INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Universal actions of locally finite groups on metric and Banach spaces by isometries

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Preprint No. 19-2017

PRAHA 2017

UNIVERSAL ACTIONS OF LOCALLY FINITE GROUPS ON METRIC AND BANACH SPACES BY ISOMETRIES

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ABSTRACT. We construct a universal action of a countable locally finite group on a separable metric space by isometries. This single action contains all actions of all countable locally finite groups on all separable metric spaces as subactions. The main ingredient is an amalgamation of actions by isometries. We also construct a universal action of a universal countable torsion abelian group on a separable Banach space by linear isometries.

We show that the restriction to locally finite groups in our results is necessary as analogous results do not hold for infinite non-locally finite groups.

INTRODUCTION

Groups acting by isometries on metric and Banach spaces are one of the active areas of research in geometry, group theory and functional analysis. In this paper, we are interested in amalgamation of group actions and constructing universal actions. It is well known from the beginnings of combinatorial group theory that one can construct an amalgam of two groups over some common subgroup. At least as old is the amalgamation of metric spaces, or amalgamation of normed vector spaces. However, to the best of our knowledge, nobody has considered yet amalgamation of actions of groups on metric or Banach spaces by isometries. In metric geometry or functional analysis, amalgamation techniques are often used to construct various universal metric or Banach spaces (consider for instance the Urysohn universal metric space [16], or the Gurarij universal Banach space [6]). The well-known Hall's universal locally finite group ([7]) is essentially made by amalgamating finite groups. Here by amalgamating actions of finite groups on finite metric spaces by isometries we obtain the following result.

2010 *Mathematics Subject Classification.* 20E06, 20F50, 20K10, 22F05, 54E50, 46B04.

Key words and phrases. group actions, actions by isometries, universal action, amalgamation of actions, Hall's group, Urysohn space, Gurarij space, metric Fraïssé theory.

The author was supported by the GAČR project 16-34860L and RVO: 67985840.

Theorem 0.1. *There exists a universal action of the Hall's locally finite group G on the Urysohn space $\mathbb{U}$ by isometries. That is, for any action of a countable locally finite group H on a separable metric space X by isometries, there exists a subgroup $H' \leq G$ isomorphic to H such that, after identifying H and H', there is an H-equivariant isometric embedding of X into $\mathbb{U}$.*

The meaning of the theorem is that there is a single action of a countable locally finite group on a separable metric space by isometries that captures/contains all actions of all countable locally finite groups on all separable metric spaces.

One of the main ingredients is the amalgamation of actions and we have the following general theorem.

Theorem 0.2. *Let G_1, G_2 be two groups (countable or not) with a common subgroup G_0. Suppose that G_1 acts on a metric space X_1 and G_2 acts on X_2, by isometries in both cases. Let X_0 be a common subspace of X_1 and X_2 such that the restriction of the two actions on G_0 and X_0 coincide. Then there is an amalgam of the action, which is an action of $G_1 *_{G_0} G_2$ on a metric space with density $\max\{|G_1|, |G_2|, \mathrm{dens}(X_1), \mathrm{dens}(X_2)\}$.*

Following the research of Rosendal in [15] and of Glasner, Kitroser and Melleray in [4] we investigate the genericity of the universal action from Theorem 0.1.

Theorem 0.3. *The universal action from Theorem 0.1 is* weakly generic *in some sense. That is, the set of those actions in the Polish space $\mathrm{Iso}\mathbb{U}^G$ that are naturally equivalent to the universal one is dense G_δ.*

We show that the restriction to locally finite groups is essential.

Theorem 0.4. *There are no analogously universal actions of infinite groups that are not locally finite.*

Finally, we investigate universal actions on Banach spaces. General actions by isometries are by affine isometries. Unfortunately, we show that no universal action by affine isometries can exist, even of finite groups. Thus we are forced to restrict to actions by linear isometries. We are not able to show the result for the Hall's group, so we restrict ourselves to abelian locally finite groups, i.e. to torsion abelian groups.

Theorem 0.5. *There exists a universal action of the group $G = \bigoplus_{n\in\mathbb{N}} \mathbb{Q}/\mathbb{Z}$ on the Gurarij space $\mathbb{G}$ by linear isometries. That is, for any action of a countable torsion abelian group H on a separable Banach space X by linear isometries, there exists a subgroup $H' \leq G$ isomorphic to H*

such that, after identifying H and H', there is an H-equivariant linear isometric embedding of X into $\mathbb{G}$.

1. Preliminaries

Let us start with our notational convention. All the group actions in this paper are by isometries. We usually denote actions by the symbol '$\alpha : G \curvearrowright X$', where G is a group and X is a metric space. However, as it is common, we usualy write $g \cdot x$ instead of $\alpha(g, x)$.

Regarding groups, we are mostly concerned with locally finite ones, where group is locally finite if every finitely generated subgroup is finite. Since we shall work solely with countable groups, it is the same as saying that the group is a direct limit of a sequence of finite groups.

Our constructions of universal objects are based on techniques commonly referred as "Fraïssé theory". We refer to Chapter 7 in [8] for more information about this subject. For a reader unfamiliar with this method we briefly and informally describe the basics of Fraïssé theory that we use in the paper.

Let $\mathcal{K}$ be some countable class of mathematical objects of some type with some notion of embedding between these objects. Suppose that direct limits of objects from $\mathcal{K}$ exist. Think of the class of finite groups for instance. We say it is a *Fraïssé class* if any two objects from $\mathcal{K}$ can be embedded into a single object from $\mathcal{K}$, such a property is called *joint embedding property*, and if whenever we have objects $A, B, C \in \mathcal{K}$ such that A embeds into both B and C, witnessed by embeddings ι_B, resp. ι_C, then there exists an object $D \in \mathcal{K}$ and embeddings ρ_B, resp. ρ_C of B into D, resp. C into D such that $\rho_C \circ \iota_C = \rho_B \circ \iota_B$; i.e we can do amalgamation with object from $\mathcal{K}$. The latter property is called *amalgamation property.* The Fraïssé theorem (see Chapter 7 in [8]) then asserts that there exists a unique object K, called the *Fraïssé limit of* $\mathcal{K}$, which is a direct limit of a sequence of objects from $\mathcal{K}$ satisfying

- every object $A \in \mathcal{K}$ embeds into K;
- whenever we have objects $A, B \in \mathcal{K}$ such that A embeds via ρ_A into K and via ι_A into B, then there exists an embedding ρ_B of B into K such that $\rho_A = \rho_B \circ \iota_A$.

The second property is called the *extension property* and will be used in our proofs of universality of certain actions. Note that whenever X is some direct limit of a sequence of objects from $\mathcal{K}$, then successive application of the extension property gives an embedding of X into K.

We note that the Fraïssé theorem stated above is the only too which we shall use and its proof is actually much shorter that the discussion on Fraïssé theory above and may be left as an exercise. Since we are going to work with Fraïssé classes which are 'metric' we note that recently a general theory for metric Fraïssé classes was developed independently in [1] and [9]. However, we shall not directly use their results in our paper.

Example 1 Consider the countable class of all finite graphs. It is easy to show it has the joint and amalgamation properties, thus by the Fraïssé theorem there exists a Fraïssé limit, a certain direct limit of a sequence of finite graphs, which is a countable graph commonly known as the random graph, or the Rado graph. The extension property allows to show that it contains as a subgraph a copy of every countable graph.

Example 2 Consider now the countable class of all finite abelian groups. It is again easy to show the joint and amalgamation properties and one can even show that the Fraïssé limit is nothing else than $\bigoplus_{n\in\mathbb{N}} \mathbb{Q}/\mathbb{Z}$.

Example 3 Consider now the countable class of all finite groups, not necessarily abelian. This is the most important example for us regarding the topic of our paper. It is less straightforward, nevertheless possible to show (see [11]), that this class has the amalgamation property, and thus also the joint embedding property. The Fraïssé limit is what is commonly known as the Hall's universal locally finite group.

Example 4 Consider the countable class of all finitely presented groups. It is again easy to show the amalgamation property. We are not aware anyone has considered the Fraïssé limit of this class yet.

Example 5 Consider the countable class of all finite metric spaces with rational distances. The amalgamation and joint embedding is again straightforward. The Fraïssé limit is what is known as the rational Urysohn space. Its completion is the Urysohn universal space (see [16]).

Example 6 As the last example, we present another 'metric Fraïssé class' recently discovered by the author in [3]. It is the class of all finitely generated free abelian groups with a 'finitely presented rational metric'. The completion of its limit gives the metrically universal abelian separable group. See the paper for details.

2. Universal actions

Definition 2.1. Let G be a group and X a metric space. A *pointed free action* of G on X by isometries is a tuple $(G \curvearrowright X, (x_i)_{i \in I})$, where $G \curvearrowright X$ is a free action of G on X by isometries and I is some index set for the orbits of the action and $(x_i)_{i \in I}$ is a selector on the orbits, i.e. $X = \bigcup_{i \in I} G \cdot x_i$ and for $i \neq j$, x_i and x_j lie in different orbits.

There is also a natural notion of embedding between two pointed free actions. Suppose we are given two such actions $(H \curvearrowright Y, (y_i)_{i \in I})$ and $(G \curvearrowright X, (x_j)_{j \in J})$. An *embedding* of $(H \curvearrowright Y, (y_i)_{i \in I})$ into $(G \curvearrowright X, (x_j)_{j \in J})$ is a pair (ϕ, ψ), where $\phi : H \hookrightarrow G$ is a group embedding and $\psi : Y \hookrightarrow X$ is an isometric embedding such that for any $i, j \in I$ and $f, h \in H$ we have

$$d_Y(f \cdot y_i, h \cdot y_j) = d_X(\phi(f) \cdot \psi(y_i), \phi(h) \cdot \psi(y_j)).$$

In particular, for any $i \in I$ there is some $j \in J$ such that $\psi(y_i) = x_j$, i.e. ψ sends the distinguished points $(y_i)_{i \in I}$ into the set of distinguished points $(x_j)_{j \in J}$.

Theorem 2.2. *The pointed free actions can be amalgamated.*

Remark 2.3. It means that for any embeddings $\psi_i : (G_0 \curvearrowright X_0, (x_j)_{j \in I_0}) \hookrightarrow (G_i \curvearrowright X_i, (x_j)_{j \in I_i})$, for $j \in \{1, 2\}$, where we assume that $G_0 \leq G_1$ and $G_0 \leq G_2$, there are a group $G_1, G_2 \leq G_3$, pointed action $(G_3 \curvearrowright X_3, (x_j)_{j \in I_3})$ and embeddings $\rho_j : (G_i \curvearrowright X_i, (x_j)_{j \in I_i}) \hookrightarrow (G_3 \curvearrowright X_3, (x_j)_{j \in I_3})$, for $j \in \{1, 2\}$, such that $\rho_2 \circ \psi_2 = \rho_1 \circ \psi_1$.

Proof. Consider such actions from the remark above, i.e. $(G_i \curvearrowright X_i, (x_j)_{j \in I_i})$, for $i \in \{0, 1, 2\}$. We may also suppose that $I_0 \subseteq I_i$, for $i = 1, 2$, and that $I_0 = I_1 \cap I_2$. Let G_3 be $G_1 *_{G_0} G_2$, i.e. the free product of G_1 and G_2 amalgamated over G_0 (we refer to [10] for constructions of amalgamated free products of groups). Let $I_3 = I_1 \cup I_2$ and set $X_3 = \bigcup_{j \in I_3} G_3 \cdot j$. Clearly, $X_i \subseteq X_3$, for $i = 1, 2$. We shall define a metric on X_3 so that the canonical action of G_3 on X_3 is by isometries and that the inclusion of X_i into X_3 is isometric (it is obviously G_i-equivariant), for $i = 1, 2$.

We define a structure of a weighted graph on X_3 that will help us define a metric there. That is, we define edges on X_3 and then associate a certain weight function w giving positive real numbers to these edges. For $g, h \in G_3$ and $i, j \in I_3$, the elements $g \cdot x_i$ and $h \cdot x_j$ are connected by an edge if and only if

- either $g^{-1}h \in G_1$ and $i, j \in I_1$, then its weight is

$$w(g \cdot x_i, h \cdot x_j) = d_{X_1}(x_i, g^{-1}h \cdot x_j);$$

- or $g^{-1}h \in G_2$ and $i,j \in I_2$, then analogously its weight is
$$w(g\cdot x_i, h\cdot x_j) = d_{X_2}(x_i, g^{-1}h\cdot x_j).$$

In case that $g^{-1}h \in G_0$ and $i,j \in I_0$ there is no ambiguity in the definition. Indeed, by assumption, in such a case we have
$$d_{X_0}(x_i, g^{-1}h\cdot x_j) = d_{X_1}(x_i, g^{-1}h\cdot x_j) = d_{X_2}(x_i, g^{-1}h\cdot x_j).$$

It is clear that this graph is connected, so we define the graph metric d on X_3 as follows: for $x,y \in X_3$ we set
$$d(x,y) = \inf\{\sum_{i=1}^{n} w(e_i) : e_1 \ldots e_n \text{ is a path from } x \text{ to } y\}.$$

In case the groups and the index sets are finite we may replace the infimum above by minimum. It follows immediately from the definition that the natural action of G_3 on X_3 is a weighted graph automorphism, i.e. it preserves the edges including their weight. It follows that G_3 acts by isometries on X_3. We shall check that the canonical embeddings (inclusions) of X_1 and X_2 into X_3 are isometric.

We shall check it for both X_1 and X_2. Thus fix some $g,h \in G$ and $i,j \in I_3$ such that either both $g,h \in G_1$ and both $i,j \in I_1$, or both $g,h \in G_2$ and both $i,j \in I_2$. We need to check that $d_{X_l}(g\cdot x_i h\cdot x_j) = d(g\cdot x_i, h\cdot x_j)$, where $l \in \{1,2\}$ depending on whether $g,h \in G_1$, $i,j \in I_1$, or $g,h \in G_2$, $i,j \in I_2$. It is clear that $d_{X_l}(g\cdot x_i, h\cdot x_j) \geq d(g\cdot x_i, h\cdot x_j)$, so suppose there is a strict inequality and we shall reach a contradiction. There is then an edge-path $e_1 \ldots e_n$ from $x = g\cdot x_i$ to $y = h\cdot x_j$. By induction on n, the length of the path, we shall show that $d_{X_l}(g\cdot x_i, h\cdot x_j) \leq w(e_1) + \ldots + w(e_n)$. The case $n=1$ is clear, so we suppose that $n \geq 2$ and we have proved it for all paths of length strictly less than n between all pairs of elements from X_1 and all pairs of elements from X_2.

Now without loss of generality we suppose that $g,h \in G_1$, $i,j \in I_1$, the other case is analogous. For $1 \leq l \leq n$, let $z_l = g_l \cdot x_{i_l}$ be the start vertex of e_l and $z_{l+1} = g_{l+1}\cdot x_{i_l+1}$ the end vertex. Set $h_l = g_l^{-1} g_{l+1}$, for $1 \leq l \leq n$. It follows that $gh_1h_2\ldots h_n = h$ and each h_l belongs to either G_1 or G_2. If all the h_l's belong to G_1 then also all the i_l's belong to I_1 and the path goes within X_1 and we can use the triangle inequalities there. So we suppose that some h_l, $1 \leq l \leq n$, is from G_2; equivalently, that the path leaves X_1 at some point. Let $1 \leq l < n$ be the least index where the path leaves X_1, i.e. $z_l \in X_1$, while $z_{l+1} \notin X_1$. It follows that $i_l \in I_0$. Then let $l < l' \leq n$ be the least index such that the path returns back to X_1, i.e. the least index $l < l'$ such that $z_{l'} \in X_1$. Again necessarily $i_{l'} \in I_0$. If $1 < l$ or $l' < n$, then the subpath

$e_l \dots e_{l'-1}$ between two elements of X_1 is strictly shorter than n and thus by the inductive hypothesis we have $d_{X_1}(z_l, z_{l'}) \leq w(e_l)+\dots+w(e_{l'-1})$. So we may replace this subpath by a single edge going from z_l to $z_{l'}$, hereby again shortening the path, so by the inductive hypothesis we get $d_{X_1}(g \cdot x_i, h \cdot x_j) \leq w(e_1) + \dots + w(e_n)$.

Thus we are left with the case $l = 1$ and $l' = n$. In such a case we have $h_1 \in G_2$ and $h_n \in G_2$, $i_1, i_2, i_{n-1}, i_n \in I_0$, and also, since n is the least number l such that $h_1 \dots h_l \in G_1$, we must actually have $h_1 \dots h_n \in G_0$. It follows that g and h lie in the same left-coset of G_0 in G_1, i.e. $g^{-1}h \in G_0$. It follows that $d(g \cdot x_i, h \cdot x_j) = d(x_i, g^{-1}h \cdot x_j)$. Thus it suffices to show that

$$d(x_i, g^{-1}h \cdot x_j) = d_{X_0}(x_i, g^{-1}h \cdot x_j) = d_{X_1}(x_i, g^{-1}h \cdot x_j),$$

where the latter equality is known and we need to show the former. In other words, we shall thus now, without loss of generality, assume that $g = 1$, so $h = h_1 \dots h_n \in G_0$ and $x_i, h \cdot x_j \in X_0$.

We have two cases:

(1) If $n = 2$, i.e. $h = h_1 h_2$, then the path $e_1 e_2$ is within X_2 between two elements from X_0. Therefore, by the triangle inequality in X_2, its length is greater or equal to the path consisting of a single edge from x_i to $h \cdot x_j$, that means we have

$$w(e_1) + w(e_2) = d_{X_2}(x_i, h_1 \cdot x_{i_2}) + d_{X_2}(h_1 \cdot x_{i_2}, h \cdot x_j) \geq$$

$$d_{X_2}(x_i, h \cdot x_j) = d_{X_0}(x_i, h \cdot x_j) = d_{X_1}(x_i, h \cdot x_j),$$

and we are done.

(2) If $n > 2$, then the non-trivial subpath $e_2 \dots e_{n-1}$ is a path of length strictly less than n between two elements from X_2 (note that $z_2 = h_1 \cdot x_{i_2} \in X_2$ and also $z_n = hh_n^{-1} \cdot x_{i_n} \in X_2$), thus by the inductive hypothesis we get that

$$w(e_2) + \dots + w(e_{n-1}) \geq d_{X_2}(z_2, z_n).$$

It follows that

$$\sum_{l=1}^{n} w(e_l) \geq d_{X_2}(x_i, z_2) + d_{X_2}(z_2, z_n) + d_{X_2}(z_n, h \cdot x_j) \geq$$

$$d_{X_2}(x_i, h \cdot x_j) = d_{X_0}(x_i, h \cdot x_j) = d_{X_1}(x_i, h \cdot x_j),$$

and we are again done.

□

Remark 2.4. The previous theorem was stated and proved for free actions. However, the proof can be modified to work for non-free actions as follows: Replace the metric by a pseudometric so that the action becomes free. Then proceed completely analogously working with pseudometrics instead of metrics and at the end make a metric quotient.

In the next theorem we shall restrict our attention to actions of finite groups on finite metric spaces.

Theorem 2.5. *The class of pointed free actions of finite groups on finite metric spaces has the amalgamation property*

Proof. Let us start as in the previous theorem with three pointed actions $(G_i \curvearrowright X_i, (x_j)_{j\in I_i})$, for $i \in \{0,1,2\}$ such that $G_0 \leq G_i$, $I_0 \subseteq I_i$, for $i = 1,2$, $I_0 = I_1 \cap I_2$. Now the difference is that all the sets are finite. Let G be now any amalgam group of G_1 and G_2 over G_0, e.g. the free product with amalgamation $G_1 *_{G_0} G_2$. Set $I_3 = I_1 \cup I_2$ and $X_G = \bigcup_{j\in I_3} G \cdot x_j$. As in the proof of Theorem 2.2 we define a weighted graph structure on X_G. That is, for $g, h \in G$ and $i, j \in I_3$, the elements $g \cdot x_i$ and $h \cdot x_j$ are connected by an edge if and only if

- either $g^{-1}h \in G_1$ and $i, j \in I_1$, then its weight is
$$w(g \cdot x_i, h \cdot x_j) = d_{X_1}(x_i, g^{-1}h \cdot x_j);$$
- or $g^{-1}h \in G_2$ and $i, j \in I_2$, then analogously its weight is
$$w(g \cdot x_i, h \cdot x_j) = d_{X_2}(x_i, g^{-1}h \cdot x_j).$$

There is no ambiguity when $g^{-1}h \in G_0$ and $i, j \in I_0$. We again define the graph metric as follows: for $x, y \in X_G$ we set

$$d_G(x,y) = \min\{\sum_{i=1}^{n} w(e_i) : e_1 \ldots e_n \text{ is a path from } x \text{ to } y\}.$$

Notice that now w assumes only finitely many values, so we may indeed use the minimum. Again, G acts on X_G by graph automorphisms preserving the weight function, thus also by isometries. In the proof of Theorem 2.2 we showed that d_G extends d_{X_1} and d_{X_2} in case $G = G_1 *_{G_0} G_2$. We shall now find a finite amalgam G with the same property. First set $G' = G_1 *_{G_0} G_2$.

Set $M = \max\{w(e) : e \text{ is an edge in } X_{G'}\}$ and $m = \min\{w(e) : w(e) \neq 0 \text{ and } e \text{ is an edge in } X_{G'}\}$. Set $K = \lceil \frac{M}{m} \rceil$. Consider the finite set $G_1 \cup G_2$ as the set of generators of G' and let $\lambda : G' \to [0,\infty)$ be the corresponding length function, i.e. the distance from the unit in G' in the Cayley graph of G' with $G_1 \cup G_2$ as the generating set. Since G' is an amalgam of finite groups, it is residually finite. Thus let G_3 be a finite group such that there is an onto homomorphism $\phi : G' \to G_3$ which is injective on the ball $\{g \in G' : \lambda(g) \leq K+1\}$. Clearly, G_1 and G_2 are subgroups of G_3 with the identified common subgroup G_0. Thus in particular, G_3 is a finite amalgamation of G_1 and G_2 over G_0. Moreover, we may suppose that $G_1 \cup G_2$ generates G_3. Let ρ be the length function on G_3 with respect to these generators. We have that ϕ is isometric with respect to λ and ρ on the ball $\{g \in G : \lambda(g) \leq K+1\}$

We now set X_3 to be the finite set $X_{G_3} = \bigcup_{j \in I_3} G_3 \cdot x_j$. We again consider X_0, X_1, X_2 to be subsets of X_3. We have a metric $d_{X_3} = d_{G_3}$ defined using the weight function. What remains to check is that the canonical inclusions of X_1, resp. X_2 into X_3 are isometric. We shall do it for X_1, for X_2 it is analogous. So take some $g, h \in G_1$ and $i, j \in I_1$. We must check that $d_{X_3}(g \cdot x_i, h \cdot x_j) = d_{X_1}(g \cdot x_i, h \cdot x_j)$. Again, it is clear that $d_{X_3}(g \cdot x_i, h \cdot x_j) \leq d_{X_1}(g \cdot x_i, h \cdot x_j)$; suppose that there is a strict inequality. It follows that there is a path $e_1 \ldots e_n$ from $g \cdot x_i$ to $h \cdot x_j$ such that $\sum_{l=1}^{n} w(e_l) < d_{X_1}(g \cdot x_i, h \cdot x_j)$. We claim that the length of the path n is less or equal to K. Suppose that $n > K$. Then since for every $1 \leq l \leq n$ we have $w(e_l) \geq m$, we get

$$\sum_{l=1}^{n} w(e_l) \geq n \cdot m > K \cdot n \geq M.$$

However, by assumption $d_{X_1}(g \cdot x_i, h \cdot x_j) \leq M$, a contradiction.

Now, it follows that the path $e_1 \ldots e_n$ lies within the finite set $\bigcup_{i \in I_3} \{g \in G_3 : \rho(g) \leq K+1\} \cdot x_i$. Since ϕ is isometric with respect to λ and ρ on the ball $\{g \in G : \lambda(g) \leq K+1\}$ it follows that the path $e_1 \ldots e_n$ from G_3 also exists in $X_{G'}$, and is, by definition, of the same length. However, we showed in the proof of Theorem 2.2 that in $X_{G'}$ its weight was greater or equal to $d_{X_1}(g \cdot x_i, h \cdot x_j)$. This finishes the proof. □

Let $(G_n \curvearrowright X_n, (x_i)_{i \in I_n})_{n \in \mathbb{N}}$ be an enumeration of all pointed free actions of finite groups on finite metric spaces with rational distances. It follows from the previous theorem that it is a Fraïssé class. Indeed, it is clear from the proof that when working with rational spaces the amalgam will be rational as well. Moreover, the joint-embedding property is just a special case of the amalgamation property (note that any two actions have a common subaction, namely the action of a trivial group on a one-point space). So it has some Fraïssé limit $(\alpha_0 : G \curvearrowright X, (x_i)_{i \in I})$, where G is some countably infinite locally finite group, X is a countably infinite rational metric space with countably infinite distinguished set of points $(x_i)_{i \in I}$ and $\alpha_0 : G \curvearrowright X$ is a free action by isometries.

It follows from the Fraïssé theorem that $(\alpha_0 : G \curvearrowright X, (x_i)_{i \in I})$ has the following extension property:

Fact 2.6. *Let $F \leq G$ be a finite subgroup, $A \subseteq I$ a finite subset, and denote by X_0 the finite metric space $\bigcup_{i\in A} F \cdot x_i$. Consider the free pointed action $(F \curvearrowright X_0, (x_i)_{i\in A})$. Let $(H \curvearrowright Y, (y_j)_{j\in B})$ be some free pointed action of a finite group on a finite rational metric space and let (ψ, ϕ) is an embedding from $(F \curvearrowright X_0, (x_i)_{i\in A})$ to $(H \curvearrowright Y, (y_j)_{j\in B})$. Then there exists an embedding $(\bar{\psi}, \bar{\phi})$ from $(H \curvearrowright Y, (y_j)_{j\in B})$ to $(G \curvearrowright X, (x_i)_{i\in I})$ such that $\bar{\psi} \circ \psi = \mathrm{id}_F$ and $\bar{\phi} \circ \phi = \mathrm{id}_{X_0}$.*

Now let $\mathbb{X}$ be the metric completion of X. The action $\alpha_0 : G \curvearrowright X$ obviously extends to the action $\alpha : G \curvearrowright \mathbb{X}$ by isometries, which is no longer free though.

Theorem 2.7. *The action $\alpha : G \curvearrowright \mathbb{X}$ is a universal action of a countable locally finite group on a separable metric space by isometries.*

Remark 2.8. That means that for any countable locally finite group H and any action $\beta : H \curvearrowright Y$ by isometries, where Y is a separable metric space, there is a subgroup $H' \leq G$ isomorphic to H and a subspace $Y' \subseteq \mathbb{X}$ isometric to Y such that the restriction of α to H' has Y' as an invariant subspace and this restriction is isometric to the action β; in other words, if we identify H and H', the isometric embedding of Y onto $Y' \subseteq \mathbb{X}$ is H-equivariant.

Before we prove the theorem we shall need few notions and lemmas.

Definition 2.9. Let X be a set equipped with two pseudometrics d and p. We define the distance $D(d, p)$ between these two pseudometrics as their supremum distance, i.e.

$$D(d, p) = \sup_{x,y\in X} |d(x, y) - p(x, y)|.$$

Lemma 2.10. *Let $(H \curvearrowright X, (x_i)_{i\in I})$ be a free pointed action by isometries of some finite group H on a finite* pseudometric *space $X = \bigcup_{i\in I} H \cdot x_i$ with pseudometric d. Then for any $\varepsilon > 0$ there exists a rational* metric *p on X such that the free action of H on (X, p) is still by isometries and $D(d, p) < \varepsilon$.*

Proof of Lemma 2.10. Enumerate by $(d_i)_{i\leq n}$ the distances from (X, d) in an increasing order. Also, we may suppose that $\varepsilon < \min\{|k - l| : k \neq l, k, l \in \{d_i : i \leq n\} \cup \{0\}\}$.

For $i \leq n$, let p_i be an arbitrary rational number from the open interval $(d_i + \frac{(n-i)\varepsilon}{n+1}, d_i + \frac{(n+1-i)\varepsilon}{n+1})$. Now for a pair $x, y \in X$ set $p(x, y) = 0$ if $x = y$ and for $x \neq y \in X$ set

$$p(x, y) = p_i \text{ iff } d(x, y) = d_i.$$

Let us check that p is a rational metric. By definition it is rational. It is clear that $p(x,y)=0$ iff $x=y$, and that it is symmetric, so we must just check the triangle inequality. Take a triple $x,y,z\in X$. We check that $p(x,z)\leq p(x,y)+p(y,z)$. If either $d(x,y)$ or $d(y,z)$ is bigger or equal to $d(x,z)$, then the same is true for $p(x,y)$, $p(y,z)$, $p(x,z)$ by definition. So we may suppose that $d(x,z)>\max\{d(x,y),d(y,z)\}$. Then by setting $d(x,z)=d_i$, $d(x,y)=d_j$ and $d(y,z)=d_k$, we have that $i>\max\{j,k\}$. We must check that $p_i\leq p_j+p_k$. However, we have

$$p_i\leq d_i+\frac{(n+1-i)\varepsilon}{n+1}\leq d_j+\frac{(n-j)\varepsilon}{n+1}+d_k+\frac{(n-k)\varepsilon}{n+1}\leq p_j+p_k,$$

and we are done. □

Lemma 2.11. *Let $H_1\leq H_2$ be two finite groups and $I\subseteq J$ two finite sets. Let d be a metric on $X=\bigcup_{i\in I}H_1\cdot x_i$ and p be a metric on $Y=\bigcup_{j\in J}H_2\cdot x_j\supseteq X$ such that the canonical corresponding actions are by isometries. Suppose further that $D(d,p\restriction X)\leq\varepsilon$. Then there exists a metric ρ on Z, the disjoint union $X\subseteq\bigcup_{i\in I}H_2\cdot x_i\coprod\bigcup_{j\in J}H_2\cdot x_j=Y$ which is equal to $\bigcup_{i\in I}H_2\cdot x_i\cup\bigcup_{j\in J}H_2\cdot y_j$ such that*

- *ρ extends both d and p on the corresponding subspaces,*
- *for every $i\in I$, $\rho(x_i,y_i)\leq\varepsilon$,*
- *the canonical action of H_2 on Z is by isometries.*

Proof of Lemma 2.11. As before, we define a weighted graph structure on Z. A pair x,y is connected by an edge if and only if

- either $x,y\in X$, resp. $x,y\in Y$, in such a case $w(x,y)=d(x,y)$, resp. $w(x,y)=p(x,y)$;
- or there are $i\in I\subseteq J$ and $h\in H_2$ such that $x=h\cdot x_i$ and $y=h\cdot y_i$ or vice versa, in such a case we set $w(x,y)=\varepsilon$;
- or $x=g\cdot x_i$, $y=h\cdot x_j$ such that $i\in I$ and $g^{-1}h\in H_1$; in such a case we set $w(x,y)=d(x_i,g^{-1}hx_j)$.

It is again immediate that the graph is connected, thus it determines a metric ρ on Z, and the canonical action of H_2 on Z is by isometries. We need to check that ρ extends d and p. We check both simultaneously.

Fix x,y such that either $x,y\in X$ or $x,y\in Y$. Suppose that $\rho(x,y)<d(x,y)$ (it is again clear that $\rho(x,y)\leq d(x,y)$), resp. $\rho(x,y)<p(x,y)$ depending on where x,y lie. Then there is an edge path $e_1\ldots e_n$ such that $w(e_1)+\ldots+w(e_n)<d(x,y)$, resp. $w(e_1)+\ldots+w(e_n)<p(x,y)$. We shall again prove the claim by induction on the length of the edge path. The case $n=1$ is clear. Suppose we have proved it for all $l<n$ and all edge paths of length at most l between all pairs $x,y\in X$ and

all pairs $x, y \in Y$. We may suppose that there are not two neighboring edges e_i and e_{i+1} such that both of them lie in X or both of them lie in Y, for otherwise we could contract them into a single edge using triangle inequality in X, resp. Y.

Suppose first that $x, y \in X$ and let $x = g \cdot x_i$ and $y = h \cdot x_j$, for some $g, h \in H_1$ and $i, j \in I$. Denote by $\bar{X}$ the set $\bigcup_{i \in I} H_2 \cdot x_i \supseteq X$. Notice that there is no edge between an element $z \in X$ and an element $z' \in \bar{X} \setminus X$. Thus we may suppose that e_1 is an edge between $x = g \cdot x_i$ and $g \cdot y_i$ and e_n is an edge between $h \cdot y_j$ and $h \cdot x_j = y$. Indeed, otherwise either e_1 is an edge within X, so we may use the inductive assumption for the subpath $e_2, \ldots, e_n$, or e_n is an edge within X and we may use the inductive assumption for the subpath $e_1, \ldots, e_{n-1}$. It follows that $e_2, \ldots, e_{n-1}$ is an edge path of length strictly less than n between two elements of Y, thus by inductive assumption we may suppose that $n = 3$ and e_2 is an edge between $g \cdot y_i$ and $h \cdot y_j$ and we have

$$w(g \cdot y_i, h \cdot y_j) = p(g \cdot y_i, h \cdot y_j) \geq d(g \cdot x_i, h \cdot x_j) - \varepsilon.$$

However, since $w(e_1) = w(e_3) = \varepsilon$, we get that

$$d(x, y) = d(g \cdot x_i, h \cdot x_j) < w(e_1) + w(e_2) + w(e_3),$$

a contradiction.

Suppose now that $x, y \in Y$ and again let $x = g \cdot y_i$ and $y = h \cdot y_j$, for some $g, h \in H_2$ and $i, j \in J$. As in the paragraph above, we may without loss of generality assume that e_1 is an edge between $x = g \cdot y_i$ and $g \cdot x_i$ and e_n is an edge between $h \cdot x_j$ and $h \cdot y_j = y$; thus in particular $i, j \in I$. If both $g, h \in H_1$ then $g \cdot x_i, h \cdot x_j \in X$ and we are done by the same argument as in the paragraph above. So suppose that at least one of g, h is in $H_2 \setminus H_1$. Say $g \in H_2 \setminus H_1$, i.e. $g \cdot x_i \in \bar{X} \setminus X$. Since there is no edge between an element from X and an element from $\bar{X} \setminus X$ there exists a minimal $l \leq n$ such that $e_2, \ldots, e_{l-1}$ is a path within $\bar{X} \setminus X$ and e_l is an edge between an element from $\bar{X} \setminus X$ and an element from Y. If $l < n$ then we use the inductive hypothesis, so suppose that $l = n$, i.e. the subpath $e_2, \ldots, e_{n-1}$ is within $\bar{X} \setminus X$. Note also that there is an edge between elements $f \cdot x_k$ and $f' \cdot x_{k'}$ in $\bar{X} \setminus X$ if and only if $f^{-1} f' \in H_1$. It follows that $g^{-1} h \in H_1$. Translating the whole path $e_1, \ldots, e_n$ by g^{-1} we does not change the distance (g^{-1} acts as an isometry). Thus we may assume that $g = 1$ and $h \in H_1$. However, then we are again done by an argument used above. $\square$

Proof of Theorem 2.7. Let $H \curvearrowright Z$ be an action of an infinite locally finite group by isometries on a separable metric space. It is sufficient to prove the theorem in case Z is countable. Indeed, in the general case we would find a countable dense H-invariant subspace Z'. Then the action on the metric completion of Z', thus in particular on Z, is uniquely determined by its behavior on Z'. Since the space $\mathbb{X}$ is complete, we are done.

So assume that both H and Z are countable. Without loss of generality we assume that Z has infinitely many H-orbits and let $(z_n)_{n\in\mathbb{N}}$ be a sequence which picks one single element from each orbit, i.e. we may write the metric space Z as $\bigcup_{n\in\mathbb{N}} H \cdot z_n$ with a pseudometric d. Also, without loss of generality we shall assume that H is infinite and write H as $H_1 \leq H_2 \leq H_3 \leq \ldots$ which is an increasing chain of finite subgroups of H whose union is H. Moreover, for every n define Z_n to be the finite pseudometric subspace $\bigcup_{i\leq n} H_n \cdot z_i \subseteq Z$.

For every n consider the free pointed action $(H_n \curvearrowright Z_n, (z_i)_{i\leq n})$. By Lemma 2.10 there exists a rational metric p_n on Z_n such that $D(d \restriction Z_n, p_n) < 1/2^{n+1}$. It follows that for every n we have $D(p_n, p_{n+1} \restriction Z_n) < 1/2^n$. By Lemma 2.11, for every n we can define a rational metric ρ_n on a disjoint union of $Z_n \coprod Z_{n+1} = \bigcup_{i\leq n} H_n \cdot z_i \cup \bigcup_{j\leq n+1} H_{n+1} \cdot z'_j$ which extends the original metrics and for $i \leq n$ we have $\rho_n(z_i, z'_i) = 1/2^n$. Now by a successive application of the extension property of $(G \curvearrowright X, (x_{i\in I})$ we obtain

- an increasing chain of finite subgroups $H'_1 \leq H'_2 \leq \ldots \leq G$ such that $H'_i \cong H_i$ for $i \in \mathbb{N}$, and thus also $H' = \bigcup_n H'_n \cong H$;
- isometric embeddings $\phi_n : Z_n \coprod Z_{n+1} \hookrightarrow X$ such that $\phi_n \restriction Z_{n+1} = \phi_{n+1} \restriction Z_{n+1}$, for every n;
- for every n, we have that the free actions $H_n \curvearrowright Z_n$ and $H'_n \curvearrowright \phi_n[Z_n]$ are isometric.

For every i we have that the sequence $(\phi_n(z_i))_{n\geq i}$ is Cauchy, since $d_X(\phi_n(z_i), \phi_{n+1}(z_i)) = 1/2^n$. Let $y_i \in \mathbb{X}$ be the limit of that sequence. Consider the subset $Z' = \bigcup_{i\in\mathbb{N}} H' \cdot y_i \subseteq \mathbb{X}$. It follows it is naturally isometric to Z. Indeed, take any $x, y \in Z$ and write them as $x = h \cdot z_i$ and $y = g \cdot z_j$ for some $h, g \in H$ and $i, j \in \mathbb{N}$. Since H and $H' \leq G$ are isomorphic, let h', g' be the corresponding elements of $H' \leq G$ and consider the elements $h' \cdot y_i, g' \cdot y_j \in Z' \subseteq \mathbb{X}$. Then

$$d(h' \cdot y_i, g' \cdot y_j) = \lim_n d(h' \cdot \phi_n(z_i), g' \cdot \phi_n(z_j)) = \lim_n d_Z(h \cdot z_i, g \cdot z_j) + o(n),$$

where $o(n) \in [0, 1/2^n]$, so the claim is proved.

Finally, consider the restriction of the action $G \curvearrowright \mathbb{X}$ on $H' \curvearrowright \bigcup_{i\in\mathbb{N}} H' \cdot y_i$. It follows from the approximation above that it is isometric to the action $H \curvearrowright Z$, and we are done. $\square$

Finally, we show that the group G is isomorphic to the Hall's universal locally finite group and that the space X is isometric to the rational

Urysohn space, so the completion $\mathbb{X}$ is isometric to the Urysohn universal space. It is just the use of the extension property of $G \curvearrowright X, (x_i)_{i\in I}$ from Fact 2.6. These are standard arguments, so we omit some details.

For the former it is necessary to show that G has the extension property. That is, whenever $F \leq G$ is some finite subgroup and $H \geq F$ is some abstract finite supergroup of F, i.e. a supergroup of F that does not in principle lie in G, then we can actually find a copy H' of H within G so that it is a supergroup of F there, i.e. $F \leq H' \leq G$.

So pick some finite subgroup $F \leq G$ and some abstract supergroup $H \geq F$. The Fraïssé limit $G \curvearrowright X, (x_i)_{i\in I}$ is a direct limit of a sequence of some finite actions $(G_n \curvearrowright X_n, (x_i)_{i\in I_n})_{n\in\mathbb{N}}$. Take n so that $F \leq G_n$ and consider the subaction $F \curvearrowright X'_n, (x_i)_{i\in I_n}$, where $X'_n = \bigcup_{i\in I_n} F \cdot x_i$. It is possible to use Lemma 2.11 to extend this action to an action of H on $\bigcup_{i\in I_n} H \cdot x_i$. Then we use the extension property of $G \curvearrowright X, (x_i)_{i\in I}$ to find the action $H \curvearrowright \bigcup_{i\in I_n} H \cdot x_i$ within the universal one, thus in particular to find a copy of H within G that is a supergroup of F.

Now for the latter, it is necessary to show that the countable rational metric space X has the extension property. That is, whenever $A \subseteq X$ is some finite subspace and $A \subseteq B$ is finite abstract extension, still a rational metric space, then we can actually find this extension within X. So take some finite $A \subseteq X$. As above, find some n so that $A \subseteq X_n$. By extending the metric by metric amalgamation if necessary we may assume that $A = X_n$. Set $I'_n = I \cup (B \setminus X_n)$ and $X'_n = \bigcup_{i\in I'_n} G_n \cdot x_i$. Clearly, $A = X_n \subseteq B \subseteq X'_n$. By using the technique with defining a weighted graph structure on X'_n we can extend the metric from B to X'_n so that G_n acts on X'_n by isometries. Then we use the extension property of $G \curvearrowright X, (x_i)_{i\in I}$ to get a copy of X'_n, thus also of B, in X so that it is an extension of A there.

2.1. **No universal actions for non-locally finite groups.** We stress that the universality of G from the previous theorem does <u>not</u> say that whenever we have an action of the group G itself on some separable metric space Z by isometries, then we can find a G-equivariant isometric embedding of Z into $\mathbb{X}$. What it does say is that we can find a subgroup $G' \leq G$ isomorphic to G itself, so that after identification of G and G' there is some G-equivariant isometric embedding of Z into $\mathbb{X}$. This is not a flaw of the proof. The universality in such a strong sense is not possible for any countably infinite group G, locally finite or not. That is the content of the following theorem.

Theorem 2.12. *Let G be a countably infinite group. Then there is no such strongly universal action of G on both metric and Banach spaces.*

Proof. Suppose the contrary, first for the metric spaces. That is, suppose there exists a separable metric space X and an action $\alpha : G \curvearrowright X$ by isometries such that for any action $\beta : G \curvearrowright Y$ by isometries on a separable metric space Y there is a G-equivariant isometric embedding of Y into X.

Fix some unbounded length function $\lambda : G \to \mathbb{N}$ and a sequence $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that $(g_n)_n$ generate G and $1 < \lambda(g_1) < \lambda(g_2) < \lambda(g_3) < \ldots$. That is possible to find for any countably infinite group.

For any $x \in 2^{\mathbb{N}}$ let $\lambda'_x : \{g_n, g_n^{-1} : n \in \mathbb{N}\} \to \mathbb{N}$ be defined as follows:

$$\lambda'_x(g) = \begin{cases} 1 & g \in \{g_n, g_n^{-1}\} \wedge x(n) = 0, \\ 2 & g \in \{g_n, g_n^{-1}\} \wedge x(n) = 1. \end{cases}$$

Finally, for any $x \in 2^{\mathbb{N}}$ define a length function $\lambda_x : G \to \mathbb{N}$ as follows: for any $g \in G$, set

$$\lambda_x(g) = \min\{\sum_{i=1}^{m} \lambda'_x(h_i) : g = h_1 \ldots h_m, (h_i)_{i=1}^m \subseteq S \cup \{g_n, g_n^{-1} : n \in \mathbb{N}\}\}.$$

We claim that λ_x extends λ'_x, i.e. for every $g \in \{g_n, g_n^{-1} : n \in \mathbb{N}\}$, $\lambda_x(g) = \lambda'_x(g)$. It suffices to show that for any n such that $x(n) = 1$ we have $\lambda_x(g_n) = 2$. Suppose the contrary. Then necessarily $\lambda_x(g_n) = 1$, so by definition $g_n = g_m$ or $g_n = g_m^{-1}$ for m such that $x(m) = 0$. However, that is in contradiction with the assumption that $1 < \lambda(g_1) < \lambda(g_2) < \lambda(g_3) < \ldots$.

Now for every $x \in 2^{\mathbb{N}}$ take the left-invariant metric d_x on G induced by λ_x. The action of G on itself by left translations is then an action of G on (G, d_x) by isometries. We claim that there is $x \in 2^{\mathbb{N}}$ such that there is no G-equivariant isometric embedding of (G, d_x) into X. Suppose otherwise that for every $x \in 2^{\mathbb{N}}$ there is a G-equivariant isometric embedding ι_x of (G, d_x) into X. For every $x \in 2^{\mathbb{N}}$ denote $\iota_x(1_G) \in X$ by z_x. Then for $x \neq y \in 2^{\mathbb{N}}$ we have $d_X(z_x, z_y) \geq 1/2$, for if $d_X(z_x, z_y) < 1/2$ and $n \in \mathbb{N}$ is such that $x(n) \neq y(n)$, say $x(n) = 1$, $y(n) = 0$, then

$$1/2 > d_X(z_x, z_y) = d_X(g_n \cdot z_x, g_n \cdot z_y) \geq$$
$$|d_X(g_n \cdot z_x, z_x) - d_X(z_x, z_y) - d_X(z_y, g_n \cdot z_y)| > 1/2,$$

a contradiction. Thus we get that $\{z_x : x \in 2^{\mathbb{N}}\} \subseteq X$ is a 1/2-separated uncountable set in X which contradicts the separability of X.

To prove the same for the category of Banach spaces, we can for example extend the action of G on (G, d_x), for every $x \in 2^{\mathbb{N}}$, to an action of G on the Lipschitz-free Banach space $F(G,d_x)$ over (G,d_x) (see [5] and [17] for information about Lipschitz-free Banach spaces). That is, consider a real vector space V_G with $G \setminus \{1_G\}$ as the free basis, and 1_G as a zero. Define a norm $\|\cdot\|_x$ on V_G as follows: for $v = \alpha_1 g_1 + \ldots + \alpha_n g_n$ set

$$\|v\|_x = \min\{\sum_{i=1}^{m} |\beta_i| \cdot d_x(h_i, h_i') : v = \sum_{i=1}^{m} \beta_i(h_i - h_i')\}.$$

Then it is easy to check (and it is a standard fact about Lipschitz-free spaces) that for any $g,h \in G$, $\|g-h\|_x = d_x(g,h)$. G acts by (affine) isometries on $(V_G, \|\cdot\|_x)$ in the following way: for $h \in G$ and $\alpha_1 g_1 + \ldots + \alpha_n g_n \in V_G$ we set $h \cdot (\alpha_1 g_1 + \ldots + \alpha_n g_n) = (\alpha_1 h g_1 + \ldots + \alpha_n h g_n) - (\alpha_1 + \ldots + \alpha_n - 1)h$. It is easy to check that this gives an action of G on $(V_G, \|\cdot\|_x)$ by isometries which extends the action of G on itself by translation. It also extends to an action of G on the completion W_x. Then arguing exactly the same as with the metric space one can show that it is not possible to embed in a G-equivariant way all the spaces $W_x, x \in 2^{\mathbb{N}}$, into a single separable Banach space with an action of G. $\square$

Second, we show that the universality result for the Hall's group from Theorem 2.7 is not beyond the locally finite case. Let us start with a definition.

Definition 2.13. Let G be a countable group. Say that G *admits a universal action by isometries* if there is an action $\alpha : G \curvearrowright X$ on some separable metric space X such that for any countable group H which is isomorphic to a subgroup of G and for any action $\beta : H \curvearrowright Y$ of H on a separable metric space Y by isometries there is a subgroup $H' \leq G$ isomorphic to H and an isometric embedding of Y into X which is, after identifying H and H', H-equivariant.

With this definition, the statement of Theorem 2.7 says that the Hall's group admits a universal action by isometries.

That is the strongest result possible of this type as shown by the following proposition.

Proposition 2.14. *Let G be a countably infinite non-locally finite group. Then G does not admit a universal action by isometries.*

Proof. Suppose there is such an action $\alpha : G \curvearrowright X$ on some separable metric space X. Since G is not locally finite it contains a finitely generated infinite subgroup $H \leq G$. By the proof of Theorem 2.12 there are continuum many somewhat different left-invariant metrics $(d_x)_{x \in 2^{\mathbb{N}}}$

on H. Note that G contains at most countably many subgroups isomorphic to H. By the pigeonhole principle there is one fixed subgroup $H' \leq G$ isomorphic to G and an uncountable subset $I \subseteq 2^{\mathbb{N}}$ such that for each $x \in I$ there is an H'-equivariant isometric embedding of (H', d_x) into X. We reach a contradiction with separability of X by the same argument as in the proof of Theorem 2.12. $\square$

3. Genericity of the action

Let us start with a general discussion. Fix some countable group G. Let X be some Polish metric space, i.e. a complete separable metric space. We want to define a space of all actions of G on X by isometries.

We have that $\mathrm{Iso}(X)$, the group of all isometries on X with the pointwise-convergence, or equivalently compact-open, topology is a Polish group, i.e. a completely metrizable second-countable topological group. Fixing a countable dense subset $\{x_i : i \in \mathbb{N}\} \subseteq X$ we may define a compatible complete metric ρ on $\mathrm{Iso}(X)$ as follows: for $\phi, \psi \in \mathrm{Iso}(X)$ we set

$$\rho(\phi, \psi) = \sum_{i=1}^{\infty} \frac{\min\{d_X(\phi(x_i), \psi(x_i)), 1\}}{2^i}.$$

Since every action $\alpha : G \curvearrowright X$ by isometries is in unique correspondence with some homomorphism $f : G \to \mathrm{Iso}(X)$, we may define the space $\mathrm{Act}_G(X)$ of all actions of G on X by isometries as the space of all homomorphisms of G into $\mathrm{Iso}(X)$. $\mathrm{Act}_G(X)$ is a closed subspace of the product space $\mathrm{Iso}(X)^G$, thus a Polish space.

There has been a recent research on investigating which countable groups admit actions, on certain spaces, which are generic in some sense. That means, fix a countable group G and a Polish metric space X. Note that there is a natural equivalence relation on the space $\mathrm{Act}_G(X)$, that is of conjugation, where two homomorphisms $f, g : G \to \mathrm{Iso}(X)$ are conjugate if there exists an autoisometry $\phi : X \to X$ such that $f = \phi^{-1} g \phi$. Say that an action f (or rather a homomorphism) is *generic* if some element $f \in \mathrm{Act}_G(X)$ has a comeager conjugacy class. In [15] Rosendal proved that every finitely generated group with the Ribes-Zalesskiĭ property ([14]), i.e. products of finitely generated subgroups are closed in the profinite topology of the group, has a generic action on the rational Urysohn space. More recently, Glasner, Kitroser and Melleray ([4]) characterized those countable groups that have generic actions on a countable set with a trivial metric (attaining just values 1 and 0).

We prove a genericity result for the universal action from Theorem 2.7. On the one hand we have the result on a much more complicated space, the Urysohn space, on the other hand we need to somehow weaken the definition of genericity. In other words, we need to extend the equivalence relation of being conjugate by also allowing group automorphisms. Let us state that precisely in the following definition.

Definition 3.1. Let G be a countable group and X a Polish metric space. Say that two homomorphisms $f, g : G \to \mathrm{Iso}(X)$ are *weakly equivalent* if there exist an autoisometry $\phi : X \to X$ and an automorphism $\psi : G \to G$ such that for all $x \in X$ and $v \in G$ we have

$$f(v)x = \phi^{-1} g(\psi(v))\phi x.$$

Moreover, we say that an element $f \in \mathrm{Act}_G(X)$ is *weakly generic* if it has a comeager equivalence class in the weak equivalence.

We shall prove the following.

Theorem 3.2. *The universal action $\alpha \curvearrowright \mathbb{U}$ from Theorem 2.7 is weakly generic.*

The rest of this section is devoted to prove the theorem.

We need some notions. When F and F' are two isomorphic finite groups, I and I' two finite bijective sets and d, resp. d' a metric on $\bigcup_{i\in I} G\cdot x_i$, resp. on $\bigcup_{i\in I'} G'\cdot y_i$, we denote by $D((G, \{x_i : i \in I\}, d), (G', \{y_i : i \in I'\}, d'))$, analogously as in the previous section, the supremum distance $\sup_{i,j\in I, g,h\in H} |d(g\cdot x_i, h\cdot x_j) - d'(g'\cdot y_{i'}, h'\cdot y_{j'})|$, where $g', h' \in G'$ are the images of $g, h \in G$ under the given isomorphism between G and G' and $i', j' \in I'$ are the images of $i, j \in I$ under the given bijection between I and I'. Such an isomorphism and a bijection will be never explicitly mentioned, it should be always clear from the context. Also, we shall often write $D((G, \{x_i : i \in I\}), (G', \{y_i : i \in I'\}))$, thus suppressing the metrics from the notation; they should also be clear from the context. The following fact follows from Lemma 2.11, however we will state it here since it will be used extensively.

Fact 3.3. *Suppose we are given two finite isomorphic groups G and G', finite bijective sets I and I', and metrics d and d' on $\bigcup_{i\in I} G\cdot x_i$, resp. on $\bigcup_{i\in I'} G'\cdot y_i$. Suppose moreover that $D((G, \{x_i : i \in I\}), (G', \{y_i : i \in I'\})) < \varepsilon$ for some $\varepsilon > 0$. Then there exists a metric ρ on $\bigcup_{i\in I\cup I'} G\cdot x_i$ such that*

- *$D((G, \{x_i : i \in I\}, d), (G, \{x_i : i \in I\}, \rho)) = 0$, i.e. ρ extends d;*
- *$D((G, \{x_i : i \in I'\}, \rho), (G', \{y_i : i \in I'\}, d')) = 0$;*
- *for every $i \in I$ we have $\rho(x_i, x_{i'}) \leq \varepsilon$.*

Remark 3.4. Conversely, notice suppose that a finite group G acts freely on some metric space Y and let $\{y_i : i \in I\}$ and $\{z_i : i \in I\}$ be two finite subsets of Y indexed by the same set such that for every $i \in I$, $d_Y(y_i, z_i) < \varepsilon$. Then $D((G, \{y_i : i \in I\}), (G, \{z_i : i \in I\})) < 2\varepsilon$.

We shall now define a subset of $\mathrm{Act}_G(\mathbb{U})$. By $\mathbb{QU}$ we denote the rational Urysohn space, a countable dense subset of $\mathbb{U}$. We shall denote the pointed free rational actions of finite groups by $(F, \{x_i :\in I\})$ and we write $(F, \{x_i : i \in I\}) \leq (H, \{x_i : i \in I'\})$ to denote that the former actions embeds into the latter. To simplify the notation, we always assume in such a case that $F \leq H$ and $I \subseteq I'$. Recall that the class $\mathcal{K}$ of all pointed free rational actions by finite groups is countable.

By $\mathbb{D}$ we denote the subset of $\mathrm{Act}_G(\mathbb{U})$ of all actions $G \curvearrowright \mathbb{U}$ satisfying:

for all $\varepsilon > \varepsilon' > 0$, for all $(F, \{x_i : i \in I\}) \leq (H, \{x_i : i \in I'\}) \in \mathcal{K}$ and for every subgroup $F' \leq G$ isomorphic to F and all $\{u_i : i \in I\} \subseteq \mathbb{QU}$ such that

$$D((G, \{x_i : i \in I\}), (G', \{u_i : i \in I\}) < \varepsilon'$$

there exist a subgroup $F' \leq H' \leq G$ isomorphic to H, and points $\{u_i : i \in I'\} \subseteq \mathbb{QU}$ such that

$$D((H, \{x_i : i \in I'\}), (H', \{u_i : i \in I'\})) < \varepsilon.$$

We shall refer to the property above as D-property. A simple computation shows that the D-property is a G_δ condition, i.e. $\mathbb{D}$ is a G_δ set. It is non-empty since the universal action from Theorem 2.7 clearly belongs to $\mathbb{D}$. Moreover, a standard argument following from the construction of the universal action gives that $\mathbb{D}$ is actually dense, so dense G_δ. We need to show that any two actions from $\mathbb{D}$ are weakly equivalent. That is, for actions $\alpha, \beta \in \mathbb{D}$ we need to find an automorphism $\phi : G \to G$ and an autoisometry of $\psi : \mathbb{U} \to \mathbb{U}$ such that for all $g \in G$ and $x \in \mathbb{U}$ we have

$$\alpha(g, x) = \beta(\phi(g), \psi(x)).$$

We now fix two actions $\alpha, \beta \in \mathbb{D}$ and show that. Let $(z_n)_{n \in \mathbb{N}}$ be some enumeration of $\mathbb{QU}$ such that for each $i_0 \in \mathbb{N}$ both sets $\{z_i : i \geq i_0, i \text{ is odd}\}$ and $\{z_i : i \geq i_0, i \text{ is even}\}$ are dense in $\mathbb{U}$. Also, write G as an increasing union $G_1 \leq G_2 \leq G_3 \leq \ldots$ of finite subgroups.

By induction, we shall find for each $n \in \mathbb{N}$:

- an increasing sequence of finite groups $H_1 \leq \ldots \leq H_n \leq G$ and $H'_1 \leq \ldots \leq H'_n$ such that for each $i \leq n$, H_i and H'_i are isomorphic by some ϕ_i and $\phi_i \supseteq \phi_{i-1}$, and for every odd $i \leq n$

we have that $G_i \leq H_i$, and for every even $i \leq n$ we have that $G_i \leq H'_i$;
- for each $i \leq n$, sequences $(u_i^j)_{j=i}^n \subseteq \mathbb{QU}$ and $(v_i^j)_{j=i}^n \subseteq \mathbb{QU}$ such that
  - for each $i \leq n$ and $i \leq j < k \leq n$, $d(u_i^j, u_i^k) \leq 1/2^{j+1}$ and $d(v_i^j, v_i^k) \leq 1/2^{j+1}$,
  - for every odd $i \leq n$, $u_i^i = z_i$, and for every even $i \leq n$, $v_i^i = z_i$;
- $D((H_n, \{u_i^n : i \leq n\}), (H'_n, \{v_i^n : i \leq n\})) < 1/2^{n+1}$.

Once the induction is finished, we have that $G = \bigcup_n H_n = \bigcup_n H'_n$, i.e $\phi = \bigcup_n \phi_n : G \to G$ is an isomorphism. Also we have that for every $i \leq n$ the sequences $(u_i^n)_n$ and $(v_i^n)_n$ are Cauchy in $\mathbb{QU}$, thus they have some limit $u_i \in \mathbb{U}$, resp. $v_i \in \mathbb{U}$. It follows from the inductive assumption that both $\{u_i : i \in \mathbb{N}\}$ and $\{v_i : i \in \mathbb{N}\}$ are dense in $\mathbb{U}$ and that the map sending u_i to v_i is an isometry which extends to an autoisometry ψ of $\mathbb{U}$. By the limit argument we get that the actions α and β are weakly equivalent witnessed by ϕ and ψ. Thus we need to describe the inductive steps to finish the proof.

The first and second step of the induction. Set $H_1 = G_1$, $u_1^1 = z_1$. By Lemma 2.10 there exists $(H_1, \{x_1\}) \in \mathcal{K}$ such that $D((H_1, \{u_1^1\}), (H_1, \{x_1\}) < 1/2$. Since β satisfies the D-property, there is a subgroup $H'_1 \leq G$ isomorphic to H_1 (via some ϕ_1) and some v_1^1 which, because of Remark 3.4 we may find in $\mathbb{QU}$, such that $D((H_1, \{x_1\}), (H'_1, \{v_1^1\}) < 1/2$, thus by triangle inequality $D((H_1, \{u_1^1\}), (H'_1, \{v_1^1\})) < 1$. That finishes the first step of the induction.

Next, set $v_1^2 = v_1^1$ and $v_2^2 = z_2$. Let $H'_2 \leq G$ be an arbitrary finite group containing both H'_1 and G_2, e.g. the subgroup generated by these two groups. Again by Lemma 2.10 there exists $(H'_2, \{x_1, x_2\}) \in \mathcal{K}$ such that $D((H'_2, \{v_1^2, v_2^2\}), (H'_2, \{x_1, x_2\}) < 1/4$. By the D-property of α, Fact 3.3 and also Remark 3.4 we can find $H_1 \leq H_2 \leq G$ isomorphic to H'_2 (via some ϕ_2 extending ϕ_1) and $u_1^2, u_2^2 \in \mathbb{QU}$ such that $d(u_1^1, u_1^2) < 1/2$ and $D((H_2, \{u_1^2, u_2^2\}), (H'_2, \{v_1^2, v_2^2\})) < 1/2$. This finishes the second step of the induction.

The general odd and even step of the induction. The general steps are treated analogously as the second step of the induction. So we only briefly show the general odd n-th step of the induction, i.e. n is now odd greater than 2. For $i < n$ we set $u_i^n = u_i^{n-1}$ and we set $u_n^n = z_n$. Let $H_n \leq G$ be an arbitrary finite subgroup containing both H_{n-1} and G_n. By Lemma 2.10 there exists $(H_n, \{x_1, \ldots, x_n\}) \in \mathcal{K}$ such that

$D((H_n,\{u_1^n,\dots,u_n^n\}),(H_n,\{x_1,\dots,x_n\})<1/2^n$. By the D-property of β, Fact 3.3 and also Remark 3.4 we can find $H'_{n-1}\leq H'_n\leq G$ isomorphic to H_n (via some ϕ_n extending ϕ_{n-1}) and $v_1^n,\dots,v_n^n\in\mathbb{QU}$ such that $d(v_i^{n-1},v_i^n)<1/2^{n-1}$, for all $i<n$, and $D((H_n,\{u_1^n,\dots,u_n^n\}),(H'_n,\{v_1^n,\dots,v_n^n\}))<1/2^{n-1}$. That finishes the inductive construction and the whole proof.

4. Universal actions on Banach spaces

In the last section we attempt to find some universal actions by isometries on Banach spaces.

First, we notice that in the full generality, as for metric spaces, it is not possible. Recall that by the theorem of Mazur and Ulam (see e.g. Theorem 14.1 in [2]) every (onto) isometry on a Banach space is affine, that means it is a linear isometry plus translation. From that, one can derive that every action $\alpha : G\curvearrowright X$ of some group G on a Banach space X by isometries is determined by an action $\alpha_0 : G\curvearrowright X$, which is by *linear* isometries, and by a *cocycle map* $b : G\to X$ which determines the corresponding translates. That is, for any $g\in G$ and $x\in X$ we have

$$\alpha(g)x=\alpha_0(g)x+b(g).$$

Conversely, whenever we have an action $\alpha_0 : G\curvearrowright X$ by linear isometries and a map $b : G\to X$ satisfying the so-called 'cocycle condition', i.e. for every $g,h\in G$, we have

$$b(gh)=\alpha_0(g)b(h)+b(g),$$

we can get an action of G on X by affine isometries. We refer the reader to Chapter 6 of [12] for more information.

We again provide some definitions of universal actions by isometries on Banach space. Theorem 2.12 excludes again universal actions in a strong sense, so we modify Definition 2.13 for actions on Banach spaces.

Definition 4.1. Say that G *admits a universal action by isometries* on a (universal) separable Banach space X if there is an action $\alpha : G\curvearrowright X$ such that for any countable group H which is isomorphic to a subgroup of G and for any action $\beta : H\curvearrowright Y$ of H on a separable Banach space Y by isometries there is a subgroup $H'\leq G$ isomorphic to H and a linear isometric embedding of Y into X which is, after identifying H and H', H-equivariant.

Proposition 4.2. *No finite or countable group G admits a universal action by (affine) isometries on a Banach space X.*

Proof. Fix an at most countable group G and suppose that it has a universal action on a Banach space X. Let $b : G\to X$ be the cocycle associated to this action. The range of b is at most countable, so in particular the set $R = \{\|b(g)\|_X : g \in G\}$ is an at most countable set of reals. One can easily find an action of some subgroup $H \leq G$ on some Banach space Y such that, if we denote by $b' : H \to Y$ the associated cocycle, we have that $\{\|b'(h)\|_Y : h \in H\} \nsubseteq R$. It follows the action was not universal. $\square$

A similar arguments show that the universality cannot be saved if one replaces the H-equivariant linear isometric embeddings in the definition by H-equivariant affine isometric embeddings or linear $(1+\varepsilon)$-isometric embeddings.

The distinguished point in Banach spaces, the zero, is what causes problems. If one is satisfied with universality for actions by linear isometries, the problems disappear. Still the proofs are much more technical than in the case of plain metric spaces without algebraic structure. We shall only prove that there is a universal action by linear isometries on Banach spaces of the group $\bigoplus_{n\in\mathbb{N}} \mathbb{Q}/\mathbb{Z}$. This is the infinite direct sum of all roots of unity and it is a universal abelian locally finite group. We replace the Urysohn space from Theorem 2.7 by a similar object in the category of Banach spaces, the Gurarij space. We refer the reader to [6] for more information about this space. We shall thus show:

Theorem 4.3. *Let $G = \bigoplus_{n\in\mathbb{N}} \mathbb{Q}/\mathbb{Z}$. There exists a universal action of G on the Gurarij space $\mathbb{X}$. That is, for any action $\beta : H \curvearrowright Y$ by linear isometries, where H is a countable abelian locally finite group and Y is a separable Banach space, there exists a subgroup $H' \leq G$ such that, after identifying H and H', there is an H-equivariant linear isometric embedding of Y into $\mathbb{X}$.*

Let F be a finite abelian group and I a non-empty finite set. By F_I we denote the finite set $F \times I = \{x_{g,i} : g \in F, i \in I\}$. Instead of $x_{0,i}$, where $i \in I$ and $0 \in F$ is the group zero, we may just write x_i. Consider now a finite-dimensional real vector space $E_{F,I}$ with F_I as a basis. The canonical action of F on F_I, where $g \cdot x_{f,i} = x_{gf,i}$ (resp. the permutation representation of F on F_I), extends to a linear action of F on $E_{F,I}$ (resp. the representation of F in $\mathrm{GL}(E_{F,I})$).

Now let $W \subseteq E_{F,I}$ be any finite subset satisfying:

- $0 \in W$; if $w \in W$, then $-w \in W$;
- for every $i \neq j \in I$, $x_i - x_j \in W$;
- for every $i \in I$, $g \in F$, $x_{g,i} \in W$;
- for any $g \in F$ and $w \in W$, $g \cdot w \in W$.

A partial F-norm $\|\cdot\|'$ on W is a partial norm on the finite set W compatible with the action of F; that is, a function satisfying

- $\|w\|' = 0$ iff $w = 0$; (positivity);
- $\|\alpha w\|' = |\alpha|\|w\|'$ provided that $w, \alpha w \in W$, for $\alpha \in \mathbb{R}$; (homogeneity)
- $\|w\|' \leq \sum_{i=1}^{n} |\alpha_i|\|w_i\|'$, where $w = \sum_{i=1}^{n} \alpha_i w_i$ $w, (w_i)_{i=1}^{n} \subseteq W$, $(\alpha_i)_{i=1}^{n} \subseteq \mathbb{R}$; (triangle inequality)
- $\|w\|' = \|g \cdot w\|'$, for $g \in F$, $w \in W$. (compatibility with the action)

Having $\|\cdot\|'$ we define a norm $\|\cdot\|$ on $E_{F,I}$ as the maximal extension of $\|\cdot\|'$ to the whole $E_{F,I}$. That is, for any $x \in E_{F,I}$ we set

$$\|x\| = \min\{\sum_{j=1}^{n} |\beta_j|\|w_j\|' : x = \sum_{j=1}^{n} \beta_j w_j, (w_j)_{i\leq n} \subseteq W\}.$$

It follows by compactness (from the finite-dimensionality) that the minimum is indeed attained.

Now it is straightforward to check that $\|\cdot\|$ is a norm that extends $\|\cdot\|'$ and that the action of F on $E_{F,I}$ with $\|\cdot\|$ is by linear isometries. Notice that there are several other equivalent ways how to define $\|\cdot\|$ using $\|\cdot\|'$. For instance one can take the closed convex hull of the set $\{w/\|w\|' : w \in W\}$ and then consider the Minkowski functional of such a set. The resulting norm will be $\|\cdot\|$. Another way is to consider the following set of functions $\mathcal{F} = \{f : F_I \cup \{0\} \to \mathbb{R} : f(0) = 0, |\tilde{f}(w)| \leq \|w\|'\}$, where $\tilde{f}$ is the unique linear extension of f on $E_{F,I}$. Then we have, for every $x \in E_{F,I}$, $\|x\| = \sup_{f\in\mathcal{F}} |\tilde{f}(x)|$.

We shall call such an action of F on such a finite-dimensional space *finitely presented*. If the partial norm $\|\cdot\|'$ is defined only on linear combinations of basis vectors with rational coefficients and it has a rational range, we shall call such a finitely presented action *rational*.

Let us have finite-dimensional spaces $E_{F,I}$ and $E_{H,J}$, where F, H are finite abelian groups and I, J finite sets. Suppose there are embeddings $\phi : F \hookrightarrow H$ and $\psi : I \hookrightarrow J$. Then they naturally induce a linear embedding of $E_{F,I}$ into $E_{H,I}$ which is also, after identifying F and $\psi[F] \leq H$, F-equivariant. If $E_{F,I}$, resp. $E_{H,J}$ are equipped with the finitely presented norm, invariant by the action, and the linear embedding given by ϕ and ψ is also isometric, we call such a pair (ϕ, ψ) an embedding between two finitely presented actions.

In most cases, unless stated otherwise, we shall implicitly assume that ϕ and ψ are just inclusions, so the determined linear (isometric) embedding is also an inclusion.

Proposition 4.4. *The class of all finitely presented rational actions has the amalgamation property.*

Proof. Assume we are given finite abelian groups G_0, G_1, G_2. We suppose (without loss of generality) that $G_0 \leq G_1$, $G_0 \leq G_2$ and $G_1 \cap G_2 = G_0$. Also, we are given finite sets I_0, I_1, I_2; again we assume that $I_0 \subset I_1$, $I_0 \subseteq I_2$ and $I_1 \cap I_2 = I_0$. And finally, we have some finitely presented rational actions of G_i on $E_i = E_{G_i,I_i}$ determined by a partial norm $\|\cdot\|'_i$ defined on finite $W_i \subseteq E_i$, for $i \leq 2$. Again, we may assume that $W_0 \subseteq W_1$, $W_0 \subseteq W_2$ and $W_1 \cap W_2 = W_0$. Moreover, we may suppose that every element $w \in W_i \setminus W_0$, $i = 1, 2$, contains a scalar multiple of a basis element not from E_0.

The proof is analogous to the proof of Theorem 2.2, though a bit more technical.

Set G_3 to be the free abelian amalgam of G_1 and G_2 over G_0, i.e. $G_3 = G_1 \oplus G_2/\{(g, -g) : g \in G_0\}$. Set $I = I_1 \cup I_2$. Let E_3 be the vector space E_{G_3,I_3}. We naturally identify the spaces E_i, $i \leq 2$, as subspaces of E_3. Set $W' = W_1 \cup W_2$ and let W be the 'closure of W with respect to the action of G_3, i.e. W is the smallest (finite) set containing W' such that for any $g \in G_3$ and $w \in W$ we have $g \cdot w \in W$. We define a function $\|\cdot\|'$ on W as the unique function which extends $\|\cdot\|'_i$, for $i \leq 2$, and satisfying that $\|g \cdot w\|' = \|w\|$, for $w \in W$ and $g \in G$. This definition is correct, for suppose that for some $g \in G_1$, $h \in G_2$ and $w_1 \in W_1$, $w_2 \in W_2$ we have $g \cdot w_1 = h \cdot w_2$. It follows that $g - h \in G_0$, thus $g, h \in G_0$ and so $w_1, w_2 \in W_0$. However we have $\|w_1\|'_1 = \|w_1\|'_0 = \|w_2\|'_0 = \|w_2\|'_2$.

We need to check that $\|\cdot\|'$ is a partial norm. Once we do that we again define a norm $\|\cdot\|$ on E_3 using $\|\cdot\|'$ as follows: for any $x \in E$ we set

$$\|x\| = \min\{\sum_{j=1}^{n} |\beta_j| \|w_j\|' : x = \sum_{j=1}^{n} \beta_j w_j, (w_j)_{i \leq n} \subseteq W\}.$$

It will follow that $\|\cdot\|$ extends $\|\cdot\|'$, thus it also extends $\|\cdot\|_1$, resp. $\|\cdot\|_2$, on $E_1 \subseteq E$, resp. $E_2 \subseteq E$.

The only condition from the definition of a partial norm which is not obviously satisfied is the triangle inequality. So suppose that for some $x \in W_1$ (the case $x \in W_2$ is analogous) there exist $w_1, \ldots, w_n \in W$ and non-zero $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $x = \sum_{i=1}^n \alpha_i w_i$ and $\sum_{i=1}^n |\alpha_i| \|w_i\|' < \|x\|'$. We shall reach a contradiction by induction on the size n of the decomposition of x on elements from W. Suppose first that $n = 2$, i.e. $x = \alpha_1 w_1 + \alpha_2 w_2$ and $|\alpha_1| \|w_1\|' + |\alpha_2| \|w_2\|' < \|x\|'$. At least one of the elements w_1 and w_2 must not lie in W_1 since otherwise

$$|\alpha_1|\|w_1\|' + |\alpha_2|\|w_2\|' = |\alpha_1|\|w_1\|'_1 + |\alpha_2|\|w_2\|'_1 \geq \|x\|_1 = \|x\|'$$

Similarly, at least one of the elements w_1 and w_2 must not lie in W_2. It is also impossible that one of the elements is from $W_1 \setminus W_0$ and the other from $W_2 \setminus W_0$ Suppose that $w_1 \notin W_1$ (recall that we assumed that every element $w \in W_i \setminus W_0$, $i = 1, 2$, contains a scalar multiple of a basis element not from E_0).

One of these elements, say w_1, must contain a scalar multiple of a basis element from E_1. Since $w_1 \notin W_1 \cup W_2$ it follows that there must exist $g \in G_1$ and $w'_1 \in W_2$ such that $w_1 = g \cdot w'_1$. In particular, the multiples of basis elements from w_1 that belong to E_1 are of the form $x_{g+a,l}$, where $a \in G_0$ and $l \in I_0$. Since $w_1 \notin W_2$, it also must contain multiples of basis elements of the form $x_{g+h,k}$, where $h \in G_2 \setminus G_0$. These elements must cancel out in w_2. Thus it follows that also w_2 is of the form $g \cdot w'_2$, where $w'_2 \in W_2$. It follows that $\alpha_1 w'_1 + \alpha_2 w'_2 = -g \cdot x \in E_0$. Thus we have

$$\|x\|_1 = \| - g \cdot x\|_1 = \| - g \cdot x\|_0 = \| - g \cdot x\|_2 \leq |\alpha_1|\|w'_1\|'_2 + |\alpha_2|\|w'_2\|'_2 \leq$$
$$|\alpha_1|\|w'_1\|' + |\alpha_2|\|w'_2\|' = |\alpha_1|\|w_1\|' + |\alpha_2|\|w_2\|' < \|x\|_1,$$

a contradiction.

Suppose now that $n > 2$ and that for every $x \in W_1 \cup W_2$ and every non-zero $(\alpha_i)i \leq l \subseteq \mathbb{R}$, $(w_i)_{i \leq l} \subseteq W$ such that $x = \sum_{i \leq l} \alpha_i w_i$ and $l < n$ we have that

$$\|x\|'_j \geq \sum_{i \leq l} |\alpha_i|\|w_i\|',$$

where $j = 1, 2$ depending on whether $x \in W_1$ or $x \in W_2$.

So fix $x \in W_1$ (the case for W_2 being analogous), $w_1, \ldots, w_n \in W$ and non-zero $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $x = \sum_{i=1}^n \alpha_i w_i$ and $\sum_{i=1}^n |\alpha_i|\|w_i\|' < \|x\|'_1$. Again we may suppose that for some $i \leq n$ we have $w_i \notin W_1$. We claim that we may moreover assume that the element w_i contains a scalar multiple of a basis element from E_1, i.e. some $x_{g,j}$, where $g \in G_1$ and $j \in I_1$. Indeed, otherwise we have that for every $j \leq n$ we have that either $w_j \in W_1$, or $w_j \in W_2$, or w_j does not contain scalar multiples of basis elements of E_1. Let $J_1 \subseteq \{1, \ldots, n\}$ be the set of indices $j \leq n$ such that $w_j \in W_1$, $J_2 \subseteq \{1, \ldots, n\}$ the set of indices $j \leq n$ such that $w_j \in W_2$ and $J_3 \subseteq \{1, \ldots, n\}$ the set of indices $j \leq n$ such that w_j does not contain scalar multiples of basis elements of E_1. We have that $\{1, \ldots, n\} = J_1 \cup J_2 \cup J_3$. Then $\sum_{j \in J_1} \alpha_j w_j \in E_1$ and $\sum_{j \in J_2 \cup J_3} \alpha_j w_j \in E_0$. If J_1 is non-empty, then $|J_2 \cup J_3| < n$ and we use the inductive hypothesis to reach the contradiction. Thus we may suppose that $J_1 = \emptyset$. Then $x \in E_0 \subseteq E_2$ and we claim that necessarily there is $i \in J_3$ such that w_i contains a scalar multiple of a basis element from E_2, since otherwise $\sum_{j\in J_3} \alpha_j w_j = 0$ and we may remove these elements from the decomposition of x. However, since the situation when $x \in E_1$ and some $w_i \notin W_1 \cup W_2$ contains a scalar multiple of a basis element from E_1, and the situation $x \in E_2$ and some $w_i \notin W_1 \cup W_2$ contains a scalar multiple of a basis element from E_2 are symmetric, we shall treat only the former.

So to repeat, we are now left to the situation where

- $x \in W_1$;
- $x = \sum_{i=1}^n \alpha_i w_i$ and $\sum_{i=1}^n |\alpha_i| \|w_i\|' < \|x\|_1'$;
- there is some $i \leq n$ such that $w_i \notin W_1 \cup W_2$ and w_i contains a scalar multiple of a basis element from E_1.

Next, since $x \in E_1$, there is a minimal subset, with respect to inclusion, $J \subseteq \{1, \ldots, n\}$ containing $\{i\}$ such that $\sum_{j\in J} \alpha_j w_j \in E_1$. Without loss of generality we may assume that $J = \{1, \ldots, n\}$. Now since $w_i \notin W_1$, however contains basis elements of E_1, we get that there exist $g \in G_1$ and $w' \in W_2$ such that $w_i = g \cdot w'$. In particular, every basis element $x_{g',l}$ from w_i, with a non-zero scalar, that belongs to E_1 is of the form $x_{g+a,l}$, where $a \in G_0$ and $l \in I_0$. We claim that by the minimality of J, actually for every $j \in J$ $(= \{1, \ldots, n\}$ by our assumption) there is some $w_j' \in W_2$ such that $w_j = g \cdot w_j'$. Let us prove it. For any $g \in G_3$ and $i \in I_3$ and any $x \in E_3$ let $p_{g,i}(x)$ be the scalar coefficient of $x_{g,i}$ in x, i.e. for any $x \in E_3$ we have

$$x = \sum_{g\in G, i\in I_3} p_{g,i} x_{g,i}.$$

Now since $w_i \notin W_1$ there is some $h \in G_2 \setminus G_0$ and $k \in I_2$ such that $p_{g+h,k}(w_i) \neq 0$. There must exist some minimal subset $\{i\} \subseteq J_0 \subseteq J$ such that for every $j \in J_0$ we have $p_{g+h,k}(w_j) \neq 0$ and

$$\sum_{j\in J_0} \alpha_j p_{g+h,k}(w_j) = 0.$$

It follows that for each $j \in J_0$ we also have that $-g \cdot w_j \in W_2$. If $x_0 := \sum_{j\in J_0} \alpha_j w_j = x$, then $J_0 = J$ and we are done. Otherwise there is some $h' \in G_2 \setminus G_0$ and $k' \in I_2$ such that $p_{g+h',k'}(x_0) \neq 0$. Then there again must exist some minimal subset $J_0 \subseteq J_1 \subseteq J$ such that for every $j \in J_1 \setminus J_0$ we have $p_{g+h',k'}(w_j) \neq 0$ and

$$\sum_{j\in J_1} \alpha_j p_{g+h,k}(w_j) = 0.$$

It again follows that for each $j \in J_1$ we still have that $-g \cdot w_j \in W_2$. If $x_1 := \sum_{j \in J_0} \alpha_j w_j = x$, then $J_1 = J$ and we are done. Otherwise we continue by the same procedure to obtain some J_m such that $J_m = J$ and proving that for every $j \in J_m = J$ we have that $-g \cdot w_j \in W_2$.

So now for every $j \leq n$ denote by w'_j the element $-g \cdot w_j$. In particular, $w'_i = w' \in W_2$. By the argument above, since we have $x = \sum_{j \leq n} \alpha_j w_j$, we have $-g \cdot x = \sum_{j \leq n} \alpha_j w'_j \in E_0$, thus $-g \cdot x \in W_0$. However, since $\| \cdot \|'_2$ is a partial norm, we have

$$\|x\|' = \|x\|'_1 = \|x\|'_0 = \|x\|'_2 \leq \sum_{i \leq n} |\alpha_i| \|w'_i\|'_2 = \sum_{i \leq n} |\alpha_i| \|w_i\|',$$

a contradiction. □

It follows that the class of finitely presented rational actions has some Fraïssé limit which is an action of locally finite abelian group G on a normed vector space $E_{G,J}$ with the corresponding extension property. Let $\mathbb{G}$ be the completion of $E_{G,J}$. The action of G by linear isometries extends to $\mathbb{G}$. We omit the proofs that $G = \bigoplus_{n \in \mathbb{N}} \mathbb{Q}/\mathbb{Z}$ and that $\mathbb{G}$ is isometric to the Gurarij space.

Proof of Theorem 4.3. Let $\beta : H \curvearrowright Y$ be some action of a countable torsion group on a separable Banach space by linear isometries. Without loss of generality, we shall assume that H is infinite and Y is infinite-dimensional. Also, we claim that without loss of generality we may assume that there is a countably infinite linearly independent set $\{e_n : n \in \mathbb{N}\} \subseteq Y$ such that

- the linear span of $\{e_n : n \in \mathbb{N}\}$ is dense in Y,
- $\{e_n : n \in \mathbb{N}\}$ is closed under the action of H, i.e. for any $h \in H$ and n there is m such that $h \cdot e_n = e_m$.

In other words, we suppose that Y has a dense normed subspace on which H acts freely.

Indeed, fix some countably infinite linearly independent set $\{e_n : n \in \mathbb{N}\} \subseteq Y$ whose linear span is dense in Y. Consider the set $\{f_{i,j,h} : i, j \in \mathbb{N}, h \in H\}$ and let F be a vector space freely spanned by this countable set. There is a canonical action of H on F by linear isometries which is determined by $g \cdot f_{i,j,h} = f_{i,j,g+h}$. We define a norm on $Y \oplus F$ such that

- it extends the norm on Y,
- F is dense in Y,
- the derived action of H on $Y \oplus F$ is still by isometries.

Once that is done, the claim is proved. Let $V = \{h \cdot e_i - f_{i,j,h} : i, j \in \mathbb{N}, h \in H\}$ and define a function $\rho : Y \cup V \to \mathbb{R}_0^+$ as follows. Set

$$\rho(v) = \begin{cases} \|v\|_Y & v \in Y; \\ 1/2^j & v \in V. \end{cases}$$

Now we define a norm $|||\cdot|||$ on $Y \oplus F$ as follows. We set

$$|||x||| = \inf\{\sum_{k=1}^{l} |\alpha_j|\rho(v_k) : x = \sum_{k=1}^{l} \alpha_k v_k, (v_k)_k \subseteq V \cup Y\}.$$

It is clearly a pseudonorm and clearly $|||x||| = |||h \cdot x|||$ for every $x \in Y \oplus F$ and $h \in H$. We show that it is a norm and that it extends the norm on Y. Then we will be done. Suppose first that for some $y \in Y$ we have $|||y||| < \|y\|_Y$. Then there exists $(\alpha_k)_k \subseteq \mathbb{R}$ and $(v_k)_k \subseteq V \cup Y$ such that $y = \sum_{k=1}^{l} \alpha_k v_k$ and $\sum_{k=1}^{l} |\alpha_k|\rho(v_k) < \|y\|_Y$. It follows that there must be some v_k from V, i.e. of the form $h \cdot e_i - f_{i,j,h}$. Let $J \subseteq \{1, \ldots, l\}$ be the set of those indices such that $v_k = h \cdot e_i - f_{i,j,h}$ for all $k \in J$. It is non-empty and since $y \in Y$ we have $\sum_{k \in J} \alpha_k = 0$. Thus we may remove the elements $\alpha_k v_k$, for $k \in J$, from the decomposition of y and decrease the sum. Continuing in this fashion we get rid of all v_k's that belong to V while decreasing the sum, a contradiction.

Now fix some non-zero $w \in Y \oplus F$ and suppose that $|||w||| = 0$. We have that $w = y + \sum_{i,j \in \mathbb{N}, h \in H} \alpha_{i,j,h} f_{i,j,h}$, where $y \in Y$, $(\alpha_{i,j,h})_{i,j,h} \subseteq \mathbb{R}$ and all but finitely many of these coefficients are zero. However, at least one of them, say $\alpha_{i,j,h}$, for some i, j, h, is non-zero since otherwise $w = y \in Y$ and $|||y||| = \|y\|_Y > 0$. Since $|||w||| = 0$ there exist $(\alpha_k)_k \subseteq \mathbb{R}$ and $(v_k)_k \subseteq V \cup Y$ such that $w = \sum_{k=1}^{l} \alpha_k v_k$ and $\sum_{k=1}^{l} |\alpha_k|\rho(v_k) < |\alpha_{i,j,h}|/2^j$. Let $J \subseteq \{1, \ldots, l\}$ be the set of those indices k such that $v_k = h \cdot e_i - f_{i,j,h}$. It is non-empty and we have $\sum_{k \in J} \alpha_k = \alpha_{i,j,h}$. Thus it follows that

$$\sum_{k=1}^{l} |\alpha_k|\rho(v_k) \geq \sum_{k \in J} |\alpha_k|\rho(v_k) \geq |\alpha_{i,j,h}|/2^j,$$

a contradiction.

So to repeat, we now may assume that we have an infinite set I and a linearly independent set $\{y_{h,i} : h \in H, i \in I\}$, on which H acts freely obviously and such that its linear span is dense in Y. We write H as an increasing union of finite abelian subgroups $(H_n)_n$ such that $H = \bigcup_n H_n$ and also we write I as an increasing union of finite subsets $(I_n)_n$ such that $I = \bigcup_n I_n$. In particular, we consider the increasing sequence of finite dimensional subspaces $(Y_n)_n$, such that $Y = \overline{\bigcup_n Y_n}$, where Y_n is spanned by $\{y_{h,i} : h \in H_n, i \in I_n\}$.

Next we need to approximate the norm restricted to Y_n by the finitely presented rational ones.

Lemma 4.5. *For every n and every $\varepsilon > 0$ there exists a finitely presented rational norm $\|\cdot\|_n^\varepsilon$ on Y_n such that bot the identity map and its inverse from $(Y_n, \|\cdot\|_Y)$ to $(Y_n, \|\cdot\|_n^\varepsilon)$ has norm less than $1+\varepsilon$.*

The proof is analogous to the proof of Lemma 2.10 and uses a standard argument from Banach space theory - using local compactness (or say total boundedness of the unit ball) of the finite-dimensional Y_n. Following the notation from the first section and the approximation there, when we have two norms $\|\cdot\|$ and $\|\cdot\|'$ on some finite-dimensional space Z, we denote by $D(\|\cdot\|, \|\cdot\|')$ the maximum of the norms $\|\mathrm{id}_{\|\cdot\|,\|\cdot\|'}\}$ and $\|\mathrm{id}_{\|\cdot\|',\|\cdot\|}\|$, where $\mathrm{id}_{\|\cdot\|,\|\cdot\|'}$ is the identity map from $(Z, \|\cdot\|)$ to $(Z, \|\cdot\|')$ and vice versa.

The following is an exact analogue of Lemma 2.11 for Banach spaces.

Lemma 4.6. *Let $H_1 \leq H_2$ be finite abelian groups and $I \subseteq J$ finite sets. Suppose we have an H_1-finitely presented rational norm $\|\cdot\|_1$ on $E_{H_1,I}$ and an H_2-finitely presented rational norm $\|\cdot\|_2$ on $E_{H_2,J}$ such that $D(\|\cdot\|_1, \|\cdot\|_2 \restriction E_{H_1,I}) < \varepsilon$. Then there exists an H_2-finitely presented rational norm $\|\cdot\|$ on $E_{H_2,I} \oplus E_{H_2,J}$ (which is isomorphic to $E_{H_2,I \coprod J}$) with basis $\{x'_{i,h} : i \in I, h \in H_2\} \cup \{x_{j,h} : j \in J, h \in H_2\}$ such that*

- *$\|\cdot\|$ restricted to the subspace $E_{H_1,I}$ of the first summand coincides with $\|\cdot\|_1$;*
- *$\|\cdot\|$ restricted on the second summand coincides with $\|\cdot\|_2$;*
- *for every $i \in I$ and $h \in H_2$ we have $\|x'_{i,h} - x_{i,h}\| \leq \varepsilon$.*

The proof is analogous to the proof of Lemma 2.11, so we only sketch it.

Sketch of the proof. Identify the first space $E_{H_1,I}$ with the subspace of the first summand, i.e. subspace spanned by $\{x'_{i,h} : i \in I, h \in H_1\}$, and the second space $E_{H_2,J}$ with the second summand, i.e. subspace spanned by $\{x_{i,h} : i \in J, h \in H_2\}$. The norm $\|\cdot\|_1$ on the first space is determined by some partial norm $\|\cdot\|'_1$ on some finite W_1 and the norm $\|\cdot\|_2$ on the second space is determined by some partial norm $\|\cdot\|'_2$ on some W_2. Extend W_1 to W'_1 by closing it under the action of H_2, i.e. $W'_1 = H_2 \cdot W_1$ and extends $\|\cdot\|'_1$ on W'_1. Then let $W = W'_1 \cup W_2 \cup \{\pm(x'_{i,h} - x_{i,h}) : i \in I, h \in H_2\}$ and define $\|\cdot\|'$ on W so that it extends $\|\cdot\|'_1$ on W'_1, it extends $\|\cdot\|'_2$ on W'_2 and for every $i \in I$,

$h \in H_2$, $\|x'_{i,h} - x_{i,h}\|' = \varepsilon$. The verification that it is a partial norm is straightforward, easier than in the case of metric spaces in Lemma 2.11. It follows that the norm $\|\cdot\|$ defined by $\|\cdot\|'$ extends $\|\cdot\|_1$ and $\|\cdot\|_2$ and is as desired. □

We now finish the proof of Theorem 4.3. By Lemma 4.5, for every n we can find an H_n-finitely presented rational norm $\|\cdot\|'_n$ on $E_{H_n,I_n} \cong Y_n$, whose basis we shall denote by $\{x^n_{h,i} : h \in H_n, i \in I_n\}$, such that $D(\|\cdot\|'_n, \|\cdot\|_Y \upharpoonright Y_n) < 1/2^n$. In particular, by triangle inequality, for every n we have $D(\|\cdot\|'_n, \|\cdot\|'_{n+1} \upharpoonright Y_n) < 1/2^{n+1}$. So by Lemma 4.6, for every n we may define an H_{n+1}-finitely presented rational norm $\|\cdot\|_n$ on $E_{H_{n+1},I_n} \oplus E_{H_{n+1},I_{n+1}}$ ($\cong H_{n+1} \cdot Y_n \oplus Y_{n+1}$) such that it extends the norms $\|\cdot\|'_n$ and $\|\cdot\|'_{n+1}$ respectively and for every $i \in I_n$ and $h \in H_{n+1}$, $\|x^n_{h,i} - x^{n+1}_{h,i}\| \leq 1/2^{n+1}$.

Now as in the proof of Theorem 2.7, by a successive application of the extension property of $(G \curvearrowright E_{G,J})$ we obtain

- an increasing chain of finite subgroups $H'_1 \leq H'_2 \leq \ldots \leq G$ such that $H'_i \cong H_i$ for $i \in \mathbb{N}$, and thus also $H' = \bigcup_n H'_n \cong H$;
- linear isometric embeddings $\phi_n : (E_{H_{n+1},I_n} \oplus E_{H_{n+1},I_{n+1}}, \|\cdot\|_n) \hookrightarrow E_{G,J}$ such that $\phi_n \upharpoonright (E_{H_{n+1},I_{n+1}}, \|\cdot\|_n) = \phi_{n+1} \upharpoonright (E_{H_{n+1},I_{n+1}}, \|\cdot\|_{n+1})$, for every n;
- for every n, we have that the actions $H_n \curvearrowright E_{H_n,I_n}$ and $H'_n \curvearrowright \phi_n[E_{H-n,I_n}]$ are isometric.

Then since for every $i \in I$ and $h \in H$, the sequence $(x^n_{h,i})$ is Cauchy in $\mathbb{G} = \overline{E_{G,J}}$, we may take the limit $z_{h,i} \in \mathbb{G}$. It is readily checked that after identifying the groups H and $H' \leq G$, the linear operator from Y into $\mathbb{G}$, determined by sending $y_{h,i} \in Y$ to $z_{h,i} \in \mathbb{G}$, is an H-equivariant linear isometry. □

Finally, let us note here that when one wants to have an action of a countable locally finite group on a universal Banach space by (general affine) isometries which is universal with respect to general actions of locally finite groups on general metric spaces, i.e. for any such an action we have a corresponding equivariant (not necessarily linear) isometric embedding, one can again use the Lipschitz-free Banach spaces.

Let F be the functor which sends a pointed metric space $(X, 0)$ to its Lipschitz-free Banach space $F(X)$. By functoriality, any autoisometry of X which preserves 0 extends to a linear autoisometry of $F(X)$. However, even every autoisometry of X extends to an affine autoisometry of $F(X)$. Indeed, let $\phi : X \to X$ be some autoisometry. In what follows, we view X as a metric subspace of $F(X)$, i.e. we view every point $x \in X$ as a point in $F(X)$ also. Then the map $x \to \phi(x) - \phi(0)$

from X to $F(X)$ is an isometric embedding of X into $F(X)$ which preserves 0, thus extends to a linear isometric embedding from $F(X)$ into $F(X)$. It is easy to check that it is actually onto. Composing it with the translation '$+\phi(0)$' gives the affine autoisometry that extends ϕ.

Moreover, it is easy to check that by the same method every action of a group G on a pointed metric space $(X, 0)$ by isometries extends to the action of G on $F(X)$ by affine isometries; i.e. the cocycle condition is satisfied. Thus from Theorem 2.7 we get the following corollary. We refer to Chapter 5 in [13] for information about the Holmes space.

Corollary 4.7. *There exists a universal action of the Hall's group on the Holmes space $F(\mathbb{U})$, the Lipschitz-free Banach space over the Urysohn space, which is universal for general actions on general metric spaces.*

4.1. **Problems.** Although the restriction to locally finite groups is necessary for our purposes there seems to be much more freedom in considering the class of metric spaces on which one wish to act. We considered general metric spaces and general Banach spaces. However, one may restrict the attention to some special subclasses.

Question 4.8. *Does there exist a universal action of a universal locally finite group on the Hilbert space?*

We expect the answer to be negative. On the other hand, the isometric universality for actions on the Hilbert space may not be the right problem and one could rather consider coarse universality in some sense.

Next, we assume that the answer to the next question is positive.

Question 4.9. *Does there exists a universal action of a universal locally finite group on a universal Banach space?*

Recall that in Theorem 4.3 we only proved the universality for abelian locally finite groups.

Acknowledgement. The author was supported by the GAČR project 16-34860L and RVO: 67985840.

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