Weak vs. strong solutions in the mathematical theory of compressible fluid flows

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Wuerzburger Mathematisches Kolloquium, 19 October 2016

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

Compressible Navier-Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Isentropic EOS, Newton's rheological law

$$p(\varrho) = a\varrho^\gamma$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

No-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0$$

Numerical method [T. Karper]

FV framework

regular tetrahedral mesh, $Q_h = \{ v \mid v = \text{piece-wise constant} \}$

FE framework - Crouzeix - Raviart

$$V_h = \left\{ v \;\middle|\; v = \text{piece-wise affine, } \tilde{v}_\Gamma \text{ continuous on face } \Gamma \right\}$$

$$\tilde{v}_\Gamma \equiv \frac{1}{|\Gamma|} \int_\Gamma v \ \mathrm{dS}_x$$

Upwind discretization of convective terms

$$\langle h\mathbf{u}; \nabla_x \varphi \rangle_E \approx \sum_\Gamma \int_\Gamma \mathrm{Up}[h, \mathbf{u}][[\varphi]] \ \mathrm{dS}_x$$

Dissipative upwind operator

Upwind operator

$$\mathrm{Up}[r_h, \mathbf{u}_h] = \underbrace{\{r_h\}\langle \mathbf{u}_h \cdot \mathbf{n}\rangle_\Gamma}_{\text{convective part}} - \frac{1}{2}\underbrace{\max\{h^\alpha; |\langle \mathbf{u}_h \cdot \mathbf{n}\rangle_\Gamma|\}[[r_h]]}_{\text{dissipative part}}$$

$$= \underbrace{r_h^{\mathrm{out}}[\langle \mathbf{u}_h \cdot \mathbf{n}\rangle_\Gamma]^- + r_h^{\mathrm{in}}[\langle \mathbf{u}_h \cdot \mathbf{n}\rangle_\Gamma]^+}_{\text{standard upwind}} - \frac{h^\alpha}{2}[[r_h]]\,\chi\left(\frac{\langle \mathbf{u}_h \cdot \mathbf{n}\rangle_\Gamma}{h^\alpha}\right)$$

Auxilliary function

$$\chi(z) = \begin{cases} 0 \text{ for } z < -1, \\ z+1 \text{ if } -1 \leq z \leq 0 \\ 1 - z \text{ if } 0 < z \leq 1 \\ 0 \text{ for } z > 1 \end{cases}$$

Numerical scheme

Discrete time derivative - implicit scheme

$$D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}$$

Continuity method

$$\int_{\Omega_h} D_t \varrho_h^k \phi \mathrm{d}x - \sum_{\Gamma \in \Gamma_{\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k, \mathbf{u}_h^k] \, [[\phi]] \, \mathrm{dS}_x = 0$$

Momentum method

$$\int_{\Omega_h} D_t \left(\varrho_h^k \left\langle \mathbf{u}_h^k \right\rangle \right) \cdot \boldsymbol{\phi} \mathrm{d}x - \sum_{\Gamma \in \Gamma_{\mathrm{int}}} \int_\Gamma \mathrm{Up}[\varrho_h^k \left\langle \mathbf{u}_h^k \right\rangle, \mathbf{u}_h^k] \cdot [[\langle \boldsymbol{\phi} \rangle]] \, \mathrm{dS}_x$$

$$- \int_{\Omega_h} p(\varrho_h^k) \mathrm{div}_h \boldsymbol{\phi} \mathrm{d}x$$

$$+\mu \int_{\Omega_h} \nabla_h \mathbf{u}_h^k : \nabla_h \boldsymbol{\phi} \mathrm{d}x + \left(\frac{\mu}{3} + \eta \right) \int_{\Omega_h} \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \boldsymbol{\phi} \mathrm{d}x = 0$$

Convergence results for Karper's scheme

Convergence to weak solutions

Karper [2013]: Convergence to a weak solution if $\boxed{\gamma > 3}$

Error estimates

Gallouet, Herbin, Maltese, Novotný [2015]

Convergence to smooth solutions + error estimates if $\boxed{\gamma > 3/2}$, Ω a polyhedral domain

Convergence for general adiabatic coefficient

EF, M. Lukáčová/Medviďová [2016]

Let $\Omega \subset R^3$ be a smooth bounded domain. Let

$$1 < \gamma < 2,\ \Delta t \approx h,\ 0 < \alpha < 2(\gamma - 1).$$

Suppose that the initial data are smooth and that the compressible Navier-Stokes system admits a smooth solution in $[0, T]$ in the class

$$\varrho,\ \nabla_x \varrho,\ \mathbf{u}, \nabla_x \mathbf{u} \in C([0, T] \times \overline{\Omega})$$

$$\partial_t \mathbf{u} \in L^2(0, T; C(\overline{\Omega}; R^3)),\ \varrho > 0,\ \mathbf{u}|_{\partial\Omega} = 0.$$

Then

$$\varrho_h \to \varrho \text{ (strongly) in } L^\gamma((0, T) \times K)$$

$$\mathbf{u}_h \to \mathbf{u} \text{ (strongly) in } L^2((0, T) \times K; R^3)$$

for any compact $K \subset \Omega$.

General strategy

Basic properties of numerical scheme

Show stability, consistency, discrete energy inequality

Measure valued solutions

Show convergence of the scheme to a

dissipative measure – valued solution

Weak-strong uniqueness

Use the weak-strong uniqueness principle in the class of measure-valued solutions. Strong and measure valued solutions emanating from the same initial data coincide as long as the latter exists

Hierarchy of solutions

Classical solutions

Solutions are (sufficiently) smooth satisfying the equations point-wise, determined uniquely by the data. Requires strong *a priori* bounds usually not available

Weak solutions

Equations satisfied in the sense of distributions. Requires *a priori* bounds to ensure equi-integrability of nonlinearities + compactness

Measure-valued solutions

Equations satisfied in the sense of distributions, nonlinearities replaced by Young measures (weak limits) $f(u)(t,x) \approx \langle \nu_{t,x}; f(\mathbf{v}) \rangle$. Requires *a priori* bounds to ensure equi-integrability of nonlinearities.

Measure-valued solutions with concentration measure

Measure-valued solutions + concentration defects. Requires *a priori* bounds to ensure integrability of nonlinearities.

Dissipative solutions

Energy (entropy) inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x \boxed{\leq} 0$$

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

Known results

- **Local strong solution for any data and global weak solutions for small data.** Matsumura and Nishida [1983], Valli and Zajaczkowski [1986], among others
- **Global-in-time weak solutions.** $p(\varrho) = \varrho^\gamma$, $\gamma \geq 9/5$, $N = 3$, $\gamma \geq 3/2$, $N = 2$ P.L. Lions [1998], $\gamma > 3/2$, $N = 3$, $\gamma > 1$, $N = 2$ EF, Novotný, Petzeltová [2000], $\gamma = 1$, $N = 2$ Plotnikov and Vaigant [2014]
- **Measure-valued solutions.** Neustupa [1993], related results Málek, Nečas, Rokyta, Růžička, Nečasová - Novotný

Bounded sequences of integrable functions

Boundedness

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(Q; R^M)$$

$$\|F(\mathbf{v}_n)\|_{L^1(Q)} \le C \;\Rightarrow\; F(\mathbf{v}_n) \to \overline{F(\mathbf{v})} \not\equiv F(\mathbf{v}) \text{ weakly-(*) in } \mathcal{M}(\overline{Q})$$

Biting limit - parameterized Young measure

$$\langle \nu_{t,x}; F_k(\mathbf{v})\rangle = \overline{F_k(\mathbf{v})}(t,x),\; F_k \in BC(R^M)$$

$$\langle \nu_{t,x}; F(\mathbf{v})\rangle = \lim_{k\to\infty} \overline{F_k(\mathbf{v})}(t,x),\; F_k \nearrow F,\; \|F(\mathbf{v}_n)\|_{L^1(Q)} \le C$$

Concentration part - defect measure

$$\overline{F(\mathbf{v})}(t,x) = \underbrace{\langle \nu_{t,x}; F(\mathbf{v})\rangle}_{integrable} + \underbrace{\left[\overline{F(\mathbf{v})}(t,x) - \langle \nu_{t,x}; F(\mathbf{v})\rangle\right]}_{concentration\ \ defect}$$

Measure-valued solutions

Parameterized (Young) measure

$$\nu_{t,x} \in L^\infty_{\rm weak}((0,T)\times\Omega; \mathcal{P}([0,\infty)\times R^N),\ [s,\mathbf{v}] \in [0,\infty)\times R^N$$

$$\varrho(t,x) = \langle \nu_{t,x}; s\rangle,\ \mathbf{u} = \langle \nu_{t,x}; \mathbf{v}\rangle \in L^2(0,T; W^{1,2}_0(\Omega; R^N))$$

Field equations revisited

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s\rangle \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v}\rangle \cdot \nabla_x \varphi \ {\rm d}x\ {\rm d}t = \langle R_1; \nabla_x \varphi\rangle$$

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s\mathbf{v}\rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v}\otimes\mathbf{v}\rangle \cdot \nabla_x \varphi + \langle \nu_{t,x}; p(s)\rangle {\rm div}_x \varphi \ {\rm d}x\ {\rm d}t$$

$$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \ {\rm d}x\ {\rm d}t + \langle R_2; \nabla_x \varphi\rangle$$

Dissipativity

Energy inequality

$$\int_{\Omega} \left\langle \nu_{\tau,x}; \left(\frac{1}{2} s|\mathbf{v}|^2 + P(s) \right) \right\rangle \, \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t + \boxed{\mathcal{D}(\tau)}$$

$$\leq \int_{\Omega} \left\langle \nu_0; \left(\frac{1}{2} s|\mathbf{v}|^2 + P(s) \right) \right\rangle \, \mathrm{d}x$$

Compatibility

$$\left| R_1[0,\tau] \times \overline{\Omega} \right| + \left| R_2[0,\tau] \times \overline{\Omega} \right| \leq \xi(\tau)\mathcal{D}(\tau), \ \xi \in L^1(0,T)$$

$$\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; |\mathbf{v} - \mathbf{u}|^2 \right\rangle \, \mathrm{d}x \, \mathrm{d}t \leq c_P \mathcal{D}(\tau)$$

Truly measure-valued solutions

Truly measure-valued solutions for the Euler system (with E.Chiodaroli, O.Kreml, E. Wiedemann)

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded L^p weak solutions to the Euler system.

Do we need measure valued solutions?

Limits of problems with higher order viscosities

Multipolar fluids with complex rheologies (Nečas - Šilhavý)

$$\mathbb{T}(\mathbf{u}, \nabla_x \mathbf{u}, \ \nabla_x^2 \mathbf{u}, \dots)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) + \delta \sum_{j=1}^{k-1} \left((-1)^j \mu_j \Delta^j (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) + \lambda_j \Delta^j \mathrm{div}_x \mathbf{u} \ \mathbb{I} \right)$$

$$+ \text{ non-linear terms}$$

Limit for $\delta \to 0$

Limits of numerical solutions

Numerical solutions resulting from Karlsen-Karper and other schemes

Sub-critical parameters

$p(\varrho) = a\varrho^\gamma$, $\gamma < \gamma_{\mathrm{critical}}$

Weak (mv) - strong uniqueness

Theorem - EF, P.Gwiazda, A.Świerczewska-Gwiazda, E. Wiedemann [2015]

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

Relative energy (entropy)

Relative energy functional

$$\mathcal{E}\left(\varrho,\mathbf{u}\ \Big|\ r,\mathbf{U}\right)(\tau)$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v}-\mathbf{U}|^2 + P(s) - P'(r)(s-r) - P(r) \right\rangle \ \mathrm{d}x$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v}|^2 + P(s) \right\rangle \ \mathrm{d}x - \int_\Omega \langle \nu_{\tau,x}; s\mathbf{v}\rangle \cdot \mathbf{U}\ \mathrm{d}x$$

$$+ \int_\Omega \frac{1}{2} \langle \nu_{\tau,x}; s\rangle |\mathbf{U}|^2\ \mathrm{d}x$$

$$- \int_\Omega \langle \nu_{\tau,x}; s\rangle P'(r)\ \mathrm{d}x + \int_\Omega p(r)\ \mathrm{d}x$$

Relative energy (entropy) inequality

Relative energy inequality

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) + \int_0^\tau \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t + \mathcal{D}(\tau)$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}_0|^2 + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle \ \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \ \mathrm{d}t$$

Remainder

$$\mathcal{R}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right)$$

$$= - \int_0^\tau \int_\Omega \langle \nu_{t,x}, s\mathbf{v} \rangle \cdot \partial_t \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_{\overline{\Omega}} [\langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(s) \rangle \, \mathrm{div}_x \mathbf{U}] \, \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega [\langle \nu_{t,x}; s \rangle \, \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U}] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left[\left\langle \nu_{t,x}; \left(1 - \frac{s}{r}\right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \left\langle R_1; \frac{1}{2} \nabla_x \left(|\mathbf{U}|^2 - P'(r) \right) \right\rangle \ \mathrm{d}t - \int_0^\tau \langle R_2; \nabla_x \mathbf{U} \rangle \, \mathrm{d}t$$

Regularity

Theorem - EF, P.Gwiazda, A. Świerczewska-Gwiazda, E. Wiedemann

Suppose that the initial data are smooth and satisfy the relevant compatibility conditions. Let $\nu_{t,x}$ be a measure-valued solution to the compressible Navier-Stokes system with a dissipation defect $\mathcal{D}$ such that

$$\mathrm{supp}\ \nu_{t,x} \subset \left\{ (s, \mathbf{v}) \ \middle| \ 0 \leq s \leq \overline{\varrho},\ \mathbf{v} \in R^N \right\}$$

for a.a. $(t, x) \in (0, T) \times \Omega$.
Then $\mathcal{D} = 0$ and

$$\nu_{t,x} = \delta_{\varrho(t,x),\mathbf{u}(t,x)}$$

where ϱ, $\mathbf{u}$ is a smooth solution.

Sketch of the proof

- The Navier-Stokes system admits a local-in-time smooth solution
- The measure-valued solution coincides with the smooth solution on its life-span
- The smooth solution density component remains bounded by $\overline{\varrho}$ as long as the solution exists
- Y. Sun, C. Wang, and Z. Zhang [2011]: The strong solution can be extended as long as the density component remains bounded

Corollary

Convergence of numerical solutions

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution