Fluids in Motion
Navier-Stokes equations and similar problems

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Motto

Johann von Neumann
[1903-1957]

In mathematics you don't understand things. You just get used to them.

A complicated mathematical theory [Is it really worth it?]

- model does not reflect the real situation
- model is not well-posed
- numerical method does not give us the right solutions
- computer implementation does not yield the "expected" results

Fluids in motion

honey

beer

tornado

Sun

plane

- wheather prediction
- ships, planes, cars, trains
- astrophysics, gaseous stars
- rivers, floods, oceans, tsunami waves
- human body, blood motion

MATHEMATICAL ISSUES

- Modeling
- Analysis of models, well-posedness, stability, determinism (?)
- Numerical analysis and implementations, computations

Do we need mathematics?

Luc Tartar
[Compensation effects in partial differential equations]

What puzzles me more is the behaviour of people who have failed to become good mathematicians and advocate using the language of engineers ... as if they were not aware of the efficiency of the engineering approach that one can control processes that one does not understand at all

Good models?

Stephen William Hawking [*1942]

A model is a good model if it:

- Is elegant
- Contains few arbitrary or adjustable elements
- Agrees with and explains all existing observation
- Makes detailed predictions about future observations that disprove or falsify the model if they are not borne out

Mathematical modeling of fluids in motion

Molecular dynamics

Fluids understood as huge families of individual particles (atoms, molecules)

Kinetic models

Large ensembles of particles in *random* motion, description in terms of averages

Continuum fluid mechanics

Phenomenological theory based on observable quantities - mass density, temperature, velocity field

Models of turbulence

Essentially based on classical continuum mechanics but description in terms of averaged quantities

Conservation/balance laws

total amount at time t_2

$\int_B D(t_2, x)\,\mathrm{d}x$

minus

$-$

total amount at time t_1

$\int_B D(t_1, x)\,\mathrm{d}x$

conservation/balance

$=$

boundary flux

$-\int_{t_1}^{t_2} \int_{\partial B} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S_x\,\mathrm{d}t$

plus

$+$

sources

$\int_{t_1}^{t_2} \int_B s\,\mathrm{d}x\,\mathrm{d}t$

Conservation laws as PDE's

Limit processes

$$t_2 \to t_1, \;\; B = B_x \to x$$

Field equation

$$\frac{\partial}{\partial t} D + \mathrm{div}_x \mathbf{F} = s$$

Constitutive relations

$$\mathbf{F} = \mathbf{F}(D), \;\; s = s(D)$$

Conclusion

The resulting equations are *partial differential equations* with *nonlinear* dependence of fields

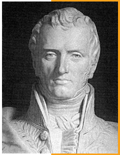
Millennium problems (?)

CLAY MATHEMATICS INSTITUTE, PROVIDENCE, RI

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equation
- P vs NP Problem
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills and Mass Gap

Navier-Stokes system - Millenium Problem

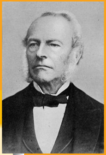
- $\mathbf{u} = \mathbf{u}(t, x)$ . . . . fluid velocity
- $\Pi = \Pi(t, x)$ . . . . pressure



Claude Louis Marie Henri Navier [1785-1836]

"Incompressibility"

$$\mathrm{div}_x \mathbf{u} = 0$$



George Gabriel Stokes [1819-1903]

Balance of momentum

$$\partial_t \mathbf{u} + \mathrm{div}_x (\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = \Delta_x \mathbf{u}$$

Linear vs. nonlinear models

Linear equations

- Solutions built up from elementary functions - modes
- Solvability by means of the symbolic calculus - Laplace and Fourier transform
- Limited applicability

Nonlinear equations

- Explicit solutions known only exceptionally: solitons, simple shock waves
- Possible singularities created by nonlinearity - blow up and/or shocks
- Almost all genuine models are nonlinear

Jacques Hadamard, [1865 - 1963]

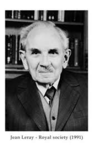
- **Existence.** Given problem is solvable for any choice of (admissible) data
- **Uniqueness.** Solutions are uniquely determined by the data
- **Stability.** Solutions depend continuously on the data

Solvability - modern way

Jacques-Louis Lions, [1928 - 2001]

- **Approximations.** Given problem admits an approximation scheme that is solvable analytically and, possibly, numerically
- **Uniform bounds.** Approximate solutions possesses uniform bounds depending solely on the data
- **Stability.** The family of approximate solutions admits a limit representing a (generalized) solution of the given problem

State of the art



Jean Leray [1906-1998] Global existence of the so-called **weak** solutions for the incompressible Navier-Stokes system (3D)

Olga Aleksandrovna Ladyzhenskaya [1922-2004] Global existence of classical solutions for the incompressible 2D Navier-Stokes system

Pierre-Louis Lions[*1956] Global existence of weak solutions for the compressible barotropic Navier-Stokes system (2,3D)

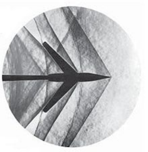
and many, many others...

Blow-up singularities - concentrations

Solutions become large (infinite) in a finite time.
There is too much energy pumped in the system

Shock waves - oscillations

Shocks are singularities in "derivatives".
Originally smooth solutions become discontinuous in a finite time



"Bad" nonlinearities

$$\partial_t U = \boxed{U^2}, \ \partial_t U + \boxed{U \partial_x U} = 0$$

Euler system (compressible inviscid)

- $\mathbf{u} = \mathbf{u}(t, x)$ . . . . fluid velocity
- $\varrho = \varrho(t, x)$ . . . . density

Leonhard Paul Euler
[1707-1783]

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Balance of momentum

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Back to integral averages

- *Pointwise* (ideal) values of functions are replaced by their *integral averages*. This idea is close to the physical concept of *measurement*

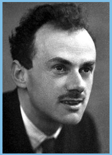
$$u \approx \left[\varphi \mapsto \int u\varphi \right]$$

- Derivatives in the equations replaced by integrals:

$$\frac{\partial u}{\partial x} \approx \left[\varphi \mapsto - \int u \partial_x \varphi \right], \ \varphi \text{ a smooth } \textit{test} \text{ function}$$

Example

Dirac distribution: $\delta_0 : \ \varphi \mapsto \varphi(0)$



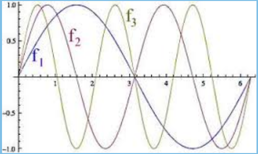
Paul Adrien Maurice Dirac [1902-1984]

Oscillatory solutions - velocity

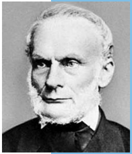
$$U(x) \approx \sin(nx), \ U \to 0 \text{ in the sense of avarages (weakly)}$$

Oscillatory solutions - kinetic energy

$$\frac{1}{2}|U|^2(x) \approx \frac{1}{2}\sin^2(nx) \to \frac{1}{4} \neq \frac{1}{2}0^2 \text{ in the sense of avarages (weakly)}$$



Do some solutions lose/produce energy?



Rudolph Clausius, [1822–1888]

First and Second law of thermodynamics

Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu

Mechanical energy balance for compressible fluid

$$\textbf{classical: } \frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \ \mathrm{d}x = 0, \ \ P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z$$

$$\textbf{weak: } \frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \ \mathrm{d}x \boxed{\leq} 0$$

Existence

Global-in-time solutions (in general) do not exist. Weak solutions may exist but may not be uniquely determined by the initial data.

Mechanical energy

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)$$

Admissibility criteria - mechanical energy dissipation

$$\partial_t E + \mathrm{div}_x(E\mathbf{u} + p(\varrho)\mathbf{u}) \boxed{\leq} 0$$

Wild solutions?

Charles Hermite
[1822-1901]

In a letter to Stieltjes

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives

- **Past:** What is not allowed is forbidden
- **Present:** What is not forbidden is allowed

Bad or good news for compressible Euler?

Camillo DeLellis [*1976]

Existence

Good news: There exists a global-in-time weak solution of compressible Euler system for "any" initial data

Bad news: There are infinitely many...

Admissible solutions?

Good news: Most of the "wild" solutions produce energy.

Bad news: There is a vast class of data for which there exist infinitely many admissible solutions

László Székelyhidi [*1977]

Viscosity solutions or maximal dissipation?

The "correct" solutions "should be" identified as limits of the viscous system

Basic ideas of De Lellis and Székelyhidi

Incompressible Euler system

$$\partial_t \mathbf{U} + \mathrm{div}_x (\mathbf{U} \otimes \mathbf{U}) + \nabla_x \Pi = 0, \ \mathrm{div}_x \mathbf{U} = 0, N = 2, 3$$

Equivalent formulation

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{V} = 0, \ \mathrm{div}_x \mathbf{U} = 0, \ \mathbf{U} \otimes \mathbf{U} - \frac{1}{N} |\mathbf{U}|^2 \mathbb{I} = \mathbb{V}$$

Subsolutions

$$\frac{1}{2} |\mathbf{U}|^2 \leq \frac{N}{2} \lambda_{\max} \left[\mathbf{U} \otimes \mathbf{U} - \mathbb{V} \right] \equiv G(\mathbf{U}, \mathbb{V}) \boxed{<} e, \ \mathbb{V} \in R^{N \times N}_{0,\mathrm{sym}}$$

