# Mathematics of fluids in motion, I

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# Compressible Navier-Stokes/Euler system

**Field equations**

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

**Newton's rheological law**

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}, \ \mu \geq 0, \ \eta \geq 0$$

**No-flux/no-slip boundary conditions**

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbf{u}]_{\mathrm{tan}}|_{\partial\Omega} = 0$$

# Thermodynamics stability

**Pressure potential**

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z$$

**Pressure-density state equation**

$$p \in C[0, \infty) \cap C^2(0, \infty), \ p(0) = 0$$

$$\boxed{p'(\varrho) > 0} \text{ for } \varrho > 0, \ \liminf_{\varrho \to \infty} p'(\varrho) > 0$$

$$\liminf_{\varrho \to \infty} \frac{P(\varrho)}{p(\varrho)} > 0$$

**Isentropic pressure-density state equation**

$$p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma \geq 1$$

# Energy balance - conservation

**Energy**

$$E = \underbrace{\frac{1}{2}\varrho|\mathbf{u}|^2}_{\text{kinetic energy}} + \underbrace{P(\varrho)}_{\text{elastic energy}} , \; P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z$$

**Energy balance equation**

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) + \mathrm{div}_x(p(\varrho)\mathbf{u}) - \mathrm{div}_x\left(\mathbb{S}(\nabla_x\mathbf{u}) \cdot \mathbf{u}\right) = -\boxed{\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}}$$

**Total energy balance**

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right) \, \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} \, \mathrm{d}x \boxed{\leq} 0$$

# Classical (strong) solutions

**Local existence**

Smooth solutions exist on a maximal time interval $(0, T_{\max})$. This is true for both Navier-Stokes and Euler system

**Global-in-time solutions for small data**

Smooth solutions of the *Navier-Stokes system* exist globally in time provided the initial data are close to an equilibrium solution (**Matsumura and Nishida, Valli and Zajaczkowski, and others**). Solutions of the *Euler system* develop singularities in a finite time no matter how smooth and/or small the initial data are.

**Global existence for the 1-D Navier-Stokes system**

The Navier-Stokes system in the 1-D geometry admits global-in-time smooth solutions (**Antontsev, Kazhikhov, Shelukhin and others**)

# Weak solutions

## Equation of continuity

$$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t=0}^{\tau} = \int_0^\tau \int_\Omega \left(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right) \ \mathrm{d}x\mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$

## Balance of momentum

$$\left[\int_\Omega \varrho\mathbf{u}\cdot\varphi \ \mathrm{d}x\right]_{t=0}^{\tau}$$

$$= \int_0^\tau \int_\Omega \left(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u}:\cdot\nabla_x\varphi + p(\varrho)\mathrm{div}_x\varphi\right) \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T]\times\overline{\Omega};R^N)$,

$\varphi|_{\partial\Omega} = 0$ for the no-slip condition in the viscous case

# Dissipative weak solutions

**Energy (entropy) inequality**

$$\left[ \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \, \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \leq 0$$

for a.a. $\tau \in (0, T)$

# Navier-Stokes system: Weak solutions

**Global existence for large data**

$$p(\varrho) \approx a\varrho^{\gamma}, \ \mu > 0$$

The Navier-Stokes system admits global-in-time weak solutions if:

- $N = 2$, $\gamma \geq 3/2$; $N = 3$, $\gamma \geq 9/5$ **P.L.Lions 1998**
- $N = 2$, $\gamma > 1$, $N = 3$, $\gamma > 3/2$ **EF et al. 2000**
- $N = 2$, $\gamma \geq 1$, $N = 3$, $\gamma \geq 3/2$ **Plotnikov and Vaigant 2014**

**Dissipative weak solutions**

The weak solutions are not known to be unique. The construction used in the existence theory yields *dissipative* weak solutions. Weak solutions can be obtained as a limit of certain numerical schemes (**Karper**)

# Euler system: Weak solutions

### Global existence for large data in 1D

The Euler system admits global-in-time weak solutions for any bounded initial data (**DiPerna, Chen et al.**). The weak solutions can be recovered as a vanishing viscosity limit of the Navier-Stokes system (**Chen and Perepelitsa**)

### Global existence for large data for $N = 2, 3$

The compressible Euler system admits *infinitely many* global-in-time weak solutions for any smooth initial data (**Chiodaroli, EF** - based on the work of **DeLellis and Székelyhidi**)

# Euler system: Dissipative weak solutions

**Dissipative weak solutions** $N = 2, 3$

- For any $\varrho_0$, there exists $\mathbf{u}_0$ (bounded measurable) such that the Euler system admits infinitely many dissipative weak solutions in a given time interval $(0, T)$ (**Chiodaroli, EF**)

- There is a vast class of initial data for which the Euler system admits infinitely many entropy (dissipative) weak solutions in a given time interval $(0, T)$ (**Chiodaroli, EF**)

- There exist Lipschitz (smooth) initial data for which the Euler system admits infinitely many entropy (dissipative) weak solutions in a given time interval $(0, T)$ (**Chiodaroli, DeLellis, Kreml**)

# Relative entropy (energy)

**Relative energy functional**

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)\right) \, \mathrm{d}x$$

**Decomposition**

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right) \, \mathrm{d}x - \int_\Omega \varrho\mathbf{u} \cdot \mathbf{U} \, \mathrm{d}x + \int_\Omega \frac{1}{2}\varrho|\mathbf{U}|^2 \, \mathrm{d}x$$

$$- \int_\Omega P'(r)\varrho \, \mathrm{d}x + \int_\Omega p(r) \, \mathrm{d}x$$

# Dissipation inequality

**Relative energy inequality**

$$\left[ \mathcal{E} \left( \varrho, \mathbf{u} \,\middle|\, r, \mathbf{U} \right) \right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \int_\Omega \left( \mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \right) : \left( \nabla_x \mathbf{u} - \nabla_x \mathbf{U} \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R} \left( \varrho, \mathbf{u} \,\middle|\, r, \mathbf{U} \right) \, \mathrm{d}t$$

**Test functions**

$$r > 0, \ \mathbf{U}|_{\partial \Omega} = 0 \ \text{(or other relevant b.c.)}$$

# Remainder

$$\int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U}\right) \, \mathrm{d}t$$

$$\int_\Omega \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\right) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x$$

$$+ \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, \mathrm{d}x + \int_\Omega \left(p(r) - p(\varrho)\right) \operatorname{div}_x \mathbf{U} \, \mathrm{d}x$$

$$+ \int_\Omega \left[(r - \varrho)\partial_t P'(r) + \nabla_x P'(r) \cdot (r\mathbf{U} - \varrho\mathbf{u})\right] \, \mathrm{d}x$$

# Applications

## Weak-strong uniqueness

Weak and strong solutions of the compressible Navier-Stokes/Euler system emanating from the same initial data coincide as long as the latter exists (**EF, Jin, Novotný, Sun [2014]**)

## Conditional regularity

Weak solution to the Navier-Stokes system with bounded density component emanating from smooth initial data are smooth (**EF, Jin, Novotný, Sun [2014]**)

# Singular limits

**Rotating fluids**

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon}\varrho \mathbf{b} \times \mathbf{u} + \frac{1}{\varepsilon^{2M}}\nabla_x p(\varrho)$$

$$= \varepsilon^R \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^{2F}}\nabla_x G$$

**Path dependent singular limit**

$$\varepsilon \to 0, \text{ certain relation between } M, R, F > 0$$

- low Mach $\Rightarrow$ compressible $\to$ incompressible
- high Rossby $\Rightarrow$ 3D $\to$ 2D
- high Reynolds $\Rightarrow$ viscous $\to$ inviscid

# Convergence to singular limit system

**Target problem - Euler system**

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \ \mathrm{div}_x \mathbf{v} = 0, \ x \in R^2$$

**Convergence results (EF, Lu, Novotný 2014)**

- **Spatial geometry - infinite strip:**

$$\Omega = R^2 \times (0, \pi)$$

- **Complete slip (Navier) boundary conditions:**

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$$

# Limits on domains with variable geometry

**Channel like domains**

$$\Omega_\varepsilon = \left\{ (\mathbf{x}, z) \, \Big| \, z \in (0,1), \, |\mathbf{x} - \varepsilon \mathbf{X}(z)|^2 < \varepsilon^2 R^2(z) \right\}, \, |\mathbf{X}(z)| < R(z)$$

**Boundary conditions**

$$\mathbf{u} \cdot \mathbf{n}|_\Sigma = 0, \, \left( \mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n} \right) \times \mathbf{n}|_\Sigma = 0$$

$$\Sigma = \partial \Omega \cap \{ z \in (0,1) \}$$

$$\mathbf{u}|_{z=0,1} = 0$$

# Target systems

**Inviscid limit**

$$\partial_t(\varrho_E A) + \partial_z(\varrho_E u_E A) = 0$$

$$\partial_t(\varrho_E u_E A) + \partial_z(\varrho_E u_E^2 A) + A\partial_z p(\varrho_E) = 0$$

**Viscous limit**

$$\partial_t(\varrho_{NS} A) + \partial_z(\varrho_{NS} u_{NS} A) = 0$$

$$\partial_t(\varrho_{NS} u_{NS} A) + \partial_z(\varrho_{NS} u_{NS}^2 A) + A\partial_z p(\varrho_{NS})$$

$$= A\nu\partial_z^2 u_{NS} + \nu\partial_z\left(R'(z)/R(z)u_{NS}\right), \ \nu = \frac{4}{3}\mu + \eta > 0$$

$$A = R^2$$

# Convergence

**Korn-Poincaré inequality**

$$\int_{\Omega_\varepsilon} |\mathbf{v}|^2 \,\mathrm{d}x \leq c_{KP} \int_{\Omega_\varepsilon} \left| \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} \right|^2 \,\mathrm{d}x$$

**Convergence (Bella, EF, Lewicka, Novotný 2015)**

- Convergence to the target Euler system with geometric terms in the inviscid limit
- Convergence to the Navier-Stokes system in the viscous limit provided the bulk viscosity in the primitive system is positive

# Navier-Stokes system driven by stochastic forces

**Navier–Stokes system with stochastic forcing**

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho\mathbf{u}) \ \mathrm{d}t = 0$$

$$\mathrm{d}(\varrho\mathbf{u}) + [\mathrm{div}_x(\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)] \ \mathrm{d}t = \mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{u}) \ \mathrm{d}t + \mathbb{G}(\varrho, \varrho\mathbf{u})\mathrm{d}W,$$

**White–noise forcing**

$$\mathbb{G}(\varrho, \varrho\mathbf{u}) \ \mathrm{d}W = \sum_{k \geq 1} \mathbf{G}_k(\varrho, \varrho\mathbf{u}) \ \mathrm{d}W_k.$$

# Relative energy inequality

**Relative energy inequality - (Breit, EF, Hofmanová 2015)**

$$-\int_0^T \partial_t \psi \, \mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \, \mathrm{d}t$$

$$+\int_0^T \psi \int_\Omega \left(\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t\right.$$

$$\leq \psi(0)\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)(0) + \int_0^T \psi \mathrm{d}M_{RE} + \int_0^T \psi \mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \mathrm{d}t$$

$$\psi \in C_c^\infty[0, T) \text{ (deterministic)}, \ \psi \geq 0.$$

**Test functions**

$$\mathrm{d} \, r = D_t^d r \, \mathrm{d}t + D_t^s r \, \mathrm{d}W, \ \mathrm{d}\mathbf{U} = D_t^d \mathbf{U} \, \mathrm{d}t + D_t^s \mathbf{U} \, \mathrm{d}W$$

# Stochastic remainder

**Remainder**

$$\mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)$$

$$= \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, \mathrm{d}x + \int_\Omega \varrho \left(D_t^d \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\right)(\mathbf{U} - \mathbf{u}) \, \mathrm{d}x$$

$$+ \int_\Omega \left((r - \varrho)H''(r)D_t^d r + \nabla_x H'(r)(r\mathbf{U} - \varrho\mathbf{u})\right) \, \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U}(p(\varrho) - p(r)) \, \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_\Omega \varrho \left|\frac{\mathbf{G}_k(\varrho, \varrho\mathbf{u})}{\varrho} - D_t^s \mathbf{U}_k\right|^2 \, \mathrm{d}x$$

$$+ \frac{1}{2} \sum_{k \geq 1} \int_\Omega \varrho H'''(r)|D_t^s r_k|^2 \, \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_\Omega p''(r)|D_t^s r_k|^2 \, \mathrm{d}x$$

# Results for stochastic Navier-Stokes system

**Weak–strong uniqueness (Breit, EF, Hofmanová 2015)**

- Pathwise weak-strong uniqueness
- Weak-strong uniqueness in law

**Inviscid–incompressible limit in the stochastic setting (Breit, EF, Hofmanová 2015)**

Convergence to the limit stochastic Euler system for vanishing viscosity and the Mach number. Results for well-prepared data.

# Possible extensions

**Numerical analysis (Gallouet, Herbin, Maltese, Novotný 2014)**

Relative energy inequality for the numerical scheme proposed by K.Karlsen and T. Karper. Error estimates.

