrt{s}}{2^{s+4}(s+1)} + \frac{\varepsilon\sqrt{s}}{2^{s+4}(s+1)} \\ &\quad + \sum_{k=s+1}^{\infty} |\alpha_k|^2 \cdot (|\langle (T^n - D^n)e_k, e_k \rangle| + r^n) + 0 + \frac{\varepsilon}{2^{s+4}} \\ &\leq \frac{\varepsilon}{2^{s+3}} + \frac{\varepsilon}{2^{s+4}} + \frac{\varepsilon}{2^{s+4}} + \left(\frac{\varepsilon}{2^{s+4}} + \frac{\varepsilon}{2^{s+4}} \right) + \frac{\varepsilon}{2^{s+4}} \\ &< \frac{\varepsilon}{2^{s+1}}. \end{aligned}$$

Thus

$$\sup\{|\langle (T^n - D^n)x, x \rangle| : x \in L, \|x\| = 1\} \leq \frac{\varepsilon}{2^{s+1}}$$

and so $\|(T)_L - D^n\| \leq \frac{\varepsilon}{2^s}$.

Hence

$$\sup_{n \geq 1} \|(T^n)_L - D^n\| \leq \varepsilon \quad \text{and} \quad \lim_{n \rightarrow \infty} \|(T^n)_L - D^n\| = 0.$$

□

Having Lemma 6.6 at hands, we are ready to prove the main result of this section on matching compressions of operator powers by powers of any strict contraction.

Theorem 6.7. *Let $T \in B(H)$ be such that $T^n \rightarrow 0$ in the weak operator topology, and let $\sigma(T) \supset \mathbb{T}$. Let $\tilde{C} \in B(H)$ and $\|\tilde{C}\| < 1$. Then for every $\varepsilon > 0$ there exists a subspace $L \subset H$ and $C \in B(L)$ unitarily equivalent to \tilde{C} such that*

$$\sup_{n \geq 1} \|(T^n)_L - C^n\| \leq \varepsilon$$

and

$$\lim_{n \rightarrow \infty} \|(T^n)_L - C^n\| = 0.$$

Proof. Let $\|\tilde{C}\| < c < 1$. Then \tilde{C} has a unitary (power) dilation $c\tilde{U}$ on a Hilbert space \tilde{K} , where $\tilde{U} \in B(\tilde{K})$ is the bilateral shift of infinite multiplicity. So it is sufficient to show the statement for the operator $c\tilde{U}$.

Find $k \in \mathbb{N}$ such that $\sup_{n \geq 1} nc^n < \frac{k\varepsilon}{4\pi}$. For $0 \leq j \leq k-1$ let $E_j \subset \tilde{K}$ be the spectral subspace of \tilde{U} corresponding to the set $\{e^{2\pi it} : j/k \leq t < (j+1)/k\}$. Consider the operator $\tilde{D} \in B(\tilde{K})$ defined by $\tilde{D}x = e^{2\pi ij/k}x$ ($x \in E_j$). Then $\|\tilde{U} - \tilde{D}\| \leq \frac{2\pi}{k}$ and similarly, $\|\tilde{U}^n - \tilde{D}^n\| \leq \frac{2\pi n}{k}$ for all k and n from \mathbb{N} . Thus

$$\sup_{n \geq 1} \|(c\tilde{U})^n - (c\tilde{D})^n\| \leq \sup_{n \geq 1} \frac{2\pi nc^n}{k} \leq \varepsilon/2$$

and

$$\lim_{n \rightarrow \infty} \|(c\tilde{U})^n - (c\tilde{D})^n\| = 0.$$

Moreover, $c\tilde{D}$ is a diagonal operator of the form considered in Lemma 6.6. Hence, by Lemma 6.6, there exists a subspace $K \subset H$ and a unitarily equivalent copy $cD \in B(K)$ of $c\tilde{D}$ such that

$$\sup_{n \geq 1} \|(T^n)_K - (cD)^n\| \leq \varepsilon/2$$

and

$$\lim_{n \rightarrow \infty} \|(T^n)_K - (cD)^n\| = 0.$$

Thus

$$\sup_{n \geq 1} \|(T^n)_K - (cU)^n\| \leq \varepsilon$$

and

$$\lim_{n \rightarrow \infty} \|(T^n)_K - (cU)^n\| = 0,$$

where $U \in B(K)$ is an isometrical copy of \tilde{U} . Hence there exists a subspace $L \subset K$ and an isometrical copy $C \in B(L)$ of \tilde{C} such that

$$\sup_{n \geq 1} \|(T^n)_L - C^n\| \leq \varepsilon \quad \text{and} \quad \lim_{n \rightarrow \infty} \|(T^n)_L - C^n\| = 0.$$

□

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