Solutions

$$\frac{1}{2} |\mathbf{U}|^2 = e \ \Rightarrow \ \mathbb{V} = \mathbf{U} \otimes \mathbf{U} - \frac{1}{N} |\mathbf{U}|^2 \mathbb{I}$$

Oscillatory lemma

Subsolution

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{V} = 0,\ |\mathbf{U}|^2 \le G(\mathbf{U}, \mathbb{V}) < e$$

Oscillatory perturbation

$$\partial_t \mathbf{u}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0,\ \mathbf{u}_\varepsilon, \mathbb{V}_\varepsilon \text{ compactly supported}$$

$$G\left(\mathbf{U} + \mathbf{u}_\varepsilon, \mathbb{V} + \mathbb{V}_\varepsilon\right) < e,\ \mathbf{u}_\varepsilon \rightharpoonup 0$$

$$\liminf_{\varepsilon \to 0} \int_B |\mathbf{u}_\varepsilon|^2 \ge \int_B \Lambda\left(e - G(\mathbf{U}, \mathbb{V})\right),\ \Lambda(Z) > 0 \text{ for } Z > 0$$

$$\Rightarrow$$

$$\liminf_{\varepsilon \to 0} \int_B |\mathbf{U} + \mathbf{u}_\varepsilon|^2 \ge \int_B |\mathbf{U}|^2 + \int_B \Lambda\left(e - G(\mathbf{U}, \mathbb{V})\right)$$

Typical results

Good news

The set of subsolutions nonempty $\Rightarrow$ the problem possesses a *global-in-time* solution for *any* initial data

Bad news

The problem possesses *infinitely many* solutions for any initial data

What's wrong? ... more bad news

"Many" solutions violate the energy conservation **but** there is a "large" set of initial data for which the problem admits infinitely many energy conserving (dissipating) solutions

E. Chiodaroli, EF et al.

Hypotheses:

$$U \subset R \times R^N, \ \ N = 2,3 \text{ bounded open set}$$

$$\tilde{\mathbf{h}} \in C(U; R^N), \ \tilde{\mathbb{H}} \in C(U; R^{N\times N}_{\mathrm{sym},0}), \ \tilde{e}, \ \tilde{r} \in C(U), \ \tilde{r} > 0, \ \tilde{e} \leq \overline{e} \text{ in } U$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{\tilde{\mathbf{h}}\otimes\tilde{\mathbf{h}}}{\tilde{r}} - \tilde{\mathbb{H}}\right] < \tilde{e} \text{ in } U.$$

Conclusion:

$$\mathbf{w}_n \in C^\infty_c(U; R^N),\ \mathbb{G}_n \in C^\infty_c(U; R^{N\times N}_{\mathrm{sym},0}),\ n = 0, 1, \dots$$

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{G}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2} \lambda_{\max} \left[\frac{(\tilde{\mathbf{h}} + \mathbf{w}_n) \otimes (\tilde{\mathbf{h}} + \mathbf{w}_n)}{\tilde{r}} - (\tilde{\mathbb{H}} + \mathbb{G}_n) \right] < \tilde{e} \text{ in } U,$$

$$\mathbf{w}_n \to 0 \text{ weakly in } L^2(U; R^N)$$

$$\liminf_{n\to\infty} \int_U \frac{|\mathbf{w}_n|^2}{\tilde{r}} \, \mathrm{d}x\mathrm{d}t \geq \Lambda(\overline{e}) \int_U \left(\tilde{e} - \frac{1}{2} \frac{|\tilde{\mathbf{h}}|^2}{\tilde{r}} \right)^2 \, \mathrm{d}x\mathrm{d}t$$

Basic ideas of proof

Localization

Localizing the result of DeLellis and Széhelyhidi to "small" cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the "small" cubes

Eliminating singular sets

Applying Whitney's decomposition lemma to the non-singular sets (e.g. out of the vacuum $\{h = 0\}$)

Energy and other coefficients depending on solutions

Applying compactness of the abstract operators in C

Abstract formulation

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

$$\mathbf{v} \odot \mathbf{w} = \mathbf{v} \otimes \mathbf{w} - \frac{1}{2} \mathbf{v} \cdot \mathbf{w} \mathbb{I}$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Abstract operators

Boundedness

b maps bounded sets in $L^\infty((0,T)\times\Omega;R^N)$ on bounded sets in $C_b(Q,R^M)$