**Measure–valued solutions**

Weak–strong uniqueness for measure-valued solutions (**EF, Gwiazda, Swierczewska-Gwiazda, Wiedemann 2015**)

# Preliminaries to measure-valued solutions

**Families of integrable solutions**

$$[\varrho_n, \mathbf{u}_n] : \underbrace{(0, T) \times \Omega}_{\text{physical space}} \mapsto \underbrace{[0, \infty) \times R^N}_{\text{phase space}}$$

$\varrho_n \to \varrho, \ \mathbf{u}_n \to \mathbf{u}$ weakly in $L^1((0, T) \times \Omega)$

**Nonlinear compositions - Young measure**

$$F(\varrho_n, \mathbf{u}_n) \to \overline{F(\varrho, \mathbf{u})} \text{ weakly in } L^1((0, T) \times \Omega)$$

$$\Rightarrow$$

$$\overline{F(\varrho, \mathbf{u})} = \langle \nu_{t,x}; F(s, \mathbf{v}) \rangle \text{ for a.a. } (t, x)$$

**Biting limit**

$$\int_0^T \int_\Omega |F(\varrho_n, \mathbf{u}_n) \ \mathrm{d}x \ \mathrm{d}t \leq c \ \Rightarrow \ \langle \nu_{t,x}; F(s, \mathbf{v}) \rangle \in L^1((0, T) \times \Omega)$$

# Biting limit decomposition

**Bounded integrable compositions**

$$\int_0^\tau \int_\Omega |F(\varrho_n, \mathbf{u}_n)| \; \mathrm{d}x \; \mathrm{d}t \leq c$$

$$\underbrace{\Rightarrow}_{\text{up to a subsequence}}$$

$$F(\varrho_n, \mathbf{u}_n) \to \overline{F(\varrho, \mathbf{u})} \text{ weakly-(*) in } \mathcal{M}([0, T] \times \overline{\Omega})$$

**Biting limit decomposition**

$$\overline{F(\varrho, \mathbf{u})} = \underbrace{\overline{F(\varrho, \mathbf{u})} - \langle \nu_{t,x}; F(s, \mathbf{v}) \rangle}_{\text{concentration part}} + \underbrace{\langle \nu_{t,x}; F(s, \mathbf{v}) \rangle}_{\text{oscillatory part}}$$

# Measure-valued solutions

**Parameterized (Young) measure**

$$\nu_{t,x} \in L^{\infty}_{\text{weak}}((0,T) \times \Omega; \mathcal{P}([0,\infty) \times R^N)), \ [s,\mathbf{v}] \in [0,\infty) \times R^N$$

$$\varrho(t,x) = \langle \nu_{t,x}; s \rangle, \ \mathbf{u} = \langle \nu_{t,x}; \mathbf{v} \rangle$$

**Navier-Stokes/Euler, velocity/momentum**

$$\text{Navier-Stokes } \mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^N)),$$

$$\text{Euler } \mathbf{u} \approx \mathbf{m} \approx \varrho \mathbf{u}$$

**Initial data**

$$\nu_0 = \nu_{0,x}$$

Regular initial data

$$\nu_{0,x} = \delta_{\varrho_0(x), \mathbf{u}_0(x)} \text{ for a.a. } x$$

# Field equations

## Equation of continuity

$$\left[ \int_\Omega \langle \nu_{t,x}, s \rangle \, \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \langle \nu_{t,x}; s \rangle \, \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t + \langle R_1; \nabla_x \varphi \rangle$$

## Momentum balance

$$\left[ \int_\Omega \langle \nu_{t,x}, s\mathbf{v} \rangle \, \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle : \nabla_x \varphi + \langle \nu_{t,x}; p(s) \rangle \, \mathrm{div}_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t + \langle R_2; \nabla_x \varphi \rangle$$

# Dissipativity

**Energy inequality**

$$\left[\int_\Omega \left\langle \nu_{\tau,x}; \left(\frac{1}{2}s|\mathbf{v}|^2 + P(s)\right)\right\rangle \, \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t + \boxed{\mathcal{D}(\tau)} \leq 0$$

**Compatibility**

$$\left|R_1[0,\tau] \times \overline{\Omega}\right| + \left|R_2[0,\tau] \times \overline{\Omega}\right| \leq \xi(\tau)\mathcal{D}(\tau), \; \xi \in L^1(0,T)$$

$$\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; |\mathbf{v} - \mathbf{u}|^2\right\rangle \, \mathrm{d}x \, \mathrm{d}t \leq c_P \mathcal{D}(\tau)$$

# Truly measure-valued solutions

**Truly measure-valued solutions for the Euler system (EF, Chiodaroli, Kreml, Wiedemann)**

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded $L^p$ weak solutions to the Euler system.

# Do we need measure valued solutions?

**Limits of problems with higher order viscosities**

Multipolar fluids with complex rheologies (Nečas - Šilhavý)

$$\mathbb{T}(\mathbf{u}, \nabla_x\mathbf{u}, \ \nabla_x^2\mathbf{u}, \dots)$$

$$= \mathbb{S}(\nabla_x\mathbf{u}) + \delta \sum_{j=1}^{k-1} \left( (-1)^j \mu_j \Delta^j (\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u}) + \lambda_j \Delta^j \mathrm{div}_x\mathbf{u} \ \mathbb{I} \right)$$

$$+ \text{ non-linear terms}$$

Limit for $\delta \to 0$

**Limits of numerical solutions**

Numerical solutions resulting from Karlsen-Karper and other schemes

**Sub-critical parameters**

$p(\varrho) = a\varrho^\gamma$, $\gamma < \gamma_{\mathrm{critical}}$

# Weak (mv) - strong uniqueness

**Theorem - EF, Gwiazda, Swierczewska-Gwiazda, Wiedemann 2015**

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

# Relative energy (entropy)

**Relative energy functional**

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right)(\tau)$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2}s|\mathbf{v} - \mathbf{U}|^2 + P(s) - P'(r)(s-r) - P(r) \right\rangle \, \mathrm{d}x$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2}s|\mathbf{v}|^2 + P(s) \right\rangle \, \mathrm{d}x - \int_\Omega \langle \nu_{\tau,x}; s\mathbf{v}\rangle \cdot \mathbf{U} \, \mathrm{d}x$$

$$+ \int_\Omega \frac{1}{2}\langle \nu_{\tau,x}; s\rangle \,|\mathbf{U}|^2 \, \mathrm{d}x$$

$$- \int_\Omega \langle \nu_{\tau,x}; s\rangle \, P'(r) \, \mathrm{d}x + \int_\Omega p(r) \, \mathrm{d}x$$

# Relative energy (entropy) inequality

**Relative energy inequality**

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) + \int_0^\tau \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t + \mathcal{D}(\tau)$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}_0|^2 + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle \ \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \ \mathrm{d}t$$

## Remainder

$$\mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)$$

$$= -\int_0^\tau \int_\Omega \langle \nu_{t,x}, s\mathbf{v} \rangle \cdot \partial_t \mathbf{U} \,\mathrm{d}x \,\mathrm{d}t$$

$$-\int_0^\tau \int_{\overline{\Omega}} [\langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(s) \rangle \operatorname{div}_x \mathbf{U}] \,\mathrm{d}x \,\mathrm{d}t$$

$$+\int_0^\tau \int_\Omega [\langle \nu_{t,x}; s \rangle \,\mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U}] \,\mathrm{d}x \,\mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \left[ \left\langle \nu_{t,x}; \left(1 - \frac{s}{r}\right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \,\mathrm{d}x \,\mathrm{d}t$$

$$+\int_0^\tau \left\langle R_1; \frac{1}{2}\nabla_x \left(|\mathbf{U}|^2 - P'(r)\right) \right\rangle \,\mathrm{d}t - \int_0^\tau \langle R_2; \nabla_x \mathbf{U} \rangle \,\mathrm{d}t$$

# Regularity

**Theorem - EF, Gwiazda, Swierczewska-Gwiazda, Wiedemann 2015**

Suppose that the initial data are smooth and satisfy the relevant compatibility conditions. Let $\nu_{t,x}$ be a measure-valued solution to the compressible Navier-Stokes system with a dissipation defect $\mathcal{D}$ such that

$$\mathrm{supp}\, \nu_{t,x} \subset \left\{ (s, \mathbf{v}) \mid 0 \le s \le \overline{\varrho},\ \mathbf{v} \in R^N \right\}$$

for a.a. $(t, x) \in (0, T) \times \Omega$.

Then $\mathcal{D} = 0$ and

$$\nu_{t,x} = \delta_{\varrho(t,x),\mathbf{u}(t,x)}$$

where $\varrho$, $\mathbf{u}$ is a smooth solution.

# Sketch of the proof

- The Navier-Stokes system admits a local-in-time smooth solution
- The measure-valued solution coincides with the smooth solution on its life-span
- The smooth solution density component remains bounded by $\overline{\varrho}$ as long as the solution exists
- Y. Sun, C. Wang, and Z. Zhang [2011]: The strong solution can be extended as long as the density component remains bounded

# Corollary

**Convergence of numerical solutions**

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution