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**Generic norms and metrics on countable
abelian groups**

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GENERIC NORMS AND METRICS ON COUNTABLE ABELIAN GROUPS

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ABSTRACT. For a countable abelian group G we investigate generic properties of the space of all invariant metrics on G . We prove that for every such an unbounded group G , i.e. group which has elements of arbitrarily high order, there is a dense set of invariant metrics on G which make G isometric to the rational Urysohn space, and a comeager set of invariant metrics such that the completion is isometric to the Urysohn space. This generalizes results of Cameron and Vershik, Niemiec, and the author.

Then we prove that for every G such that $G \cong \bigoplus_{\mathbb{N}} G$ there is a comeager set of invariant metrics on G such that all of them give rise to the same metric group after completion. If moreover G is unbounded, then using a result of Melleray and Tsankov we get that the completion is extremely amenable.

INTRODUCTION

For an unbounded countable abelian group G , J. Melleray and T. Tsankov prove in [9] that the set of all invariant metrics on G which make G extremely amenable is comeager in the Polish space of all invariant metrics on G . That result motivated the work presented in this paper. We focus on two main themes:

Groups isometric to the Urysohn space. In [2], Cameron and Vershik prove that there is an invariant metric on \mathbb{Z} which makes it isometric to the rational Urysohn space. In particular, the completion is isometric to the Urysohn space and thus the Urysohn space has a structure of a monothetic abelian group. Niemiec in [10] proves that the Shkarin's universal abelian Polish group is isometric to the Urysohn space and its canonical countable dense subgroup (which is $\bigoplus_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}$) is isometric to the rational Urysohn space. Here we generalize these

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results by proving that actually every unbounded countable abelian group, i.e. group having elements of arbitrarily high order, admits an invariant metric which makes it isometric to the rational Urysohn space. In particular, this covers the cases of \mathbb{Z} and $\bigoplus_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}$. We prove something more general stated in the theorem below.

Theorem 0.1. *For any unbounded countable abelian group G , the set of all invariant metrics which make G isometric to the rational Urysohn space is dense in the space of all invariant metrics on G . Moreover, the set of all invariant metrics on G with which the completion is isometric to the Urysohn space is dense G_{δ} .*

Generic metrics. We motivate the next topic by the following facts. We recall that in the Polish space of all countable graphs, those that are isomorphic to the random graph form a comeager subset. One cannot expect literally the same property for a metric structure, however it is true that in the Polish space of all metrics on a countable set, those whose completion is isometric to the Urysohn universal space form a comeager set. Analogously, one can show that in the Polish space of all norms on the countable infinite-dimensional vector space over \mathbb{Q} , those whose completion is isometric to the Gurarij space ([5]) form a comeager set.

Our aim is to find generic metrics on countable abelian groups. There are of course many countable abelian groups to consider. However, we can prove the following result.

Theorem 0.2. *Let G be a countable abelian group such that $G \cong \bigoplus_{\mathbb{N}} G$. Then there is a comeager set of invariant metrics on G such that all of them give rise to the same metric group after the completion.*

When the group G is moreover unbounded we get the following.

Corollary 0.3. *Let G be an unbounded countable abelian group such that $G \cong \bigoplus_{\mathbb{N}} G$. Then there is a comeager set of invariant metrics on G such that all of them give rise to the same metric group after the completion which is isometric to the Urysohn space and extremely amenable.*

We also derive the extreme amenability of a universal abelian Polish group.

Corollary 0.4. *The universal abelian Polish group of Shkarin [13] and Niemiec [10] is extremely amenable.*

1. DEFINITIONS AND PRELIMINARIES

Throughout this paper, since we shall work only with abelian groups, we adopt the additive notation.

A metric d on an abelian group G is invariant if for every $a, b, c \in G$ we have $d(a, b) = d(a + c, b + c)$. A norm on G is a function $\lambda : G \rightarrow \mathbb{R}_0^+$ from G to non-negative reals which attains zero only at $0_G \in G$ and satisfies for every $a, b \in G$, $\lambda(a) = \lambda(-a)$ and $\lambda(a + b) \leq \lambda(a) + \lambda(b)$. There is a one-to-one correspondence between invariant metrics and norms on abelian groups: for a norm λ , $d_\lambda(a, b) := \lambda(a - b)$ defines an invariant metric, and for an invariant metric d , $\lambda(a) := d(a, 0)$ defines a norm.

Consider the set of all norms, respectively invariant metrics, on some countable abelian group G . It can be viewed as a subset of \mathbb{R}^G , resp. $\mathbb{R}^{G \times G}$. In both cases, one can easily check it is a closed set, thus a Polish space (we refer the reader to [7] for facts needed about Polish spaces). It turns out it is more convenient for us to work with norms, rather than invariant metrics, so we shall do so mostly in the sequel.

Let us denote the Polish space of norms on G by \mathcal{N}_G . Later, when the group is known from the context, or it is fixed, we shall just write \mathcal{N} . Clearly, the space \mathcal{N}_G is homeomorphic with the Polish space of all invariant metrics on G via the formula above.

The content of this paper is to investigate generic properties of the space of norms on countable abelian groups. We recall one important result of Melleray and Tsankov in that direction that we shall apply in our paper.

The “official” definition of an unbounded abelian group is postponed till later when it is used. Let us say it means the group contains elements of arbitrarily high order.

Theorem 1.1 (Melleray, Tsankov, Theorem 6.4 in [9]). *Let G be a countable unbounded abelian group. Then the set $\mathbb{E} = \{\lambda \in \mathcal{N}_G : (G, \lambda) \text{ is extremely amenable}\}$ is dense G_δ in \mathcal{N}_G .*

We note that Melleray and Tsankov formulated the theorem with invariant metrics rather than norms which is, as noted above, however equivalent.

Definition 1.2. Let G be an abelian group and $A \subseteq G$ a symmetric subset, i.e. $A = -A$, containing zero. A *partial norm* on A is a function $\lambda : A \rightarrow \mathbb{R}_0^+$ satisfying the following requirements:

- $\lambda(x) = 0$ iff $x = 0$, for $x \in A$,
- $\lambda(x) = \lambda(-x)$ for $x \in A$,

- $\lambda(x) \leq \sum_{i=1}^n \lambda(x_i)$, where n is arbitrary, $x, x_1, \dots, x_n \in A$ and $x = \sum_{i=1}^n x_i$.

If λ satisfies all the conditions except the first one, then it is called a partial *seminorm*. If it satisfies the first and the second condition we call it a partial pre-norm.

Lemma 1.3. *Let G be an abelian group, $A \subseteq G$ some symmetric subset containing zero and $\lambda_A : A \rightarrow \mathbb{R}_0^+$ a partial norm on A . Then for any subset B with $A \subseteq B \subseteq \langle A \rangle \leq G$, where $\langle A \rangle$ is the subgroup of G generated by A , there exists a partial seminorm $\lambda_B : B \rightarrow \mathbb{R}_0^+$ which extends λ_A . In particular, if A is finite and $B = \langle A \rangle = G$, then λ_B is a norm on G extending λ_A .*

Moreover, if A is finite and λ_A is rational-valued, then so is λ_B .

Proof. Take $b \in B$ and set

$$\lambda_B(b) = \inf \left\{ \sum_{i=1}^n \lambda_A(a_i) : (a_i)_{i=1}^n \subseteq A, b = \sum_{i=1}^n a_i \right\}.$$

Since $B \subseteq \langle A \rangle$, there exist $(a_i)_{i=1}^n \subseteq A$ such that $b = \sum_{i=1}^n a_i$. It directly follows from the definition that λ_B satisfies all the conditions of a partial seminorm and extends λ_A . Moreover, if A is finite, then the infimum from the definition of λ_B might be replaced by the minimum. It follows that λ_B in that case is a partial norm and if λ_A was rational-valued, so is now λ_B . \square

Remark 1.4. Notice that λ_B is actually the greatest extension of λ_A to a partial seminorm on B ; i.e. if $\lambda : B \rightarrow \mathbb{R}_0^+$ is any partial seminorm on B that extends λ_A , then $\lambda \leq \lambda_B$; and if A is finite, then λ_B is the greatest partial norm extending λ_A .

When we are given a symmetric subset $A \subset G$ containing zero of some abelian group and also some pre-norm $\rho' : A \rightarrow \mathbb{R}_0^+$ then we can get, using the same formula as in the proof above, a greatest partial seminorm determined by ρ' ; or, in the case A is finite, a greatest partial norm determined by ρ' . We state that explicitly in the next fact and omit its easy proof.

Fact 1.5. *Let $A \subseteq G$ be a symmetric subset containing zero of some abelian group. Let $\rho' : A \rightarrow \mathbb{R}_0^+$ be a pre-norm. Then the formula, applied for every $x \in A$,*

$$\rho(x) = \inf \left\{ \sum_{i=1}^n \rho'(x_i) : x = \sum_{i=1}^n x_i, (x_i)_{i=1}^n \subseteq A \right\}$$

gives a partial seminorm on A , resp. partial norm on A if A is finite.

In the text below we shall be interested in the possibilities how to extend partial norms. We recall the notion of a Katětov function on a metric space which corresponds, in the terminology of continuous model theory, to the quantifier-free type over a metric space. Let X be a metric space and $f : X \rightarrow \mathbb{R}_0^+$ a function. It is called *Katětov* if it satisfies for every $x, y \in X$

$$|f(x) - f(y)| \leq d_X(x, y) \leq f(x) + f(y).$$

A Katětov function f without zeros can be then viewed as a prescription of distances of a new point to the points of X in the sense that we can define a one-point extension $X \cup \{x_f\}$ and define the distance of x_f to a point $y \in X$ as $f(y)$.

We start with some algebraic definition of elements in abelian groups that have the potential to realize Katětov functions. Fix G an abelian group and $A \subseteq G$ a subset. We recall that *an oriented Cayley graph* C_G^A of G with respect to the set A is an oriented graph such that

- the set of vertices is G ;
- the set of oriented edges is the set $\{(g, g + a) : g \in G, a \in A\}$.

An *oriented path* from $g \in G$ to $h \in G$ is a sequence $g = x_0, \dots, x_n = h$ from G such that for every $0 \leq i < n$, (x_i, x_{i+1}) is an oriented edge. In other words, there are elements $a_1, \dots, a_n \in A$ such that $x_i = x_0 + \sum_{j=1}^i a_j$, for $1 \leq j \leq n$. The length of this path is n .

Definition 1.6. Let now G be again an abelian group and $A \subseteq G$ a symmetric subset (containing zero). Let $g \in G \setminus A$. We define the *distance* of g from A in G , denoted by $\text{dist}_G(g, A)$, as the length of the shortest oriented path between g and A in $C_G^{A \cup \{g\}}$.

Now let G be an abelian group, $A \subseteq G$ a finite symmetric subset containing zero and λ_A a partial norm on A . Set $\bar{A} = \{a - b : a, b \in A\} = A - A = A + A$. Note that $A \subseteq \bar{A}$ and that \bar{A} is again a finite symmetric subset containing zero. Let $\bar{\lambda}_A$ be the greatest extension of λ_A onto \bar{A} guaranteed by Lemma 1.3, and note that it induces a metric d_A on A defined as $d_A(a, b) = \bar{\lambda}_A(a - b)$ for $a, b \in A$. Let $f : A \rightarrow \mathbb{R}^+$ be a Katětov function with respect to the metric d_A . We are now interested whether it is possible to find an element $g \in G \setminus A$ and a partial norm λ on $\{g - a, a - g : a \in A\} \cup \bar{A}$ which extends $\bar{\lambda}_A$ and such that for every $a \in A$ we have $f(a) = \lambda(g - a)$.

Set $m = \min\{\min \bar{\lambda}_A(\bar{A} \setminus \{0\}), \min f(A)\}$ and $M = \max\{\max \bar{\lambda}_A(\bar{A}), \max f(A)\}$. Then we have the following proposition.

Proposition 1.7. *Under the setting above, suppose that there exists $g \in G \setminus \bar{A}$ such that $\text{dist}(g, \bar{A}) > 2\frac{M}{m}$. Then there exists a partial norm λ on $\{g - a, a - g : a \in A\} \cup \bar{A}$ which extends $\bar{\lambda}_A$ and such that for every $a \in A$ we have $f(a) = \lambda(g - a)$.*

Proof. Set $B = \bar{A} \cup \{g - a, a - g : a \in A\}$. Note that B is also a symmetric set containing zero. Define λ on B as follows: for any $b \in B$ set

$$\lambda(b) = \begin{cases} \bar{\lambda}_A(b) & b \in \bar{A}, \\ f(x) & \text{if } b = g - x, \text{ or } b = x - g. \end{cases}$$

Notice that there is no collision in the definition above since the sets \bar{A} and $\{g - a, a - g : a \in A\}$ are disjoint. Indeed, if for some $a \in A$ and $\bar{a} \in \bar{A}$ we have $g - a = \bar{a}$, then there is an oriented edge between g and \bar{a} in $C_G^{\bar{A} \cup \{g\}}$, and that is a contradiction with $\text{dist}(g, \bar{A}) > 2\frac{M}{m} \geq 2$.

Now it suffices to check that λ is a partial norm on B . The first two conditions of the definition of a partial norm are easily checked. We claim that also the last condition is satisfied. Suppose otherwise. Then there are $b \in B$ and $(b_i)_{i=1}^n \subseteq B \setminus \{0\}$ such that $b = \sum_{i=1}^n b_i$ and $\lambda(b) > \sum_{i=1}^n \lambda(b_i)$. We have that $n < \frac{M}{m}$ because $M \geq \lambda(b) > n \cdot m$.

Moreover, without loss of generality we may also suppose that for no $i \neq j \leq n$ we have $b_i = g - a_i$ and $b_j = a_j - g$, for some $a_i, a_j \in A$. Indeed, suppose otherwise. Then since $g - a_i + a_j - g = a_j - a_i \in \bar{A}$ and the function f is Katětov we have that $\lambda(g - a_i) + \lambda(a_j - g) = f(a_i) + f(a_j) \geq d_A(a_j, a_i) = \bar{\lambda}_A(a_j - a_i)$. Thus we can replace the pair b_i, b_j by $a_j - a_i$.

Case 1: $b \in \bar{A}$. We claim that there must be some $i \leq n$ such that b_i is equal to $g - a$ or $a - g$ for some $a \in A$. Indeed, otherwise we get into a contradiction since $\bar{\lambda}_A$ is a partial norm. By the argument in the paragraph above there is a single sign $\varepsilon \in \{1, -1\}$ such that for every $b_i \notin \bar{A}$ we have $b_i = \varepsilon \cdot g - \varepsilon \cdot a_i$ for some $a_i \in A$. Suppose that $\varepsilon = 1$, the other case is analogous. It follows that there is some $0 < k \leq n$ such that $b = k \cdot g + \sum_{i=1}^n c_i$, where $(c_i)_{i=1}^n \subseteq \bar{A}$. Thus there is an oriented path from g to b in $C_G^{\bar{A} \cup \{g\}}$ of length $k - 1 + n < 2n$ which contradicts that $\text{dist}(g, \bar{A}) > 2\frac{M}{m} > 2n$. Indeed, set $e_0 = 0$ and $e_i = g$, for $1 \leq i \leq k - 1$, and $e_j = c_{j-k+1}$, for $k \leq j \leq k - 1 + n$. Then $(x_i = g + \sum_{j=0}^i e_j)_{i=0}^{k-1+n}$ is the desired path.

Case 2: $b = g - a$, or $b = a - g$, for some $a \in A$. Let us say $b = g - a$, the other case is analogous. Suppose at first that for all $i \leq n$ we have $b_i \in \bar{A}$. Since $g - \sum_{i=1}^n b_i = a$, if we set $x_0 = g$ and $x_j = g - \sum_{i=1}^j b_i$,

for $1 \leq j \leq n$, we get that x_0, \dots, x_n is an oriented path from g to a in $C_G^{\bar{A} \cup \{g\}}$ which is again a contradiction with $\text{dist}(g, \bar{A}) > 2\frac{M}{m}$.

Thus we suppose that for some $i \leq n$ we have $b_i \notin \bar{A}$. As in Case 1 we may suppose that there is a single sign $\varepsilon \in \{1, -1\}$ such that for every $b_i \notin \bar{A}$ we have $b_i = \varepsilon \cdot g - \varepsilon \cdot a_i$ for some $a_i \in A$.

Suppose at first that $\varepsilon = -1$. Then there is some $1 < k \leq n+1$ such that $k \cdot g = \sum_{i=1}^n c_i$, where $(c_i)_{i=1}^n \subseteq \bar{A}$. Thus we get an oriented path of length $k-1+n-1 < 2n$ from g to c_n in $C_G^{\bar{A} \cup \{g\}}$ which again contradicts that $\text{dist}(g, \bar{A}) > 2\frac{M}{m}$.

Now suppose that $\varepsilon = 1$. Suppose without loss of generality that $b_1 = g - a_1$, for some $a_1 \in A$. Then, since $n > 1$, $\sum_{i=2}^n b_i = a_1 - a \in \bar{A}$. If there are indices $2 \leq i \leq n$ such that $b_i = g - a_i$, for some $a_i \in A$, then there is some $0 < k \leq n-1$ so that $a_1 - a = k \cdot g + \sum_{i=1}^{n-1} c_i$, where $(c_i)_{i=1}^{n-1} \subseteq \bar{A}$. As above, this again leads to a contradiction with $\text{dist}(g, \bar{A}) > 2\frac{M}{m}$.

Thus suppose that for every $2 \leq i \leq n$ we have $b_i \in \bar{A}$. Then we claim that we may suppose that $n = 2$. Indeed, we have that $\sum_{i=2}^n b_i = a_1 - a \in \bar{A}$ and thus $\bar{\lambda}_A(a_1 - a) \leq \sum_{i=2}^n \bar{\lambda}_A(b_i)$. So we are left with the case that

$$\begin{aligned} \lambda(g - a) = f(a) &> \lambda(g - a_1) + \lambda(a_1 - a) = f(a_1) + \bar{\lambda}_A(a_1 - a) = \\ &f(a_1) + d_A(a_1, a), \end{aligned}$$

which is again a contradiction with the fact that f is Katětov. \square

Definition 1.8. We shall call an abelian group G unbounded if it either contains an element of infinite order or it contains elements of arbitrarily high finite orders.

Lemma 1.9. *Let G be an unbounded abelian group, $A \subseteq G$ a finite symmetric subset containing zero and $R > 0$ a real number. Then there exists $g \in G$ such that $\text{dist}(g, A) > R$.*

Proof. Set $B = \{n_1 \cdot a_1 + \dots + n_i \cdot a_i : a_1, \dots, a_i \in A, n_1, \dots, n_i \geq 0, n_1 + \dots + n_i \leq R\}$. Note that B is again finite symmetric and containing zero. It suffices to show that there is $g \in G$ such that $n \cdot g \notin B$, for every $0 < n \leq R$. Indeed, suppose we have found such $g \in G$, yet still $\text{dist}(g, A) \leq R$. Then there exists a sequence $(c_i)_{i=1}^n \subseteq A \cup \{g\}$, where $n \leq R$, such that $g + \sum_{i=1}^n c_i \in A$. Without loss of generality, we may suppose that there is some $1 \leq j \leq n+1$ such that $c_i = g$ if and only if $i < j$. Then $b = \sum_{i=j}^n c_i \in B$ and $j \cdot g = -b$, a contradiction.

If there is $b \in B$ such that b has infinite order, then it clearly suffices to take $N \cdot b$, for N sufficiently large, as g . So suppose that every $b \in B$

has finite order. Let N be the maximum of orders of elements from B . Suppose there is no such $g \in G$, thus for every $g \in G$ there are $n \leq R$ and $b \in B$ such that $n \cdot g = b$. However, then the order of every $g \in G$ is bounded by $R \cdot N$. That is a contradiction with unboundedness of G . \square

2. GROUPS ISOMETRIC TO THE RATIONAL URYSOHN SPACE

We recall here that the Urysohn universal metric space \mathbb{U} is the unique Polish metric space (i.e. complete and separable) containing isometrically every separable metric space (containment of every finite metric space is enough) and satisfying the property that a partial isometry between two finite subsets extends to an autoisometry of the whole space. It was constructed by Urysohn in [14]. We refer to Chapter 5 in [12] for more information about this space.

The Urysohn space has one distinguished countable dense set which is called *the rational Urysohn space* and denoted by \mathbb{QU} . It is the unique countable metric space with rational distances that contains isometrically every finite rational metric space and again has the property that any partial isometry between two finite subsets extends to an autoisometry of the whole space. The following well-known fact gives another characterization of \mathbb{QU} which we will use.

Fact 2.1. *Let X be a countable metric space with rational distances. Then X is isometric to \mathbb{QU} iff for every finite subset $F \subseteq X$ and every rational Katětov function $f : F \rightarrow \mathbb{Q}$ there exists $x \in X$ realizing f , i.e. $\forall y \in F (d(x, y) = f(y))$.*

Lemma 2.2. *Let G be an abelian group, $F \subseteq G$ a finite symmetric subset containing zero and let ρ be a partial norm on F . Then for any $\varepsilon > 0$ there exists a rational partial norm ρ_R on F such that for any $f \in F$ we have $|\rho(f) - \rho_R(f)| < \varepsilon$.*

Proof. Enumerate F as $(f_i)_{i=1}^n$ such that for $i < j \leq n$ we have $\rho(f_i) \geq \rho(f_j)$, and $f_n = 0$. Let $\delta = \min\{\varepsilon, \min\{|\rho(f) - \rho(g)| : f, g \in F, \rho(f) \neq \rho(g)\}\}$. Moreover, choose an (not necessarily strictly) increasing sequence of positive real numbers $(r_i)_{i=1}^{n-1}$ such that

- for any $i, j < n$ if $\rho(f_i) = \rho(f_j)$, then $r_i = r_j$;
- for any $i < n$ we have $r_i < \delta$ and $\rho(f_i) + r_i \in \mathbb{Q}$.

Now for $i < n$ we define $\rho_R(f_i) = \rho(f_i) + r_i$, and $\rho_R(f_n) = 0$. We claim ρ_R is as desired. Clearly, it is rational and for any $i \leq n$ we have $\varepsilon > r_i = \rho_R(f_i) - \rho(f_i) \geq 0$. If we check that ρ_R is a partial norm then we will be done.

First, for any $i \leq n$ we have $\rho_R(f_i) = 0$ if and only if $f_i = f_n = 0$, and also for any $i < j < n$, if $f_i = -f_j$, then $\rho_R(f_i) = \rho_R(f_j)$. So it remains to check that for any $i_1, \dots, i_k, i < n$ such that $f_i = f_{i_1} + \dots + f_{i_k}$ we have $\rho_R(f_i) \leq \rho_R(f_{i_1}) + \dots + \rho_R(f_{i_k})$. Notice that we have $\rho_R(f_1) \geq \dots \geq \rho_R(f_n) = 0$ since for $j < l \leq n$, $\rho(f_j) = \rho(f_l)$ if and only if $\rho_R(f_j) = \rho_R(f_l)$, and otherwise we have $\rho(f_j) > \rho(f_l)$, so $\rho_R(f_l) = \rho(f_l) + r_l \leq \rho(f_j) < \rho_R(f_j)$. So if for some $j \leq k$ we have that $i_j \leq i$, then $\rho_R(f_i) \leq \rho_R(f_{i_j})$ and we are done. Otherwise, for all $j \leq k$ we have $i < i_j$. Then since $\rho_R(f_i) - \rho(f_i) = r_1 \leq r_{i_1} = \rho_R(f_{i_1}) - \rho(f_{i_1})$, so

$$\rho_R(f_i) = \rho(f_i) + r_i \leq r_{i_1} + \sum_{j=1}^k \rho(f_{i_j}) \leq \sum_{j=1}^k \rho_R(f_{i_j}).$$

□

Theorem 2.3. *Let G be a countable unbounded abelian group. Then there exists a norm λ on G such that (G, λ) is isometric to the rational Urysohn space.*

Moreover, the set of all norms on G which make G isometric to the rational Urysohn space is dense.

Proof. The proof uses Lemmas 1.9 and 1.3, and Proposition 1.7 and follows the standard construction of the rational Urysohn space. Enumerate G as $\{g_n : n \in \mathbb{N}\}$ and let $\{(A_i, f_i) : i \in \mathbb{N}\}$ be an enumeration with infinite repetition of all pairs (A, f) , where A is a finite rational metric space and $f : A \rightarrow \mathbb{Q}$ a rational Katětov function over A .

By induction, we shall produce finite symmetric sets F_i , $i \in \mathbb{N}_0$, containing zero, with partial rational norm λ_i on F_i such that for every n , $F_n \subseteq F_{n+1}$ and $\lambda_n \subseteq \lambda_{n+1}$, $G = \bigcup_n F_n$, and such that G with the metric induced by the norm $\lambda = \bigcup_n \lambda_n$ is isometric to the rational Urysohn space.

Set $F_0 = \{0\}$ and let λ_0 be the trivial norm on F_0 . Now suppose that for some even n , F_n and λ_n on F_n have been defined. We define F_{n+1} and λ_{n+1} . Take the element $g = g_{n/2+1}$. If $g \in F_n$ then we do nothing, i.e. set $F_{n+1} = F_n$ and $\lambda_{n+1} = \lambda_n$. So suppose that $g \notin F_n$. We set $F_{n+1} = F_n \cup \{g, -g\}$. We need to extend λ_n . We distinguish cases:

- $g \in \langle F_n \rangle$: then we use Lemma 1.3 to extend the partial norm λ_n on F_n to a partial norm λ_{n+1} on F_{n+1} ,
- $g \notin \langle F_n \rangle$: then we can set
 - (1) $\lambda_{n+1}(a) = \lambda_n(a)$ if $a \in F_n$,

- (2) let m be the minimal positive integer such that for some $f \in F_n \setminus \{0\}$, $m \cdot g = f$; if no such m exists (then in particular, g has an infinite order), or $f = 0$, then we set $\lambda_{n+1}(g) = \lambda_{n+1}(-g) = 1$; otherwise, we set $\lambda_{n+1}(g) = \lambda_{n+1}(-g) = \lambda_n(f)/m$.

In any case, it is easy to check that λ_{n+1} is a partial norm on F_{n+1} .

Now suppose that for some odd n , F_n and λ_n on F_n have been defined. Set $G_n = \bar{F}_n = \{a - b : a, b \in F_n\}$ and extend λ_n to ρ_n on G_n by Lemma 1.3. Then ρ_n induces a metric d_n on F_n , as usual, by $d_n(a, b) = \rho_n(a - b)$, for $a, b \in F_n$.

Set $(A, f) = (A_{\frac{n+1}{2}}, f_{\frac{n+1}{2}})$. If there is no subset of (F_n, d_n) isometric to A then do nothing and set $F_{n+1} = F_n$ and $\lambda_{n+1} = \lambda_n$. Otherwise, take some subset $B \subseteq F_n$ isometric to A and consider f to be defined on $B \cong_{\text{iso}} A$. We can clearly extend f to the whole F_n , still denoted by f , so that it is still Katětov and rational-valued. Just set for instance $f(x) = \min\{f(a) + d_n(a, x) : a \in B\}$, for $x \in F_n$.

Set $m = \min\{\min \rho_n(G_n \setminus \{0\}), \min f(F_n)\}$ and $M = \max\{\max \rho_n(G_n), \max f(F_n)\}$. By Lemma 1.9 we can find some element $g \in G$ such that $\text{dist}(g, G_n) > 2\frac{M}{m}$. Then by Proposition 1.7 we can extend G_n to $F'_{n+1} = G_n \cup \{g - a, a - g : a \in F_n\}$ and ρ_n to a partial norm λ'_{n+1} on F'_{n+1} such that $f(x) = \lambda'_{n+1}(g - x)$ for every $x \in F_n$. If there was a unique isometric embedding of A into (F_n, d_n) then we are done. Otherwise, we consecutively repeat the above procedure for all other isometric embeddings of A into F_n enlarging F'_{n+1} to F''_{n+1} , then to F'''_{n+1} , etc. The last obtained set is F_{n+1} .

When the induction is finished we have that $G = \bigcup_n F_n$ since at the n -th step, for even n , we have guaranteed that $g_{n/2+1}$ is contained in F_{n+1} . Moreover, G with the metric induced by the norm $\lambda = \bigcup_n \lambda_n$ is isometric to the rational Urysohn space. By Fact 2.1 it suffices to check it satisfies the rational one-point extension property. However if we take some finite $B \subseteq G$ and one-point extension determined by a rational Katětov function f on B , then we can find n such that $B \subseteq F_{2n-1}$, B is isometric to A_n and f on B corresponds to f_n on A_n . Then we have guaranteed that the Katětov function is realized in F_{2n} .

Finally, we show how to get the “moreover part” from the statement of the theorem, i.e. that the set of norms with which G is isometric to the rational Urysohn space is actually dense.

Take any finite symmetric subset $F \subseteq G$ containing zero, let ρ be an arbitrary partial norm on F and $\varepsilon > 0$ arbitrary. Then using Lemma 2.2 we get a partial rational norm ρ_R on F such that for every $f \in F$ we have $|\rho(f) - \rho_R(f)| < \varepsilon$. We just set $F_0 = F$ and $\lambda_0 = \rho_R$. Then we continue the induction as above and obtain at the end a norm λ with which G is isometric to the rational Urysohn space and is ε -close on a finite subset F to the partial norm ρ . \square

Corollary 2.4. *If G is unbounded, then the set of norms $\{\lambda : \overline{(G, \lambda)} \text{ is isometric to } \mathbb{U}\}$ is comeager.*

Proof. Consider the set of norms λ on G satisfying the following condition:

$$(2.1) \quad \forall \varepsilon > 0 \forall F \subseteq G \text{ finite symmetric and containing zero,} \\ \forall f : F \rightarrow \mathbb{Q} \text{ Katětov with respect to } \bar{F} \\ \exists g \in G \forall a \in F (|f(a) - \lambda(g - a)| < \varepsilon).$$

It is well-known and straightforward to prove using standard arguments that for any λ satisfying (2.1) we have that the completion $\overline{(G, \lambda)}$ is isometric to the Urysohn space.

Moreover, an immediate computation gives that (2.1) is a G_δ condition. Any norm λ with which G is isometric to the rational Urysohn space certainly satisfies (2.1), thus it follow from Theorem 2.3 that the condition defines a dense G_δ set. \square

The proofs of Theorem 2.3 and Corollary 2.4 also conclude the proof of Theorem 0.1.

Remark 2.5. We note that there is one example in the literature of a countable abelian group which admits a norm which make it isometric to the rational Urysohn space, yet it is not unbounded. It is the countable Boolean group (see [11]). The case of exponent 2 is obviously special and it is open whether other bounded countable abelian groups admit such a norm (see the open problems in [10], where it is proved that groups of exponent 3 do not admit such a norm). We conjecture that they do not.

3. GENERIC NORMS

For a norm λ on G denote by $\overline{(G, \lambda)}$ the completion. We shall call a norm λ on a countable abelian group G *generic* if the set $\{\rho \in \mathcal{N}_G : \overline{(G, \lambda)} \cong \overline{(G, \rho)}\}$ is comeager. In other words, a countable abelian group G admits a generic norm if all the norms on G except those coming from a meager set give rise to the same normed group after the

completion. It follows from Theorem 1.1 that if λ is a generic norm on a countable unbounded abelian group G , then (G, λ) is extremely amenable.

Let us start with the following easy to check and well known observation. If (G, λ) is a normed abelian group and $g \in G$ then $\lim_{n \rightarrow \infty} \frac{\lambda(n \cdot g)}{n}$ exists and is equal to $\inf_n \frac{\lambda(n \cdot g)}{n}$. Following Niemiec in [10], by \mathcal{O}_0 we denote the class of those abelian normed groups (G, λ) such that for all $g \in G$, $\lim_n \frac{\lambda(n \cdot g)}{n} = 0$. The next lemma shows that if there is a generic norm λ on G , then necessarily $(G, \lambda) \in \mathcal{O}_0$.

Lemma 3.1. *For every countable abelian group G the set $N_0 \subseteq \mathcal{N}_G$ of those norms λ on G such that $(G, \lambda) \in \mathcal{O}_0$ is dense G_δ .*

Proof. First we check that N_0 is G_δ . We have $\lambda \in N_0$ if and only if for all $g \in G$ and every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{\lambda(n \cdot g)}{n} < \varepsilon$, which is certainly a G_δ condition.

To show that it is dense it suffices to show that the set of bounded norms is dense, since bounded norms clearly belong to N_0 . That is however immediate. Take some basic open neighborhood of some λ' given by a finite set $A \subseteq G$ and some $\varepsilon > 0$ (the ε will be however irrelevant). Let $M = \max\{\lambda'(g) : g \in A\} + 1$ and define a norm λ as $\min\{\lambda', M\}$. \square

From now on, we shall call norms λ (on some G) such that $(G, \lambda) \in \mathcal{O}_0$, N_0 -norms.

Definition 3.2. Let G be a countable abelian group. We call G *infinitely-summed* if $G \cong \bigoplus_{n \in \mathbb{N}} G$.

In particular, notice that if non-trivial G is infinitely-summed then it is not finitely generated.

Let G be a non-trivial infinitely-summed countable abelian group. Write G as $\bigoplus_n G_n$, where for each $n \in \mathbb{N}$, $G_n \cong G$. For each $i \in \mathbb{N}$, let $\phi_i : G_1 \rightarrow G_i$ be an isomorphism. Fix some infinite sequence $(d'_n)_n$ of G_1 such that for every n , $d'_{n+1} \notin \langle d'_i : i \leq n \rangle$, and $G_1 = \langle d'_n : n \in \mathbb{N} \rangle$. Let $D = \{\phi_i(d'_n) : i, n \in \mathbb{N}\}$, i.e. D generates G . Fix also an enumeration $\{d_n : n \in \mathbb{N}\}$ of D . For every set of integers (finite in all cases in the sequel) $A \subseteq \mathbb{N}$, let $F_A \leq G$ be the subgroup $\langle d_i : i \in A \rangle$. Note that for any finite $A \subseteq \mathbb{N}$ there exists finite $A' \subseteq \mathbb{N}$ disjoint from A and a bijection $\phi : A \rightarrow A'$ which uniquely determines an isomorphism between $\bar{\phi} : F_A \rightarrow F_{A'}$ determined by sending d_i to $d_{\phi(i)}$ for $i \in A$. Let Φ be the set of all isomorphisms between finitely generated subgroups of the form F_A and $F_{A'}$ which are determined by some bijection between

A and A' . Obviously, for most choices of generating sets D , not all bijections between two finite subsets $A, A' \subseteq \mathbb{N}$ of the same size give rise to isomorphisms between F_A and $F_{A'}$, which do not have to be isomorphic at all. We write $A \equiv A'$ if there does exist a bijection between A and A' which gives rise to an isomorphism between F_A and $F_{A'}$. If we want to specify the isomorphism, we write $A \equiv_\phi A'$, where ϕ is the bijection between A and A' and $\bar{\phi}$ the corresponding isomorphism between F_A and $F_{A'}$.

Also, for each finite $A \subseteq \mathbb{N}$, let $|\cdot|_A : F_A \rightarrow \mathbb{N}$ be the length function (i.e. norm) associated to the generating set $\{d_i, -d_i : i \in A\}$, i.e. the distance from 0 in the graph metric of the Cayley graph of F_A with $\{d_i, -d_i : i \in A\}$ as a generating set. If there is no danger of confusion then we write just $|\cdot|$ instead of $|\cdot|_A$.

The next definition introduces a certain distance between two norms ρ , resp. ρ' on F_A , resp. $F_{A'}$, where $A \equiv A'$.

Definition 3.3. Suppose that $A, A' \subseteq \mathbb{N}$ are two finite subsets such that $A \equiv_\phi A'$ for some $\phi \in \Phi$. Suppose also that F_A is equipped with a norm ρ and $F_{A'}$ with a norm ρ' . Then we say that (A, ρ) and (A', ρ') are ϕ, ε -close and write $(A, \rho) \equiv_{\phi, \varepsilon} (A', \rho')$ if

$$\sup_{f \in F_A \setminus \{0\}} \frac{|\rho(f) - \rho'(\bar{\phi}(f))|}{|f|_A} < \varepsilon.$$

Note that if $(A, \rho) \equiv_{\phi, \varepsilon} (A', \rho')$ and $(A', \rho') \equiv_{\psi, \delta} (A'', \rho'')$, then $(A, \rho) \equiv_{\psi \circ \phi, \varepsilon + \delta} (A'', \rho'')$.

We shall also need a notion of ‘closeness’ of two subgroups $F_A, F_{A'} \leq G$ with respect to some norm λ on G and some $\phi \in \Phi$ such that $A \equiv_\phi A'$.

Definition 3.4. Suppose that G is equipped with a norm λ . Let A, A' be two finite subsets of naturals such that for some $\phi \in \Phi$ we have $A \equiv_\phi A'$ and let $\varepsilon > 0$. Then we write $\delta_\phi^\lambda(A, A') < \varepsilon$ if for every $a \in A$ we have $\lambda(d_a - \bar{\phi}(d_a)) = \lambda(d_a - d_{\phi(a)}) < \varepsilon$. Notice that this is equivalent to saying that for every non-zero $f \in F_A$ we have $\lambda(f - \bar{\phi}(f)) < \varepsilon \cdot |f|$. Again, we shall suppress the upper index λ from δ_ϕ^λ when it is clear from the context.

Next lemma shows that the condition $(A, \rho) \equiv_{\phi, \varepsilon} (A', \rho')$ is determined on finite sets for N_0 -norms.

Lemma 3.5. *Let $A, A' \subseteq \mathbb{N}$ be two finite subsets such that $A \equiv_\phi A'$. Then for any N_0 -norm ρ on F_A and any $\varepsilon > 0$ there exist a finite subset $C \subseteq F_A$ and $\delta > 0$ such that for any N_0 -norm ρ' on $F_{A'}$, if for every $a \in C$ we have $|\rho(a) - \rho'(\bar{\phi}(a))| < \delta$, then $(A, \rho) \equiv_{\phi, \varepsilon} (A', \rho')$.*

Proof. To simplify the notation, we shall assume that ϕ is the identity. Therefore we look for a finite subset $C \subseteq F_A$ and $\delta > 0$ such that for any N_0 -norm ρ' on F_A , if for every $a \in C$ we have $|\rho(a) - \rho'(a)| < \delta$, then $\frac{|\rho(x) - \rho'(x)|}{|x|} < \varepsilon$, for every $x \in F$.

Since F_A is a finitely generated abelian group it is isomorphic to a sum of a finitely generated free abelian group F and a finite abelian group K . Suppose first that F_A is F , i.e. it is a finitely generated free abelian group, and moreover suppose that the generators $(d_i)_{i \in A}$ are the generators. Any $x \in F_A$ can be thus uniquely written as $\sum_{i \in A} k_i \cdot d_i$. For $i \in A$ denote by $|x|_i$ the absolute value of k_i . Now for any $n \in \mathbb{N}$ set $A'_n = \{x \in F_A : \forall i \in A (|x|_i \leq n)\}$ and $A_n = \{x \in F_A : |x| \leq n\}$.

We also set (see Claim 3.3 in [3]), for any $k \in \mathbb{N}$

$$B'_k = \sup_{x \in F_A} \min \left\{ \frac{|x - l \cdot y|}{|x|} : l \in \mathbb{N}, y \in A'_k, |l \cdot y| \leq 2|x| \right\}$$

and

$$B_k = \sup_{x \in F_A} \min \left\{ \frac{|x - l \cdot y|}{|x|} : l \in \mathbb{N}, y \in A_k, |l \cdot y| \leq 2|x| \right\}.$$

Claim 3.6. *There exists K and $\delta > 0$ such that for every N_0 -norm ρ' on F_A , if $|\rho(x) - \rho'(x)| < \delta$, for every $x \in A_K$, then for every $x \in F_A$ such that $|x| > K$ we have $\frac{\rho'(x)}{|x|} < \varepsilon/2$.*

Once the claim is proved we are done (with the case when F_A is a free abelian group with free generators). Indeed, use $\delta > 0$ from the previous claim and set as C the set A_K . Then if ρ' is a N_0 -norm on F_A which is δ -close to ρ on C , then for any $x \in F_A \setminus \{0\}$

- if $x \in C$, then trivially $\frac{|\rho(x) - \rho'(x)|}{|x|} < \varepsilon$ as we may assume that $\delta < \varepsilon$;
- if $|x| > K$, then by the claim we have $\frac{\rho(x)}{|x|} < \varepsilon/2$ and $\frac{\rho'(x)}{|x|} < \varepsilon/2$, so $\frac{|\rho(x) - \rho'(x)|}{|x|} \leq \frac{\rho(x) + \rho'(x)}{|x|} < \varepsilon$.

Proof of the claim. Set $M = \max\{\rho(d_i) : i \in A\}$. By Claim 3.3 from [3] there exists $k' \in \mathbb{N}$ so that $B'_k < \varepsilon/(4M + \varepsilon/8)$. Since for every k' there is k so that we have $A_{k'} \subseteq A'_{k'} \subseteq A_k$ we get that $B_k < \varepsilon/(4M + \varepsilon/8)$.

Now let $K' > k$ be such that for every $y \in A_k$ we have

$$\frac{\rho(m \cdot y)}{m} < \varepsilon/32$$

for every $m \geq K'$, which is possible since ρ is an N_0 -norm. Finally, let $K > 2K' \cdot k$ be such that for every $x \in A_{K' \cdot k}$ we have $\frac{\rho(x)}{K} < \varepsilon/16$.

We claim that K and $\delta = \varepsilon/32$ are as desired. Take any N_0 -norm ρ' on F_A which is $\varepsilon/32$ -close to ρ on A_K . Take any $x \notin A_K$. We may find some $y \in A_k$ and l , such that $l|y| \leq 2|x|$, so that $\frac{|x-l \cdot y|}{|x|} < \varepsilon/(4M + \varepsilon/8)$. We claim that $l \geq K'$ since otherwise $|l \cdot y| < K' \cdot k$, so $|x - l \cdot y| \geq |x|/2$, so $\frac{|x-l \cdot y|}{|x|} > 1/2$.

We have

$$\frac{\rho'(x)}{|x|} \leq \frac{\rho'(l \cdot y) + \rho'(x - l \cdot y)}{|x|}$$

where

$$\frac{\rho'(x - l \cdot y)}{|x|} \leq \frac{(M + \varepsilon/32)|x - l \cdot y|}{|x|} < (M + \varepsilon/32)\varepsilon/(4M + \varepsilon/8) = \varepsilon/4.$$

So it suffices to show that $\frac{\rho'(l \cdot y)}{|x|} < \varepsilon/4$. Write l as $tK' + r$, where $r < K'$. Note that $t \geq 1$. Then we have

$$\frac{\rho'(l \cdot y)}{|x|} \leq \frac{t \cdot \rho'(K' \cdot y) + \rho'(r \cdot y)}{|x|} \leq \frac{2t \cdot \rho'(K' \cdot y)}{t \cdot |K' \cdot y|} + \frac{\rho'(r \cdot y)}{K}.$$

Note that

$$\frac{2t \cdot \rho'(K' \cdot y)}{t \cdot |K' \cdot y|} \leq 2 \frac{\rho'(K' \cdot y)}{K'} \leq 2 \frac{\rho(K' \cdot y) + \varepsilon/32}{K'} < 2(\varepsilon/32 + \varepsilon/32) = \varepsilon/8$$

by the definition of K' and since ρ' is $\varepsilon/32$ -close to ρ on A_K , and

$$\frac{\rho'(r \cdot y)}{K} \leq \frac{\rho(r \cdot y) + \varepsilon/32}{K} \leq \varepsilon/16 + \varepsilon/32 < \varepsilon/8$$

by the definition of K and again since ρ' is $\varepsilon/32$ -close to ρ on A_K .

Now we suppose that F_A is in general a sum of a finitely generated free abelian group F and a finite group K . Let us however assume that the given generators of F_A are free generators of F together with all the non-zero elements of K . Denote the length function on F by $|\cdot|_F$ and the length function on F_A determined by the generators above by $|\cdot|'$. First we apply the result above for F with its free generators, $\rho \upharpoonright F$ and $\varepsilon/4$. We get some corresponding finite $C' \subseteq F$ and $\delta > 0$ such that $\delta < \varepsilon/2$. Now let $M = (\max \rho(K)) + \delta$. Take some $k \in \mathbb{N}$ such that $C' \subseteq C = \{x \in F_A : |x|' \leq k\}$ and $M/k < \varepsilon/8$. We claim that now $C \subseteq F_A$ and $\delta > 0$ are as desired. Let ρ' be an arbitrary norm on F_A such that $|\rho(a) - \rho'(a)| < \delta$ for all $a \in C$. Take any $z \in F_A \setminus \{0\}$. If $|z|' \leq k$, then $|\rho(z) - \rho'(z)| < \delta$, so $\frac{|\rho(z) - \rho'(z)|}{|z|'} < \varepsilon$, since $\delta \leq \varepsilon/2$. So suppose that $|z|' > k$ and write z as $x + u$, where $x \in F$ and $u \in K$.

We have $|x + u|' - |x|_F \in \{0, 1\}$ and

$$\frac{|\rho(x + u) - \rho'(x + u)|}{|x + u|'} \leq \frac{|\rho(x) - \rho'(x)| + \rho(u) + \rho'(u)}{|x|_F + 1} \leq$$

$$\frac{|\rho(x) - \rho'(x)|}{|x|_F + 1} + \frac{\rho(u) + \rho'(u)}{k} < \varepsilon/4 + 2\varepsilon/8 + \delta < \varepsilon.$$

Now suppose that the generators of F_A are arbitrary. It is a well-known and easy to observe that the length function $|\cdot|_A$ and the length function $|\cdot|'$ from the paragraph above are bi-Lipschitz equivalent. In particular, there is some L such that $|\cdot|_A \leq L|\cdot|'$. Thus we may apply the result from the paragraph above for F_A with the generators above and with ε/L to obtain the result for F_A with generators $\{d_a : a \in A\}$ and with ε . \square

We present one more definition of a certain easily definable norm.

Definition 3.7. Let us denote a norm on some F_C , for C finite, *finitely generated* if it is obtained as an extension using Lemma 1.3 of some partial norm defined on a finite symmetric subset of F_C containing zero.

Moreover, call a norm on some F_C , where again C is finite, *bounded finitely generated*, if it is a bounded norm which is defined as a minimum over a finitely generated norm and a positive constant. This constant is called a bound of the norm.

Note that there are only countably many rational bounded finitely-generated norms on a fixed countable abelian group, i.e. norms that are defined as a minimum over a rational bound and a rational finitely generated norm. We shall call them *BRFG norms*.

Claim 3.8. *Let $C \subseteq \mathbb{N}$ be finite and let ρ be an N_0 -norm on F_C . Then for every $\varepsilon > 0$ there exists a BRFG norm ρ_R on F_C such that $(C, \rho) \equiv_{\text{id}, \varepsilon} (C, \rho_R)$.*

Proof of Claim 3.8. Fix C , an N_0 -norm ρ on F_C and $\varepsilon > 0$. We use Lemma 3.5 to find a finite set $F \subseteq F_C$ and $\delta > 0$ such that for any N_0 -norm ρ' on F_C , if $|\rho(f) - \rho'(f)| < \delta$, for every $f \in F$, then $(C, \rho) \equiv_{\text{id}, \varepsilon} (C, \rho')$. We may suppose that F is finite symmetric containing zero. Then we use Lemma 2.2 to find a partial rational norm ρ'_R on F such that $|\rho(f) - \rho'_R(f)| < \delta$, for every $f \in F$. Then we take the extension of ρ'_R to the whole F_C , obtained by Lemma 1.3. Finally we bound this extension by $\max_{f \in F} \rho'_R(f)$. This is the desired BRFG norm ρ_R . \square

We now restate Theorem 0.2 here for the convenience of the reader and start with its proof.

Theorem 3.9. *Let G be an infinitely-summed group. Then G admits a generic norm λ .*

From now on, fix a non-trivial infinitely-summed group G , the enumerated set of generators $D = \{d_n : n \in \mathbb{N}\}$ as above, and the set of bijections Φ as above.

Let $\mathcal{G} \subseteq \mathcal{N}$ denote the set of all N_0 -norms λ on G satisfying the following condition:

$$(3.1) \quad \forall \varepsilon > \varepsilon' > 0, \forall A_0 \subseteq A \subseteq \mathbb{N} \text{ finite } \forall \rho_A \text{ BRFNG norm on } F_A \\ \text{if } (A_0, \rho_A) \equiv_{\text{id}, \varepsilon} (A_0, \lambda) \text{ then } \exists A' \subseteq \mathbb{N}, \exists \phi \in \Phi \text{ such that} \\ (A, \rho_A) \equiv_{\phi, \varepsilon'} (A', \lambda) \text{ and } \delta_\phi^\lambda(A_0, \phi[A_0]) < \varepsilon.$$

In order to prove Theorem 3.9, we shall prove that \mathcal{G} is dense G_δ and that for any $\lambda, \rho \in \mathcal{G}$ we have $\overline{(G, \lambda)} = \overline{(G, \rho)}$. That will give that G admits a generic norm. Moreover in case G is unbounded, we show that there is $\lambda \in \mathcal{G}$ such that (G, λ) is isometric to the rational Urysohn space. That will complete the proof of Theorem 3.9.

The first step showed in the next lemma is simple.

Lemma 3.10. *\mathcal{G} is G_δ .*

Proof. Being N_0 -norm is a G_δ condition by Lemma 3.1, so we check that the condition (3.1) is also G_δ .

That follows from the following observations. Fix $\varepsilon > \varepsilon' > 0$, $A_0 \subseteq A \subseteq \mathbb{N}$ finite and a BRFNG norm ρ_A on F_A . Then the implication from (3.1), after rewriting it as a disjunction, gives a union of the following two sets

$$\{\lambda \in \mathcal{N} : \neg(A_0, \rho_A) \equiv_{\text{id}, \varepsilon} (A_0, \lambda)\}$$

and

$$\{\lambda \in \mathcal{N} : \exists A' \subseteq \mathbb{N} \exists \phi \in \Phi ((A, \rho_A) \equiv_{\phi, \varepsilon'} (A', \lambda) \wedge \delta_\phi^\lambda(A_0, \phi[A_0]) < \varepsilon)\}.$$

Using Lemma 3.5, which says that the relation $\equiv_{\phi, \varepsilon}$ is determined on finite subsets for N_0 -norms, we get that the first set is closed and the second one is open. In particular, both are G_δ since \mathcal{N} is a Polish space, so closed sets are G_δ . Since a union of two G_δ sets is again G_δ we get that for fixed $\varepsilon > \varepsilon' > 0$, $A_0 \subseteq A \subseteq \mathbb{N}$ finite and a BRFNG norm ρ_A on F_A the condition (3.1) determines a G_δ set.

Finally one can check that all the universal quantifiers in (3.1) can be taken over countable sets which shows that (3.1) defines a G_δ subset. \square

Next we want to show that all the norms from \mathcal{G} give rise to the same normed group after the completion. Note that the condition (3.1) is similar to the condition on vector space norms which give rise to the Gurarij space, the separable Banach space of almost universal disposition constructed by Gurarij in [5]. The following proposition is thus similar to the main result of [8] where the authors prove the uniqueness of the Gurarij space.

Proposition 3.11. *For any two $\lambda, \lambda' \in \mathcal{G}$ we have that $\overline{(G, \lambda)}$ and $\overline{(G, \lambda')}$ are isometrically isomorphic.*

Proof. Consider two norms $\lambda, \lambda' \in \mathcal{G}$. Let $(i_j)_j$ be an enumeration of \mathbb{N} with an infinite repetition.

By induction, for every $j \in \mathbb{N}$ we shall construct two finite sequences $(a_i^j)_{i=0}^{2j-1} \subseteq \mathbb{N}$ and $(b_i^j)_{i=0}^{2j} \subseteq \mathbb{N}$ such that

- (1) for every $j \in \mathbb{N}$, $i_j \in \{a_i^j : i \leq 2j - 1\} \cap \{b_i^j : i \leq 2j\}$, i.e. there are k, k' such that $i_j = a_k^j = b_{k'}^j$;
- (2) for every $j \in \mathbb{N}$ there are some $\phi_j \in \Phi$ such that $\phi_j(a_i^j) = b_i^j$, for every $i \leq 2j - 1$, and

$$((a_i^j)_{i=0}^{2j-1}, \lambda) \equiv_{\phi_j, 1/2^{2j-1}} ((b_i^j)_{i=0}^{2j-1}, \lambda'),$$

and $\psi_j \in \Phi$ such that $\psi_j(b_i^j) = a_i^{j+1}$, for every $i \leq 2j$, and

$$((b_i^j)_{i=0}^{2j}, \lambda') \equiv_{\psi_j, 1/2^{2j}} ((a_i^{j+1})_{i=0}^{2j}, \lambda);$$

- (3) for every $j \in \mathbb{N}$ we have

$$\delta_{\psi_j \circ \phi_j}^\lambda((a_i^j)_{i=0}^{2j-1}, (a_i^{j+1})_{i=0}^{2j-1}) < 1/2^{2j-1}$$

and

$$\delta_{\phi_{j+1} \circ \psi_j}^{\lambda'}((b_i^j)_{i=0}^{2j+1}, (b_i^{j+1})_{i=0}^{2j+1}) < 1/2^{2j}.$$

Note that in particular for every $j \in \mathbb{N}$ we have

$$(3.2) \quad a_i^{j+1} = \psi_j \circ \phi_j(a_i^j), \forall i \leq 2j - 1,$$

$$b_i^{j+1} = \phi_{j+1} \circ \psi_j(b_i^j), \forall i \leq 2j.$$

Suppose at first that such sequences have been constructed. Denote by \mathbb{G} the completion of (G, λ) and by \mathbb{G}' the completion of (G, λ') . By (3), for each $i \in \mathbb{N}$ we have that the sequence $(g_i^j)_j$, where $g_i^j = d_{a_i^j}$ for all i, j , is Cauchy in (G, λ) , thus it has the limit, denoted by g_i , in \mathbb{G} . Analogously by (3), for each $i \in \mathbb{N}$ the sequence $(h_i^j)_j$, where $h_i^j = d_{b_i^j}$

for all i, j , is Cauchy in (G, λ') and we denote by h_i the limit in \mathbb{G}' . We claim that $\langle (g_i)_i \rangle$ is a dense subgroup in \mathbb{G} and $\langle (h_i)_i \rangle$ is a dense subgroup in \mathbb{G}' . We prove the former, the latter is analogous. Since G is dense in \mathbb{G} it suffices to show that for any $g \in G$ and any $\varepsilon > 0$ there exists $g' \in \langle (g_i)_i \rangle$ such that $\lambda(g' - g) < \varepsilon$. Take some finite $C \subseteq \mathbb{N}$ such that $g \in F_C$ and let $k = |g|_C$. There exists N such that for every $j \geq N$ and $i \leq 2j - 1$ we have

$$\lambda(g_i^j - g_i) < \varepsilon/k.$$

Also, by (1), for each $c \in C$ we can find i_c and $j_c \geq N$ such that $g_{i_c}^{j_c} = d_c$. Since $|g|_C = k$, $\lambda(g_{i_c}^{j_c} - g_{i_c}) < \varepsilon/k$, for every $c \in C$, it follows there is an element $g' \in \langle g_{i_c} : c \in C \rangle$ such that $\lambda(g' - g) < \varepsilon$, and the claim is proved.

Next we claim that the map sending g_i to h_i , for each $i \in \mathbb{N}$, can be extended to an isometric isomorphism $\Psi : \langle g_i : i \in \mathbb{N} \rangle \rightarrow \langle h_i : i \in \mathbb{N} \rangle$. Take a finite subset $S \subseteq \mathbb{N}$ and integers $(k_i)_{i \in S} \subseteq \mathbb{Z}$. Then

$$\begin{aligned} |\lambda(\sum_{i \in S} k_i \cdot g_i) - \lambda'(\sum_{i \in S} k_i \cdot h_i)| &= \lim_{j \rightarrow \infty} |\lambda(\sum_{i \in S} k_i \cdot g_i^j) - \lambda'(\sum_{i \in S} k_i \cdot h_i^j)| \leq \\ &\lim_{j \rightarrow \infty} \sum_{i \in S} |k_i|/2^{2j-1} = 0. \end{aligned}$$

The first equality follows from the definition, the second inequality follows from (2).

It follows that we may uniquely extend Ψ to \mathbb{G} , which we shall still denote by Ψ and which is an isometric isomorphism between \mathbb{G} and \mathbb{G}' . It remains to find the sequences.

We will proceed by induction. We show the first odd and even steps of the induction and then the general odd and even steps of the induction.

Set $a_1^1 = i_1$. Then by Claim 3.8 there exists a BRFN norm ρ_1 on $F_{\{a_1^1\}}$ such that $(\{a_1^1\}, \lambda) \equiv_{\text{id}, 1/4} (\{a_1^1\}, \rho_1)$. By (3.1), using $A_0 = \emptyset$ and $A = \{a_1^1\}$, $\varepsilon' = 1/4$, ε arbitrary bigger than ε' and ρ_1 , there exists b_1^1 such that $(\{a_1^1\}, \rho_1) \equiv_{\phi_1, 1/4} (\{b_1^1\}, \lambda')$ for some $\phi_1 \in \Phi$, thus $(\{a_1^1\}, \lambda) \equiv_{\phi_1, 1/2} (\{b_1^1\}, \lambda')$. This finishes the first odd step.

Now if $b_1^1 = i_1$ then take as b_2^1 an arbitrary natural number; otherwise, take $b_2^1 = i_1$. By Claim 3.8 there exists a BRFN norm ρ_2 on $F_{\{b_1^1, b_2^1\}}$ such that $(\{b_1^1, b_2^1\}, \lambda') \equiv_{\text{id}, \delta} (\{b_1^1, b_2^1\}, \rho_2)$, where $\delta < 1/8$ is sufficiently small so that we still have $(\{a_1^1\}, \lambda) \equiv_{\phi_1, 1/2} (\{b_1^1\}, \rho_2)$. Then by (3.1), using $A_0 = \{b_1^1\}$ and $A = \{b_1^1, b_2^1\}$, $\varepsilon = 1/2$, $\varepsilon' = 1/8$ and ρ_2 , there exist $a_1^2, a_2^2 \in \mathbb{N}$ such that

- $(\{a_1^2, a_2^2\}, \lambda) \equiv_{\phi_2, 1/8} (\{b_1^1, b_2^1\}, \rho_2)$, for some $\phi_2 \in \Phi$,
thus $(\{a_1^2, a_2^2\}, \lambda) \equiv_{\phi_2, 1/4} (\{b_1^1, b_2^1\}, \lambda')$,
- $\delta_{\phi_2^{-1} \circ \phi_1}(\{a_1^1\}, \{a_1^2\}) < 1/2$; in other words, $\lambda(d_{a_1^1} - d_{a_1^2}) < 1/2$.

Now suppose that we have found sequences $(a_i^{n-1})_{i=0}^{2n-3}$ and $(b_i^{n-1})_{i=0}^{2n-2}$. We shall find $(a_i^n)_{i=0}^{2n-1}$ and $(b_i^n)_{i=0}^{2n}$. Since by assumption we have that $((a_i^{n-1})_{i=0}^{2n-3}, \lambda) \equiv_{\phi_{n-1}, 1/2^{2n-3}} ((b_i^{n-1})_{i=0}^{2n-2}, \lambda')$, again using first Claim 3.8 and then (3.1) we can find $(a_i^n)_{i=0}^{2n-2}$ and $\psi_{n-1} \in \Phi$ such that

$$((b_i^{n-1})_{i=0}^{2n-2}, \lambda') \equiv_{\psi_{n-1}, 1/2^{2n-2}} ((a_i^n)_{i=0}^{2n-2}, \lambda),$$

and moreover

$$\delta_{\psi_{n-1} \circ \phi_{n-1}}((a_i^{n-1})_{i=0}^{2n-3}, (a_i^n)_{i=0}^{2n-3}) < 1/2^{2n-3}.$$

If $i_n \in (a_i^n)_{i=0}^{2n-2}$ then we set a_{2n-1}^n to be any natural number. Otherwise, we set $a_{2n-1}^n = i_n$.

Then analogously, using Claim 3.8 and (3.1), we find $(b_i^n)_{i=0}^{2n-1}$ and $\phi_n \in \Phi$ such that

$$((a_i^n)_{i=0}^{2n-1}, \lambda) \equiv_{\phi_n, 1/2^{2n-1}} ((b_i^n)_{i=0}^{2n-1}, \lambda'),$$

and moreover

$$\delta_{\phi_n \circ \psi_{n-1}}((b_i^{n-1})_{i=0}^{2n-2}, (b_i^n)_{i=0}^{2n-2}) < 1/2^{2n-2}.$$

Again, if $i_n \in (b_i^n)_{i=0}^{2n-1}$ then we set b_{2n}^n to be any natural number. Otherwise, we set $b_{2n}^n = i_n$. This finishes the induction and the proof. \square

To finish the proof of Theorem 3.9 we need to prove that \mathcal{G} is dense. Notice that so far we have not yet even proved that \mathcal{G} is non-empty. The next proposition will do.

Proposition 3.12. *\mathcal{G} is dense.*

Moreover, if G is unbounded, then the subset $\{\lambda \in \mathcal{G} : (G, \lambda) \cong_{\text{iso}} \mathbb{Q}\mathbb{U}\} \subseteq \mathcal{G}$ is dense.

Proof. Fix a basic open set in \mathcal{N} . It is given by some partial norm ρ on some, without loss of generality, finite symmetric subset F of G containing zero, and some $\varepsilon > 0$. We may suppose that F is such that for some finite $C \subseteq \mathbb{N}$ we have $\langle F \rangle = F_C$. We use Lemma 2.2 to get a rational partial norm ρ'_R on F such that $|\rho(f) - \rho'_R(f)| < \varepsilon$ for all $f \in F$. Then we use Lemma 1.3 to extend it to a rational finitely generated norm on $\langle F \rangle = F_C$ and finally we make it bounded by some rational constant to get a BRFNG norm ρ_R on F_C that agrees with ρ'_R on F .

Let us now enumerate all triples $T_n = (B_n, A_n, \rho_n)$, where $B_n \subseteq A_n \subseteq \mathbb{N}$ are finite and ρ_n is a BRFN norm on F_{A_n} . Moreover, suppose that there is an infinite repetition of each such a triple in the enumeration.

By induction, we shall construct an increasing sequence of finite sets $(C_n)_n$, i.e. $C_n \subseteq C_m$, for $n < m$, and an increasing sequence of bounded rational norms $(\lambda_n)_n$, i.e. $\lambda_n \subseteq \lambda_m$, for $n < m$, such that

- (1) $C_1 = C$ and $\lambda_1 = \rho_R$;
- (2) $\bigcup_n C_n = \mathbb{N}$;
- (3) for each n , λ_n is a norm on F_{C_n} ;
- (4) for every n , if there are $B'_n \subseteq C_n$ and ϕ' such that

$$(B'_n, \lambda_n) \equiv_{\phi', 1/2^n} (B_n, \rho_n),$$

then there are $A'_n \subseteq C_{n+1}$ and ϕ such that

$$(A_n, \rho_n) \equiv_{\phi} (A'_n, \lambda_{n+1}),$$

i.e. ϕ induces an isometric isomorphism between (F_{A_n}, ρ_n) and $(F_{A'_n}, \lambda_{n+1})$, and

$$\delta_{\phi \circ \phi'}(B'_n, \phi \circ \phi'[B'_n]) < 1/2^n.$$

We shall now proceed to the induction. The first step has been already done, i.e. we set $C_1 = C$ and $\lambda_1 = \rho_R$ as obtained from the claim above.

Let us now describe the general step. Suppose we have produced a finite set C_n and a norm λ_n on F_{C_n} for $n \geq 1$. Consider now the triple $T_n = (B_n, A_n, \rho_n)$. Suppose that there are some $B'_n \subseteq C_n$ and $\phi' \in \Phi$ such that $\phi'[B'_n] = B_n$ and $(B'_n, \lambda_n) \equiv_{\phi', 1/2^n} (B_n, \rho_n)$. There can be at most finitely many such B'_n 's. To simplify the notation and proof, we shall suppose there is just one such $B'_n \subseteq C_n$, and actually $B'_n = B_n$, and thus $\phi' = \text{id}$. If there are more such finite subsets of C_n , the procedure that follows is repeated (finitely many times). If there is no such a finite subset B'_n , then we set $C'_{n+1} = C_n$ and $\lambda'_{n+1} = \lambda_n$ and use the procedure below to extend C'_{n+1} to C_{n+1} and λ'_{n+1} to λ_{n+1} .

Thus we suppose that $B_n \subseteq C_n$ and $(B_n, \lambda_n) \equiv_{\text{id}, 1/2^n} (B_n, \rho_n)$. Since G is infinitely-summed we can find A'_n such that there is some $\phi \in \Phi$ which is a bijection between A_n and A'_n inducing an isomorphism $\bar{\phi}$ between F_{A_n} and $F_{A'_n}$, and $F_{C_n} \cap F_{A'_n} = \{0\}$. Set $C'_{n+1} = C_n \cup A'_n$. We shall now define a bounded rational norm λ'_{n+1} on $F_{C'_{n+1}}$ which extends λ_n .

For each $c \in B_n$, set h_c to be $\min\{1/2^n, \lambda_n(d_c) + \rho_n(d_c)\}$. For $x \in F_{C_n} \cup F_{A'_n} \cup \{d_c - \bar{\phi}(d_c), \bar{\phi}(d_c) - d_c : c \in B_n\}$ we set

$$\chi(x) = \begin{cases} \lambda_n(x) & \text{if } x \in F_{C_n}, \\ \rho_n(y) & \text{if } x = \bar{\phi}(y), y \in F_{A_n}, \\ h_c & \text{if } x = \varepsilon(d_c - \bar{\phi}(d_c)), \text{ where } c \in B_n, \varepsilon \in \{1, -1\}. \end{cases}$$

We use Fact 1.5 to get a partial norm λ'_{n+1} on $F_{C_n} \cup F_{A'_n} \cup \{d_c - \bar{\phi}(d_c), \bar{\phi}(d_c) - d_c : c \in B_n\}$. Note that it is bounded by some K . Also note that for each $c \in B_n$ we have $\lambda'_{n+1}(d_c - \bar{\phi}(d_c)) \leq 1/2^n$. We need to check that for each $x \in F_{C_n} \cup F_{A'_n}$ we have $\lambda'_{n+1}(x) = \chi(x)$. Then we could use Lemma 1.3 again to extend λ'_{n+1} to a norm bounded by K on $F_{C'_{n+1}}$ still denoted by λ'_{n+1} . It will follow that $(A_n, \rho_n) \equiv_\phi (A'_n, \lambda'_{n+1})$ and that $\delta_\phi(B_n, \phi[B_n]) \leq 1/2^n$.

So fix some $x \in F_{C_n} \cup F_{A'_n}$. We need to check that for any $x_1, \dots, x_k \in F_{C_n} \cup F_{A'_n} \cup \{d_c - \bar{\phi}(d_c), \bar{\phi}(d_c) - d_c : c \in B_n\}$ such that $x = \sum_{i=1}^k x_i$ we have $\chi(x) \leq \sum_{i=1}^k \chi(x_i)$.

We have two cases: $x \in F_{C_n}$ and $x \in F_{A'_n}$. We shall treat only the first one, the second is analogous.

So we suppose that $x \in F_{C_n}$. Since G is abelian we may suppose that there are k_1, k_2 such that $0 \leq k_1 \leq k_2 \leq k$, for every $1 \leq i \leq k_1$ we have $x_i \in F_{C_n}$, for every $k_1 < i \leq k_2$ we have $x_i \in F_{A'_n}$ and for every $k_2 < i \leq k$ we have $x_i \in \{d_c - \bar{\phi}(d_c), \bar{\phi}(d_c) - d_c : c \in B_n\}$. Moreover, for every $k_2 < i \leq k$ we may suppose that $\chi(x_i) = 1/2^n$. Otherwise, $\chi(x_i) = \lambda_n(d_c) + \rho_n(d_c)$, for some $c \in B_n$, i.e. x_i is equal to $d_c - \bar{\phi}(d_c)$ or $\bar{\phi}(d_c) - d_c$. In that case we would replace x_i by a pair $d_c, -\bar{\phi}(d_c)$, resp. $\bar{\phi}(d_c), -d_c$ without increasing the sum $\sum_{i=1}^k \chi(x_i)$. Set $z_1 = \sum_{i=1}^{k_1} x_i$, $z_2 = \sum_{i=k_1+1}^{k_2} x_i$ and $z_3 = \sum_{i=k_2+1}^k x_i$. Since $x = z_1 + z_2 + z_3$, it follows that $z_3 = -z_2 + \bar{\phi}^{-1}(z_2)$. Since $(B_n, \lambda_n) \equiv_{\text{id}, 1/2^n} (B_n, \rho_n)$ we get that $\bar{\phi}^{-1}(z_2) \in F_{B_n}$ and $|\lambda_n(\bar{\phi}^{-1}(z_2)) - \rho_n(\bar{\phi}^{-1}(z_2))| < |z_2|/2^n$. Since

$$\sum_{i=k_2+1}^k \chi(x_i) \geq |z_2|/2^n,$$

we get that

$$\sum_{j=1}^k \chi(x_j) \geq \lambda_n(z_1) + \rho_n(\bar{\phi}^{-1}(z_2)) + |z_2|/2^n \geq \lambda_n(x)$$

because $x - z_1 = z_2 + z_3 = \bar{\phi}^{-1}(z_2)$, and we are done.

Finally, set $C_{n+1} = C'_{n+1} \cup \{n\}$ and extend λ'_{n+1} to a bounded rational norm λ_{n+1} on $F_{C_{n+1}}$ arbitrarily.

When the induction is finished we get $\lambda = \bigcup_n \lambda_n$ is a norm on G . We check that $\lambda \in \mathcal{G}$. First, since it is a direct limit of bounded norms, clearly it is an N_0 -norm. Now take any $\varepsilon > \varepsilon' > 0$, finite subsets $A_0 \subseteq A \subseteq \mathbb{N}$. Let ρ_R be some BRF norm on F_A such that $(A_0, \rho_R) \equiv_{\text{id}, \varepsilon} (A_0, \lambda)$. Then by the construction, we can find n such that $A \subseteq C_n$ and $T_n = (A_0, A, \rho_R)$, where $1/2^n < \varepsilon$. By the inductive construction, there is some ϕ such that $A \equiv_\phi \phi[A]$, $\phi[A] \subseteq C_{n+1}$ and

$$(A, \rho_R) \equiv_{\phi, \varepsilon'} (\phi[A], \lambda)$$

and

$$\delta_\phi(A, \phi[A]) < 1/2^n < \varepsilon,$$

and we are done.

Also, since at the beginning the finite symmetric subset F and a partial norm ρ on F were arbitrary, it shows that \mathcal{G} is dense.

Finally, we show how to get the “moreover” part from the statement of the proposition. If G is unbounded then we combine the two induction procedures from this proof and the proof of Theorem 2.3 into one. Besides the enumeration of triples $(T_n)_n$ as above, consider also the enumeration $\{(A_i, f_i) : i \in \mathbb{N}\}$ (again with infinite repetition) of all pairs (A, f) , where A is a finite rational metric space and $f : A \rightarrow \mathbb{Q}$ a rational Katětov function over A . Then we divide the induction procedure into odd and even steps. During odd steps, we take care of triples $(T_n)_n$ as above. During even steps, we take care of pairs $(A_n, f_n)_n$ as in the proof of Theorem 2.3, just ensuring that the norm is bounded after every step. It follows that after the induction we get a norm $\lambda \in \mathcal{G}$ such that (G, λ) is isometric to the rational Urysohn space. \square

Corollary 3.13. *Let G be an unbounded infinitely-summed countable Abelian group. There exists an Abelian Polish metric group \mathbb{G} which is extremely amenable and isometric to the Urysohn space such that for comeager-many norms λ on G we have*

$$\mathbb{G} = \overline{(G, \lambda)}.$$

In particular, for every such G there is a norm λ such that (G, λ) is extremely amenable and isometric to the rational Urysohn space.

Remark 3.14. Although the results above show that for every infinitely-summed countable Abelian group G there is a corresponding Abelian Polish metric group \mathbb{G} , one might ask whether there is actually *a single*

generic Abelian Polish metric group \mathbb{H} . That is, whether for every infinitely-summed countable Abelian group G and a generic metric ρ on G , we have $\overline{(G, \rho)} = \mathbb{H}$.

Clearly, if G_1 and G_2 are two infinitely-summed countable abelian groups of bounded torsion, where the bounds are different for G_1 and G_2 respectively, then the corresponding generic metrics cannot give the same group after completion. However, it is reasonable to expect that the groups $\mathbb{G}_\infty(N)$, where $N \in \{0, 2, 3, 4, \dots\}$ from [10], are the only generic abelian Polish groups.

3.1. Extremely amenable universal abelian group. In this section, we observe that the universal abelian Polish group of Shkarin from [13], further investigated by Niemiec in [10], is extremely amenable, and provide another proof that it is, with its norm, isometric to the Urysohn space. More precisely, we formulate a certain extension property of this group (analogous to the extension property of the Gurarij Banach space) and show that this extension property is a dense G_δ property. Then we use our results and results of Melleray and Tsankov to show that the extension property defines this group uniquely up to isometric isomorphism, and that the group is extremely amenable and isometric to the Urysohn space.

Since the group is essentially constructed using Fraïssé theoretic methods, we shall assume here that the reader has a basic knowledge of this area. We refer the reader to Chapter 7 in [6] as a reference to the Fraïssé theory.

We note that the group is universal in the sense that every abelian Polish group, or every second-countable abelian Hausdorff group, embeds via topological isomorphism as a subgroup. The group was constructed as a normed group, however it is universal only in the topological sense. It does not contain *isometrically* every separable abelian normed group as a subgroup. A normed group universal in this stronger sense was constructed by the author in [3].

Let us describe the construction of the Shkarin's group, further denoted by \mathbb{G}_S : First one considers the class of all finite groups equipped with rational-valued norms. Then one can check that this class is a Fraïssé class, thus it has a Fraïssé limit. It is straightforward to check that the limit is a countable abelian group, denoted by G_S , which is algebraically isomorphic to $\bigoplus_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}$, equipped with a rational norm λ_S . In particular, G_S is an infinitely-summed unbounded group. Then one takes the completion $\overline{(G_S, \lambda_S)}$ to obtain the group \mathbb{G}_S .

In the sequel, we shall use the following notation. For two finite groups G, H which are isomorphic via some ϕ , and which are equipped with norms λ_G and λ_H respectively, and for some $\varepsilon > 0$ we shall write

$$(G, \lambda_G) \equiv_{\phi, \varepsilon} (H, \lambda_H)$$

to express that they are ε -isomorphic via ϕ , i.e. for every $g \in G$ we have $|\lambda_G(g) - \lambda_H(\phi(g))| < \varepsilon$.

Consider now the following set \mathbb{S} of norms λ on $G_S = \bigoplus_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}$, i.e. a subset of \mathcal{N}_{G_S} :

$$(3.3) \quad \forall \varepsilon > 0 \forall G_0 \leq G_1 \leq G_S \text{ finite, } \forall \rho \text{ a rational norm on } G_1 \\ \text{if } (G_0, \rho) \equiv_{\text{id}, \varepsilon} (G_0, \lambda) \text{ then } \exists G'_1, G_0 \leq G'_1 \leq G_S, \text{ isomorphic to } G_1 \\ \text{via some } \phi \text{ such that } (G'_1, \lambda) \equiv_{\phi, \varepsilon} (G_1, \rho).$$

First, it is clear, by the definition of a Fraïssé limit, that $\lambda_S \in \mathbb{S}$; moreover, that the set \mathbb{S} is dense. Secondly, one can check as in the proof of Lemma 3.10 that (3.3) defines a G_δ subset of \mathcal{N}_{G_S} . Thus \mathbb{S} is a dense G_δ set.

It also follows from the proofs of Shkarin and Niemiec that for any $\lambda \in \mathbb{S}$, the group $\overline{(G_S, \lambda)}$ is also universal replicating the approximation arguments for λ_S . It is analogous to the case of metrics on a countable set satisfying (2.1) from the proof of Corollary 2.4. Then applying Theorem 3.9, Corollary 2.1 and Theorem 1.1 of Melleray and Tsankov we get the following corollary.

Corollary 3.15. *There exists a generic norm λ on $G_S = \bigoplus_{\mathbb{N}} \mathbb{Q}/\mathbb{Z}$ such that $\overline{(G_S, \lambda)}$ is the Shkarin's group, which is thus extremely amenable and isometric to the Urysohn space.*

Remark 3.16. On the other hand, the metrically universal abelian group constructed by the author in [3] is not generic. This can be immediately seen as the norm on that group cannot be an N_0 -norm.

4. PROBLEMS

The result of Melleray and Tsankov cannot be used to prove that the metrically universal group from [3] is extremely amenable. That does not mean it is not possible though. We note that although the results of this paper cannot be directly used to show that the metrically universal group is isometric to the Urysohn space, it was nevertheless proved in [3]. So we ask:

Question 4.1. *Is the metrically universal group from [3] extremely amenable?*

Another challenging problem is to investigate similar properties of the spaces of metrics on non-abelian countable groups. There one can distinguish two cases, which coincide in the case of abelian groups: the space of all *continuous* left-invariant metrics (we comment on the ‘continuity’ below) and the space of bi-invariant metrics. In the case of bi-invariant metrics, one can again easily check that for any countable group G , the set of all bi-invariant metrics, is a closed subset $\mathbb{R}^{G \times G}$, thus a Polish space.

We do not know if there is some countable non-abelian group which admits a generic bi-invariant metric. Also, we do not know whether for some countable non-abelian group the subset of bi-invariant metrics with which this group is extremely amenable is comeager; i.e. we do not know whether it is possible to generalize the Melleray and Tsankov’s result to the non-abelian situation.

Regarding the general left-invariant metrics, first thing to observe is that while bi-invariant metrics make the group operations continuous, in fact Lipschitz, this is no longer true for general left-invariant metrics. The corresponding general norms on groups (that do not necessarily make the group topological) were considered in the literature, see e.g. [1]. However, in most cases it is reasonable to consider only such metrics that do make the group operations continuous. For a group G and a left-invariant metric d on G , the group operations are continuous if and only if for every $g \in G$ and every $\varepsilon > 0$ there is $\delta > 0$ such that for every $h \in G$, if $\lambda_d(h) < \delta$, then $\lambda_d(g^{-1} \cdot h \cdot g) < \varepsilon$, where λ_d is the corresponding norm. Let us call such norms and metrics continuous.

The main problem is that we do not know how to code the continuous norms and metrics on a non-abelian group as a Polish space. Indeed, the straightforward computation gives that they form an $F_{\sigma\delta}$ subset of \mathbb{R}^G (\mathbb{R}^{G^2}), thus not necessarily a space with Polish topology. Even some special subsets of continuous norms such as uniformly discrete norms seem not to be Polish, but rather F_σ subsets of \mathbb{R}^G . A special subclass of uniformly discrete norms, often considered in geometric group theory, that is, the class of all proper norms again seems to be $F_{\sigma\delta}$.

However, we do not exclude the possibility that a better computation reveals that these are Polish spaces — in a natural way.

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