Continuity

$$b[\mathbf{v}_n]\to b[\mathbf{v}] \text{ in } C_b(Q;R^M) \text{ (uniformly for } (t,x)\in Q \text{)}$$

whenever

$$\mathbf{v}_n\to\mathbf{v} \text{ in } C_{\rm weak}([0,T];L^2(\Omega;R^N))$$

Causality

$\mathbf{v}(t,\cdot)=\mathbf{w}(t,\cdot)$ for $0\le t\le\tau\le T$ implies $b[\mathbf{v}]=b[\mathbf{w}]$ in $[(0,\tau]\times\Omega]$

Results

Result (A)

The set of subsolutions is non-empty $\Rightarrow$ there exists infinitely many weak solutions of the problem with the same initial data

Initial energy jump

$$\frac{1}{2}\frac{|\mathbf{v_0} + \mathbf{H}[\mathbf{v_0}]|^2}{h[\mathbf{v_0}]} \boxed{<} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Result (B)

The set of subsolutions is non-empty $\Rightarrow$ there exists a dense set of times such that the values $\mathbf{v}(t)$ give rise to non-empty subsolution set with

$$\frac{1}{2}\frac{|\mathbf{v_0} + \mathbf{H}[\mathbf{v_0}]|^2}{h[\mathbf{v_0}]} \boxed{=} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Example I: Savage-Hutter model for avalanches

Unknowns

flow height .... $h = h(t, x)$
depth-averaged velocity .... $\mathbf{u} = \mathbf{u}(t, x)$

$$\partial_t h + \mathrm{div}_x(h\mathbf{u}) = 0$$

$$\partial_t(h\mathbf{u}) + \mathrm{div}_x(h\mathbf{u} \otimes \mathbf{u}) + \nabla_x(ah^2) = h\left(-\gamma\frac{\mathbf{u}}{|\mathbf{u}|} + \mathbf{f}\right)$$

Periodic boundary conditions

$$\Omega = \left([0,1]|_{\{0,1\}}\right)^2$$

Results Savage-Hutter model

Theorem (with P.Gwiazda and A.Swierczewska-Gwiazda [2015])

(i) Let the initial data

$$h_0 \in C^2(\Omega),\ \mathbf{u}_0 \in C^2(\Omega; R^2), h_0 > 0 \text{ in } \Omega$$

be given, and let $\mathbf{f}$ and a be smooth.
Then the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$.

(ii) Let $T > 0$ and

$$h_0 \in C^2(\Omega),\ \ h_0 > 0$$

be given.
Then there exists

$$\mathbf{u}_0 \in L^\infty(\Omega; R^2)$$

such that the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$ satisfying the energy inequality.

Example II, Euler-Fourier system

(joint work with E.Chiodaroli and O.Kreml [2014])

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\Big] - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x \mathbf{u}$$

Example III, Euler-Korteweg-Poisson system

(joint work with D.Donatelli and P.Marcati [2014])

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)$$

$$= \boxed{\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho)|\nabla_x \varrho|^2 \right)} - \varrho \mathbf{u} + \varrho \nabla_x V$$

Poisson equation

$$\Delta_x V = \varrho - \overline{\varrho}$$

Example IV, Euler-Cahn-Hilliard system

Model by Lowengrub and Truskinovsky

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard equation

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right)$$

Example V, models of collective behavior

(joint work with J.A. Carrillo, P.Gwiazda, A.Swierczewska–Gwiazda)

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

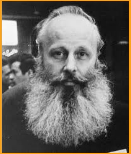
Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})$$

$$= -\nabla_x p(\varrho) + \left(1 - H\left(|\mathbf{u}|^2\right)\right) \varrho \mathbf{u}$$

$$-\varrho \nabla_x K * \varrho + \varrho \psi * \left[\varrho \Big(\mathbf{u} - \mathbf{u}(x) \Big) \right]$$

Measure-valued solutions



Young measures

$$U(t,x) \approx \nu_{t,x}[U]$$

$\nu(B), B \subset R^3$ probability that **U** belongs to the set B

Laurence Chisholm Young [1905-2000]

Siddhartha Mishra

Numerical results

Certain numerical solutions of "inviscid" problems exhibit scheme independent oscillatory behavior

What to do?

However beautiful the strategy, you should occasionally look at the results...
Sir Winston Churchill
[1874-1965]

Some good news to finish...

Navier-Stokes system

- Wild oscillatory solutions are (sofar) not known for problems with *viscosity*, in particular, the Navier-Stokes system (compressible/incompresible)
- Most of the used *numerical schemes* is based on viscous approximation, at least implicitly
- What we compute is mostly the correct solution (??)

Synergy analysis-numerics

- Certain numerical schemes converge to *weak* solutions
- Convergence is unconditional and even error estimates are available if the limit solution is smooth
- Bounded weak solutions *are* smooth
- Bounded solutions of the numerical scheme converge (with error estimates) to the smooth solution