On the motion of compressible inviscid fluids driven by stochastic forcing

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Driven Euler system

Field equations

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho)\mathrm{d}t = \varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W,$$

Stochastic forcing

$$\varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \sum_{k=1}^{\infty} \varrho \mathbf{G}_k(\varrho, \varrho \mathbf{u})\mathrm{d}W_k$$

Iconic examples

$$\varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \varrho \sum_{k=1}^{\infty} \mathbf{G}_k(x)\mathrm{d}W_k, \ \ \varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \lambda \varrho \mathbf{u}\mathrm{d}W$$

Data, initial and boundary conditions

(Random) initial data

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0$$

$W \approx \{W_k\}_{k=1}^{\infty}$ mutually independent Wiener processes

Periodic boundary conditions

$$\Omega = \mathcal{T}^N = \left([0,1]|_{\{0,1\}}\right)^N,\ N = (1), 2, 3$$

Concepts of solutions

Strong solution

Solutions are smooth in space, spatial derivatives exist in the classical sense. Equations satisfied for Itô's stochastic integral

Weak (PDE) solution

Spatial derivatives understood in the sense of distributions

Weak martingale solution

Spatial derivatives understood in the sense of distributions. Data understood in terms of stochastic distribution - law.

$$\varrho_0 \sim \widetilde{\varrho}_0, \ \mathbf{u}_0 \sim \widetilde{\mathbf{u}}_0, \ W \sim \widetilde{W}$$

Dissipative martingale solution

Martingale solutions satisfying a suitable form of energy inequality

Weak (PDE) formulation

Field equations

$$\left[\int_\Omega \varrho\phi \ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \varrho\mathbf{u}\cdot\nabla_x\phi \ \mathrm{d}x\mathrm{d}t,$$

$$\left[\int_\Omega \varrho\mathbf{u}\cdot\phi \ \mathrm{d}x\right]_{t=0}^{t=\tau} - \int_0^\tau \int_\Omega \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\phi + p(\varrho)\mathrm{div}_x\phi \ \mathrm{d}x\mathrm{d}t$$

$$= \boxed{\int_0^\tau \left(\int_\Omega \varrho\mathbf{G}\cdot\phi \ \mathrm{d}x\right)\mathrm{d}W}$$

$\phi = \phi(x) -$ a smooth test function

Stochastic integral (Itô's formulation)

$$\int_0^\tau \left(\int_\Omega \varrho\mathbf{G}\cdot\phi \ \mathrm{d}x\right)\mathrm{d}W = \sum_{k=1}^\infty \int_0^\tau \left(\int_\Omega \varrho\mathbf{G}_k\cdot\phi \ \mathrm{d}x\right)\mathrm{d}W_k$$

Admissibility - dissipative solutions

Energy inequality

$$-\int_0^T \partial_t \psi \left(\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right] \mathrm{d}x \right) \mathrm{d}t$$

$$\leq \psi(0) \int_\Omega \left[\frac{|(\varrho \mathbf{u})_0|^2}{2\varrho_0} + H(\varrho_0) \right] \mathrm{d}x$$

$$+\frac{1}{2} \int_0^T \psi \left(\int_\Omega \sum_{k \geq 1} \frac{|\mathbf{G}_k(\varrho, \varrho \mathbf{u})|^2}{\varrho} \mathrm{d}x \right) \mathrm{d}t + \int_0^T \psi \mathrm{d}M_E$$

$$\psi \geq 0, \ \psi(T) = 0, \ H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \mathrm{d}z$$

Relative energy inequality

Relative energy

$$\mathcal{E}\left(\varrho,\mathbf{u}\middle| r,\mathbf{U}\right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r)\right] \mathrm{d}x$$

Relative energy inequality

$$-\int_0^T \partial_t\psi\ \mathcal{E}\left(\varrho,\mathbf{u}\middle| r,\mathbf{U}\right)\ \mathrm{d}t$$

$$\leq \psi(0)\mathcal{E}\left(\varrho,\mathbf{u}\ \middle| r,\mathbf{U}\right)(0) + \int_0^T \psi \mathrm{d}M_{RE} + \int_0^T \psi\mathcal{R}\left(\varrho,\mathbf{u}\middle| r,\mathbf{U}\right)\mathrm{d}t$$

Test functions

$$\mathrm{d}r = D_t^d r\ \mathrm{d}t + \mathbb{D}_t^s r\ \mathrm{d}W,\ \mathrm{d}\mathbf{U} = D_t^d\mathbf{U}\ \mathrm{d}t + \mathbb{D}_t^s\mathbf{U}\ \mathrm{d}W$$

Remainder

Remainder term

$$\mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) = \int_\Omega \varrho\Big(D_t^d \mathbf{U} + \mathbf{u}\cdot\nabla_x \mathbf{U}\Big)(\mathbf{U} - \mathbf{u})\ \mathrm{d}x$$

$$+\int_\Omega \big((r-\varrho)H''(r)D_t^d r + \nabla_x H'(r)(r\mathbf{U} - \varrho\mathbf{u})\big)\ \mathrm{d}x$$

$$-\int_\Omega \mathrm{div}_x \mathbf{U}(p(\varrho) - p(r))\ \mathrm{d}x$$

$$+\frac{1}{2}\sum_{k\geq 1}\int_\Omega \varrho\left|\frac{\mathbf{G}_k(\varrho, \varrho\mathbf{u})}{\varrho} - [\mathbb{D}_t^s \mathbf{U}]_k\right|^2\ \mathrm{d}x$$

$$+\frac{1}{2}\sum_{k\geq 1}\int_\Omega \varrho H'''(r)|[\mathbb{D}_t^s r]_k|^2\ \mathrm{d}x + \frac{1}{2}\sum_{k\geq 1}\int_\Omega p''(r)|[\mathbb{D}_t^s r]_k|^2\ \mathrm{d}x$$

Existence theory

Local existence of strong solutions [Kim [2011]], [Breit, EF, Hofmanová [2017]]

If the initial data are smooth, then the problem admits local-in-time smooth solutions. Solutions exist up to a (maximal) positive *stopping time*. The life-span is a random variable.

Weak–strong uniqueness [Breit, EF, Hofmanová [2016]]

Pathwise uniqueness.
A weak and strong solutions defined on the same probability space and emanating from the same initial data coincide as long as the latter exists

Uniqueness in law.
If a weak and strong solution are defined on a different probability space, then their *laws* are the same provided the laws of the initial data are the same

Weak (PDE) solutions

Infinitely many weak (PDE) solutions, Breit, EF, Hofmanová [2017]

Let $T > 0$ and the initial data

$$\varrho_0 \in C^3(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in C^3(\Omega)$$

be given.
There exists a sequence of *strictly positive* stopping times

$$\tau_M > 0,\ \tau_M \to \infty \text{ a.s.}$$

such that the initial–value problem for the compressible Euler system possesses infinitely many weak (PDE) solutions defined in $(0, T \wedge \tau_M)$. *Solutions are adapted to the filtration associated to the Wiener process W.*

Semi-deterministic approach - additive noise

“Additive noise” problem

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \varrho \sum_{k=1}^{\infty} \mathbf{G}_k \mathrm{d}W_k$$

$$\varrho \sum_{k=1}^{\infty} \mathbf{G}_k \mathrm{d}W_k = \varrho \mathbf{G} \mathrm{d}W$$

Additive noise, Step I

Step I

$$\partial_t(\varrho\mathbf{u}-\varrho\mathbf{G}W)+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p(\varrho)=-\partial_t\varrho\mathbf{G}W=\mathrm{div}_x(\varrho\mathbf{u})\mathbf{G}W$$

Transformed system I

$$\mathbf{w}=\varrho\mathbf{u}-\varrho\mathbf{G}W$$

$$\partial_t\varrho+\mathrm{div}_x(\mathbf{w}+\varrho\mathbf{G}W)=0$$

$$\partial_t\mathbf{w}+\mathrm{div}_x\left(\frac{(\mathbf{w}+\varrho\mathbf{G}W)\otimes(\mathbf{w}+\varrho\mathbf{G}W)}{\varrho}\right)+\nabla_x p(\varrho)$$

$$=\mathrm{div}_x(\mathbf{w}+\varrho\mathbf{G}W)\mathbf{G}W$$

Additive noise, Step II

Step II

$$\mathbf{w} = \mathbf{v} + \mathbf{V} + \nabla_x \Phi, \ \mathrm{div}_x \mathbf{v} = 0, \int_\Omega \mathbf{v} \ \mathrm{d}x = 0, \ \mathbf{V} = \mathbf{V}(t)$$

Transformed system II

$$\mathbf{w} = \varrho \mathbf{u} - \varrho \mathbf{G} W$$

$$\partial_t \varrho + \mathrm{div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W)}{\varrho} \right)$$

$$+ \nabla_x p(\varrho) + \nabla_x \partial_t \Phi = \mathrm{div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) \mathbf{G} W - \partial_t \mathbf{V}$$

Additive noise, Step III

Step III

Fix Φ, ϱ, $\mathbf{V}$ so that

$$\varrho(0,\cdot) = \varrho_0,\ \ \mathbf{V}(0) = \frac{1}{|\Omega|}\int_\Omega \mathbf{u}_0 \ {\rm d}x,\ \ \nabla_x \Phi(0,\cdot) = \mathbf{H}^\perp[\mathbf{u}_0]$$

$$\partial_t \varrho + {\rm div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) = 0$$

$$\partial_t \mathbf{V} = \frac{1}{|\Omega|} {\rm div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) \mathbf{G} W$$

$${\rm div}_x \left(\nabla_x \mathbf{M} + \nabla_x \mathbf{M}^\perp - \frac{2}{N} {\rm div}_x \mathbf{M} \right)$$

$$= {\rm div}_x (\nabla_x \Phi + \varrho \mathbf{G} W) \mathbf{G} W - \partial_t \mathbf{V}$$

Additive noise, Step IV

Step IV

Fix $\mathbf{h}$, $\mathbb{H}$ so that

$$\mathbf{h} = \mathbf{V} + \nabla_x \Phi + \varrho \mathbf{G} W, \ \mathbb{H} = \nabla_x \mathbf{M} + \nabla_x^t \mathbf{M} - \frac{2}{N} \mathrm{div}_x \mathbf{M} \mathbb{I} \in R^{N \times N}_{0,\mathrm{sym}}$$

Tranformed system III

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{\varrho} - \mathbb{H} + p(\varrho)\mathbb{I} + \partial_t \Phi \mathbb{I} \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] - \frac{1}{|\Omega|} \int_\Omega \mathbf{u}_0 \ \mathrm{d}x$$

Additive noise, Step V

Prescribing the kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e = \Lambda - \frac{N}{2}\left(p(\varrho) + \partial_t \Phi\right), \ \Lambda = \Lambda(t)$$

Abstract Euler system

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} - \mathbb{H} \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0, \ \frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0$$

Random parameters

The functions $\mathbf{v}_0$, h and $\mathbb{H}$ are random variables, the energy e can be taken deterministic.

Subsolutions

Field equations, differential constraints

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0, \ \mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Non-linear constraint

$$\mathbf{v} \in C([0, T] \times \Omega; R^N), \ \mathbb{F} \in C([0, T] \times \Omega; R^{N \times N}_{\mathrm{sym},0}),$$

$$\frac{N}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{\varrho} - \mathbb{F} + \mathbb{M} \right] < e$$

Subsolution relaxation

Algebraic inequality

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} \leq \frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{h})\otimes(\mathbf{v}+\mathbf{h})}{\varrho} - \mathbb{F} + \mathbb{M}\right] < e$$

Solutions

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = e$$

$$\Rightarrow$$

$$\mathbb{F} = \frac{(\mathbf{v}+\mathbf{h})\otimes(\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} + \mathbb{M}$$

Augmenting oscillations

Oscillatory lemma

If

$$\varrho, e, \mathbf{h} \in C(Q; R^N), \varrho, e > 0,\ \mathbb{H} \in C(Q; R^{N\times N}_{\mathrm{sym},0})$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{\mathbf{h}\otimes\mathbf{h}}{\varrho} - \mathbb{H}\right] < e \text{ in } Q,$$

then there exist

$$\mathbf{w}_n \in C^\infty_c(Q; R^N),\ \mathbb{G}_n \in C^\infty_c(Q; R^{N\times N}_{\mathrm{sym},0}),\ \ n = 0, 1, \dots$$

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{G}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{h}+\mathbf{w}_n)\otimes(\mathbf{h}+\mathbf{w}_n)}{\varrho} - (\mathbb{H}+\mathbb{G}_n)\right] < e$$

$$\mathbf{w}_n \rightharpoonup 0,\ \liminf_{n\to\infty}\int_Q \frac{|\mathbf{w}_n|^2}{\varrho}\,\mathrm{d}x\mathrm{d}t \ge \Lambda(\max_\Omega e)\int_Q \left(e - \frac{1}{2}\frac{|\mathbf{h}|^2}{\varrho}\right)^2\,\mathrm{d}x\mathrm{d}t$$

Basic ideas of proof [DeLellis and Székelyhidi]

Basic result

Unit cube and constant coefficients ϱ, e, $\mathbf{h}$, $\mathbb{H}$

Scaling

Localizing the basic result to "small" cubes by means of scaling arguments

Approximation

Replacing all continuous functions by their means on any of the "small" cubes

Difficulties in the stochastic world

Adaptiveness

All quantities must be adapted to the filtration associated to the Wiener process W

Geometric setting

Continuous functions approximated in a similar way as in the definition of Itô's integral
Admissible directions for oscillations selected by the Kuratowski, Ryll–Nardzewski theorem

Space–time localization

Stopping the Wiener process by its Hölder norm

Stochastic version of the oscillatory lemma

Fixing parameters

Problem restricted to intervals small cubes $[t_k, t_{k+1}] \times B_k(x)$. All random parameters replaced by their values at t_k

Constructing oscillations

Adapting the procedure by De Lellis and Székelyhidi using Ryll–Nardzewski theorem on measurable selection

Cutting off oscillatory increments

The difference $W(t_k) - W(t)$ must remain small on $[t_k, t_{k+1}]$ - requires knowledge of the Hölder constant of W on $[t_k, t_{k+1}]$ at t_k - in general not predictable unless W is replaced by uniformly Hölder function - the necessity of stopping times τ_k.