INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Preprint No. 42-2017

PRAHA 2017

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1 Introduction

The paper deals with the well-posedness of the strong solution of fluid-structure interaction problem when the mixed boundary conditions are considered.

We shall investigate the motion of a rigid body inside of a viscous incompressible fluid. The fluid and the body occupy a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3) with the boundary $\partial\Omega \in C^{0,1}$. Let the body be an open connected set $S_0 \subset \Omega$ at the initial time $t = 0$, $\partial S_0 \in C^2$. The fluid fills the domain $\Omega_F(0) = \Omega \backslash \overline{S_0}$ at $t = 0$.

The Cartesian coordinates $\mathbf{y}$ of points of the body at $t = 0$ are called the Lagrangian coordinates. The motion of any material point $\mathbf{y} = (y_1, .., y_N)^T \in S_0$ is described by two functions

$$t \to \mathbf{q}(t) \in \mathbb{R}^N \quad \text{and} \quad t \mapsto \mathbb{Q}(t) \in SO(N) \qquad \text{for} \quad t \in [0, T],$$

where $\mathbf{q} = \mathbf{q}(t)$ is the position of the body mass center at a time t and $SO(N)$ is the rotation group in $\mathbb{R}^N$, i.e. the $\mathbb{Q} = \mathbb{Q}(t)$ is a matrix, satisfying $\mathbb{Q}(t)\mathbb{Q}(t)^T = \mathbb{I}$, $\mathbb{Q}(0) = \mathbb{I}$ with $\mathbb{I}$ being the identity matrix. Therefore, the trajectories of all points of the body are described by a orientation preserving isometry

$$(\mathbf{B})(t, \mathbf{y}) = \mathbf{q}(t) + \mathbb{Q}(t)(\mathbf{y} - \mathbf{q}(0)) \qquad \text{for any} \quad \mathbf{y} \in S_0 \tag{1.1}$$

and the body occupies the set

$$S(t) = \{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} = \mathbf{B}(t, \mathbf{y}), \quad \mathbf{y} \in S_0\} = \mathbf{B}(t, S_0) \tag{1.2}$$

at any time t. The velocity of the body, called *rigid velocity*, is defined as

$$\frac{d}{dt}\mathbf{X}(t, \mathbf{y}) = \mathbf{u}_s = \mathbf{a}(t) + \mathbb{P}(t)(\mathbf{x} - \mathbf{q}(t)) \qquad \text{for all} \quad \mathbf{x} \in S(t), \tag{1.3}$$

where $\mathbf{a} = \mathbf{a}(t) \in \mathbb{R}^N$ is the translation velocity and $\mathbb{P} = \mathbb{P}(t)$ is the angular velocity. The velocity $\mathbf{u}_s$ has to be compatible with $\mathbf{B}$ in the sense

$$\frac{d\mathbf{q}}{dt} = \mathbf{a} \quad \text{and} \quad \frac{d\mathbb{Q}}{dt}\mathbb{Q}^T = \mathbb{P} \quad \text{in } [0, T]. \tag{1.4}$$

The angular velocity $\mathbb{P}$ is a skew–symmetric matrix, i.e. there exists a vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t) \in \mathbb{R}^N$, such that

$$\mathbb{P}(t)\mathbf{x} = \boldsymbol{\omega}(t) \times \mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{R}^N. \tag{1.5}$$

We define $\Omega_F(t) = \Omega \setminus \overline{S(t)}$. We consider the following problem modeling the motion of the rigid body in viscous incompressible fluids:
Find $(\mathbf{u}, p, \mathbf{q}, \mathbb{Q})$ such that

$$\rho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}) = \nabla \cdot \mathbb{T}(\mathbf{u}, p), \qquad \nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega_F(t) \times (0, T), \tag{1.6}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

$$\begin{aligned} \tfrac{d^2}{dt^2}\mathbf{q} &= -\textstyle\int_{\partial S(t)} \mathbb{T}(\mathbf{u}, p)\mathbf{n} d\mathcal{O}, \\ \tfrac{d}{dt}(J\boldsymbol{\omega}) &= -\textstyle\int_{\partial S(t)} (\mathbf{x} - \mathbf{q}(t)) \times \mathbb{T}(\mathbf{u}, p)\mathbf{n} d\mathcal{O}, \end{aligned} \tag{1.8}$$

$$(\mathbf{u}-\mathbf{u}_s)\cdot\mathbf{n}=0,\ \beta(\mathbf{u}_s-\mathbf{u})\cdot\boldsymbol{\tau}=\mathbb{T}(\mathbf{u},p)\mathbf{n}\cdot\boldsymbol{\tau}\quad \text{on } \partial S(t), \tag{1.9}$$

$$\mathbf{u}(.,0)=\mathbf{u}_0,\ \mathbf{q}(0)=\mathbf{q}_0,\ \mathbf{q}'(0)=\mathbf{a}_0,\ \boldsymbol{\omega}(0)=\boldsymbol{\omega}_0, \tag{1.10}$$

where $\mathbf{n}(\mathbf{x})$ is the unit *interior* normal at $\mathbf{x}\in\partial S(t)$, i.e. the vector $\mathbf{n}$ is directed inside of $S(t)$.

The matrix $\mathbb{J}$

$$\mathbb{J}=\int_{S(t)}\rho_s(|\mathbf{x}-\mathbf{q}(t)|^2\mathbb{I}-(\mathbf{x}-\mathbf{q}(t))\otimes(\mathbf{x}-\mathbf{q}(t)))\,d\mathbf{x}$$

is the matrix of the inertia moments of the body $S(t)$ related to its mass center, ρ_s is the constant density on the body. In (1.6) $\mathbf{u}$ is the fluid velocity;

$$\mathbb{T}=-pI+2\mu\,\mathbb{D}\mathbf{u}\qquad\text{and}\qquad\mathbb{D}\mathbf{u}=\frac{1}{2}\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right),$$

where $\mathbb{T}$ is the stress tensor, $\mathbb{D}$ is the deformation-rate tensor; ρ is the constant density of fluid, p is the fluid pressure; $\mu>0$ is the constant viscosity of the fluid.

Let us mention that the problem of the motion of one or several rigid bodies in a viscous fluid filling a bounded domain was investigated by several authors [5, 6, 7, 16]. In all mentioned articles a non-slip boundary condition has been considered on the boundaries of the bodies and of the domain. Hesla [14], Hillairet [15] have been shown that this condition gives a very paradoxical result of no collisions between the bodies and the boundary of the domain.

Our article is devoted to the problem of the motion of the rigid body in the viscous fluid when a slippage is allowed at boundaries. The slippage is prescribed by the Navier boundary condition, having only the continuity of the velocity just in the normal component. To our knowledge the first solvability result was done by Neustupa, Penel [24], [25], in a particular situation, where they have considered a prescribed collision of a ball with a wall, when the slippage was allowed on both boundaries. Their pioneer result shown that the slip boundary condition cleans the no-collision paradox. Recently Gérard-Varet, Hillairet [12] have proved a local-in-time existence result: up to collisions. Gérard–Varet et al. [13] have been investigated the free fall of a sphere above a wall, that is when the boundaries are C^∞-smooth, in a viscous incompressible fluid in two different situations: *Mixed case:* the Navier boundary condition is prescribed on the boundary of the body and the non-slip boundary condition - on the boundary of the domain; *Slip case:* the Navier boundary conditions are prescribed on both boundaries as of the body and of the domain.

The result of them is interesting, saying that in the *Mixed case* the sphere never touches the wall and in the *Slip case* the sphere reaches the wall during a finite time period.

Recently, the global existence result was proven in the mixed case see [4], even if the collisions of the body with the boundary of domain occur in a finite time under a lower regularity of the body and domain than in work of [13]. Our article deals with the strong solution of the mixed case. The existence of strong solution was studied by Takahashi, Takahashi and Tucsnak [28, 29] in the no-slip boundary conditions and in the slip case by Wang [31] in the 2D case.

The plan of the paper is as follows. In section 2 we define the functional framework at the basis of our work, we recall also the main result of this work. Next in Section 3 we define the local change of coordinates as in Inoue and Wakimoto [17] and we prove the existence of solution to the linearized problem. Finally in Section 4 we consider the nonlinear problem and we prove the existence of solution using a fixed point argument.

2 Preliminaries and the main theorem

We will use the following function spaces on the moving domain:

$$L^2(0,T;H^2(\Omega_F(t)),H^1(0,T;L^2(\Omega_F(t)), \tag{2.1}$$
$$C([0,T];H^1(\Omega_F(t))),L^2(0,T;H^1(\Omega_F(t)). \tag{2.2}$$

To precisely define these functon spaces we follow approach of Takahashi [28], Wang [31] and suppose that there exists a C^∞ diffeomorphism $X(t,x,y)$ from $\Omega_F(0)$ to $\Omega_F(t)$ such that

$$\frac{\partial^{i+\alpha_1+\alpha_2}X(t,x,y)}{\partial t^i \partial x^{\alpha_1}\partial y^{\alpha_2}},\ i\le 1,\ \forall\,\alpha_1,\alpha_2\ge 0 \tag{2.3}$$

exist and are continuous.

For all function $\mathbf{u}(t,\cdot):\Omega_F(t)\to\mathbb{R}^3$ we define $\mathbf{U}(t,x,y)$ as

$$\mathbf{U}(t,x,y)\equiv\mathbf{u}(t,X(t,x,y)).$$

The function spaces which have been introduced in (2.1) can be redefined in the fixed domain

$$L^2(0,T;H^2(\Omega_F(t)))\equiv\{u:U\in L^2(0,T;H^2(\Omega_F(0)))\}, \tag{2.4}$$
$$C(0,T;H^1(\Omega_F(t)))\equiv\{u:U\in C(0,T;H^1(\Omega_F(0)))\}. \tag{2.5}$$

Theorem 2.1 *Suppose that*

$$\mathbf{u}_0\in H^1(\Omega_F(0)),\mathbf{u}_{s,0}=\mathbf{a}_0+\boldsymbol{\omega}_0\times(\mathbf{y}-\mathbf{q}_0)\in H^1(S(0)),$$

they satisfy

$$\begin{gathered}(\mathbf{u}_0-\mathbf{u}_{s,0})\cdot n|_{\partial S(0)}=0,\\ \mathbf{u}_0|_{\partial\Omega}=0,\\ \operatorname{div}\,\mathbf{u}_0=0\quad\text{in}\,\Omega,\\ \operatorname{dist}(S_0,\partial\Omega)>0.\end{gathered} \tag{2.6}$$

Then there exists a maximal $T_0>0$ such that (1.6)–(1.9) has a unique solution which satisfies for all $T<T_0$

$$\mathbf{u},p,\mathbf{a}(t),\boldsymbol{\omega}(t)\in\mathcal{U}_T(\Omega_F(t))\times L^2(0,T;H^1(\Omega_F(t)))\times H^1(0,T)\times H^1(0,T), \tag{2.7}$$

where

$$\mathcal{U}_T(\Omega_F(t))=L^2(0,T;H^2(\Omega_F(t))\cap C(0,T;H^1(\Omega_F(t))\cap H^1(0,T,L^2(\Omega_F(t))).$$

3 Strong solution

3.1 Local transformation

Since the domain depends on the motion of the rigid body, we transform the problem to a fixed domain. We define as in Takahaski [28] the local transformation introduced by Inoue and Wakimoto [17]. Let us $\delta(t) = dist(S(t), \partial\Omega)$. We fix δ_0 such that $\delta(t) > \delta_0$ and define the solenoidal velocity field $\Lambda(t,x)$ such that $\Lambda = 0$ in the $\delta_0/4$ neighborhood of $\partial\Omega$, $\Lambda = \mathbf{a}(t) + \boldsymbol{\omega}(t) \times (x - \mathbf{q}(t))$ in the $\delta_0/4$ neighborhood of $S(t)$. Then the flow X is defined

$$\begin{aligned} &\mathbf{X}(t) : \Omega \to \Omega, \\ &\tfrac{d}{dt}\mathbf{X}(t,\mathbf{y}) = \Lambda(t, \mathbf{X}(t,\mathbf{y})), \mathbf{X}(0,\mathbf{y}) = \mathbf{y} \quad \forall\ \mathbf{y} \in \Omega. \end{aligned} \tag{3.1}$$

We denote Y the inverse of X.

Now we introduce the new unknown functions

$$\begin{aligned} P(t,\mathbf{y}) &= p(t, \mathbf{X}(t,\mathbf{y})), \\ \mathbf{U}(\mathbf{t},\mathbf{y}) &= \mathcal{J}_Y(t, \mathbf{X}(t,\mathbf{y}))\mathbf{u}(t, \mathbf{X}(t,\mathbf{y})), \\ \Xi(t) &= \mathbb{Q}^t(t)\boldsymbol{\omega}(t), \\ \xi(t) &= \mathbb{Q}^t(t)\mathbf{a}(t), \\ \mathcal{T}(\mathbf{U}(\mathbf{t},\mathbf{y}), \mathbf{P}(\mathbf{t},\mathbf{y})) &= \mathbb{Q}^T(t)\mathbb{T}(Q(t)\mathbf{U}(t,\mathbf{y}), P(t,\mathbf{y}))Q(t), \end{aligned} \tag{3.2}$$

for $t \in [0,T]$ and $\mathbf{y} \in \Omega_0$ and

$$\mathcal{J}_Y(t, \mathbf{X}(t,\mathbf{y})) = \Big(\frac{\partial Y_i}{\partial x_j}\Big).$$

Before deriving the transformed equations we also introduce the metric covariant tensor

$$g_{ij} = X_{k,i}X_{k,j}, \tag{3.3}$$

the metric covariant tensor

$$g^{ij} = Y_{i,k}Y_{j,k} \tag{3.4}$$

and the Christoffel symbol (of the second kind)

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}). \tag{3.5}$$

It is easy to observe that in particular it holds

$$\Gamma^k_{ij} = Y_{k,l}X_{l,ij}. \tag{3.6}$$

The transformation of the rigid body

$$\int_{\partial S(t)} \mathbb{T}(\mathbf{u},p)\mathbf{n}(t)d\mathcal{O} = \mathbb{Q}\int_{\partial S(0)} \mathcal{T}(\mathbf{U},P)\mathbf{N}d\sigma,$$

$$\int_{\partial S(t)} (x - \mathbf{q}(t)) \times \mathbb{T}(\mathbf{u}, p)\mathbf{n}(t) d\mathcal{O} = \mathbb{Q} \int_{\partial S(0)} \mathbf{y} \times \mathcal{T}(\mathbf{U}, P)\mathbf{N} d\sigma.$$

After transformation we get the following system

$$\mathbf{U}_t + (\mathcal{M} - \mathcal{L})\mathbf{U} = -\mathcal{N}(\mathbf{U}) - \mathcal{G}p, \tag{3.7}$$

$$\text{div } \mathbf{U} = 0, \tag{3.8}$$

$$\mathbf{U}(t, y) \cdot \mathbf{N} = (\mathbf{\Xi}(t) \times \mathbf{y} + \xi(t)) \cdot \mathbf{N}, \tag{3.9}$$

$$m\frac{d}{dt}\boldsymbol{\xi} = -m(\mathbf{\Xi} \times \boldsymbol{\xi}) - \int_{\partial S(0)} \mathcal{T}(\mathbf{U}, P)\mathbf{N} d\sigma, \tag{3.10}$$

$$I\frac{d}{dt}\mathbf{\Xi} = \mathbf{\Xi} \times (I\mathbf{\Xi}) - \int_{S(0)} y \times \mathcal{T}(\mathbf{U}, P)\mathbf{N} d\sigma, \tag{3.11}$$

$$(\mathbf{U} - \mathbf{U}_S) \cdot \mathbf{N}|_{\partial S(0)} = 0, \tag{3.12}$$

$$\beta(\mathbf{U} - \mathbf{U}_S) \cdot \tau|_{\partial S(0)} = -2(\mathbb{D}(\mathbf{U})\mathbf{N} \cdot \tau|_{\partial S(0)}, \tag{3.13}$$

$$\mathbf{U} = 0 \text{ on } \partial\Omega, \tag{3.14}$$

$$\boldsymbol{\xi}(0) = \mathbf{a}(0) \text{ and } \mathbf{\Xi}(0) = \boldsymbol{\omega}(0), \tag{3.15}$$

where $I = \mathbb{Q}^t \mathbb{J} \mathbb{Q}$ is the transformed inertia tensor which no longer depends on time ([21]). The operator $\mathcal{L}$ is the transformed Laplace operator and it is given by

$$(\mathcal{L}\mathbf{u})_{ij} = \sum_{j,k+1}^{n} \partial_j (g^{jk} \partial \mathbf{u}_i) + 2 \sum_{j,k,l=1}^{n} g^{kl} \Gamma^i_{jk} \partial_l \mathbf{u}_j \tag{3.16}$$

$$+ \sum_{j,k,l=1}^{n} \Big(\partial_k (g^{kl} \Gamma^i_{kl}) + \sum_{m=1}^{n} g^{kl} \Gamma^m_{jl} \Gamma^i_{km} \Big) \mathbf{u}_j. \tag{3.17}$$

The convection term is transformed into

$$(\mathcal{N}\mathbf{u})_i = \sum_{j=1}^{n} \mathbf{u}_j \partial_j \mathbf{u}_i + \sum_{j,k+1}^{n} \Gamma^i_{jk} \mathbf{u}_j \mathbf{u}_k. \tag{3.18}$$

The transformation of time derivative and gradient are given by

$$(\mathcal{M}\mathbf{u})_i = \sum_{j=1}^{n} \dot{\mathbf{Y}}_j \partial_j \mathbf{u}_i + \sum_{j,k=1}^{n} \Big(\Gamma^i_{jk} \dot{\mathbf{Y}}_k + (\partial_k \mathbf{Y}_i)(\partial_j \dot{\mathbf{X}}_k) \Big) \mathbf{u}_j. \tag{3.19}$$

The gradient of pressure is transform as follows

$$(\mathcal{G}p)_i = \sum_{j=1}^{n} g^{ij} \partial_j p. \tag{3.20}$$

3.2 Stokes problem

We will consider a linear system coupling Stokes type equations in a fixed domain to a system of ordinary differential equations:

$$\begin{aligned}
\rho\partial_t \mathbf{U} - \nu\Delta\mathbf{U} + \nabla P &= \mathbf{F} \text{ in } \Omega_F(0)\times[0,T],\\
\text{div }\mathbf{U} &= 0 \text{ in } \Omega_F(0)\times[0,T],\\
\mathbf{U}(y,t) &= 0,\ \mathbf{y}\in\ \partial\Omega_F(0),\ \ t\in[0,T],\\
\mathbf{U}(y,t)\cdot\mathbf{N} &= \mathbf{U}_s\cdot\mathbf{N} \text{ on } \partial S_0,\\
(\mathbb{T}(\mathbf{U},P)\mathbf{N})\cdot\tau &= \beta(\mathbf{U}_s-\mathbf{U})\cdot\tau \text{ on }\ \partial S_0,\\
m\boldsymbol{\chi}' &= -\int_{\partial S(0)}\mathbb{T}(\mathbf{U},P)\mathbf{N}d\sigma + \mathbf{F}_M,\\
I\boldsymbol{\Omega}' &= -\int_{\partial S(0)}(\mathbf{x}-\mathbf{q}(t))\times\mathbb{T}(\mathbf{U},P)\mathbf{N}d\sigma + \mathbf{F}_J.
\end{aligned} \tag{3.21}$$

Let us mention the "classical" steady Stokes equations

$$\begin{cases} -\Delta\mathbf{v}+\nabla p = f, \quad \text{div}\,\mathbf{v}=0 \quad \text{in } \Omega\\ \qquad\qquad\qquad\quad \mathbf{v}=\mathbf{v}_* \quad \text{on } \partial\Omega \end{cases} \tag{3.22}$$

Theorem 3.1 *For any* $\mathbf{f}\in L^2(\Omega)$, $v_*\in H^{3/2}(\partial\Omega)$, $\Omega\subset C^{1,1}$, $\int_{\partial\Omega} v_* n\, d\Gamma = 0$. *Then problem* (3.22) *has a unique strong solution* $(\mathbf{v},p)\in H^2(\Omega)\times H^1(\Omega)$ *satisfying*

$$\|\mathbf{v}\|_{H^2(\Omega)}+\|p\|_{H^1(\Omega)}\le C(\|\mathbf{f}\|_{L^2(\Omega)}+\|\mathbf{v}_*\|_{H^{3/2}(\Omega)}).$$

Proof. see [11], Theorem 5.1, p. 232.

The steady Stokes equations with slip boundary boundary was investigated by B. da Veiga in Hilbertian case [2]

$$\begin{cases} -\Delta\mathbf{v}+\nabla p = f, \quad \text{div}\,\mathbf{v}=0 \quad \text{in } \Omega\\ (\mathbf{v}-\boldsymbol{g})\cdot\mathbf{n}=0 \qquad \text{on }\ \partial\Omega\\ \beta(\mathbf{v}-\boldsymbol{g})\cdot\tau = -2(\mathbb{D}(\mathbf{v})\cdot N)\cdot\tau \qquad \text{on } \partial\Omega \end{cases} \tag{3.23}$$

Theorem 3.2 *For any* $\mathbf{f}\in L^2(\Omega)$, $g\in H^{3/2}(\partial\Omega)$, $\Omega\subset C^{1,1}$, *the problem* (3.23) *has a unique strong solution* $(\mathbf{v},p)\in H^2(\Omega)\times H^1(\Omega)$ *satisfying*

$$\|\mathbf{v}\|_{H^2(\Omega)}+\|p\|_{H^1(\Omega)}\le C(\|\mathbf{f}\|_{L^2(\Omega)}+\|\boldsymbol{g}\|_{H^{3/2}(\Omega)}).$$

Proof. see [2, 27].

We recall a well-known result see Takahashi [28].

Proposition 3.1 *Let H be a Hilbert space. Let $A : D(A) \to H$ be a self-adjoint and accretive operator. If $\mathbf{f} \in L^2(0,T;H)$, $\mathbf{u}_0 \in D(A^{1/2})$, the problem*

$$\mathbf{u}' + A\mathbf{u} = f, \quad \mathbf{u}(0) = \mathbf{u}_0$$

has a unique solution $u \in L^2(0,T;D(A)) \cap C([0,T];D(A^{1/2})) \cap H^1(0,T;H)$ which satisfies

$$\|\mathbf{u}\|_{L^2(0,T;D(A))} + \|\mathbf{u}\|_{L^\infty(0,T;D(A^{1/2}))} + \|\mathbf{u}\|_{H^1(0,T;H)} \leq C(\|\mathbf{u}_0\|_{D(A^{1/2})} + \|\mathbf{f}\|_{L^2(0,T;H)}),$$

with a constant C depending on the operator A and time T. Moreover, the constant C is a nondecreasing function of T.

We define the functional spaces:

$$\mathcal{H} \equiv \{\phi \in L^2(\Omega) : \operatorname{div}\phi = 0, \text{ there exists } \phi_F \in \mathcal{D}'(\Omega_F(0)), \ \phi_S \in \mathcal{R} \\ \text{such that } \ \phi = \phi_F \text{ on } \Omega_F(0), \phi = \phi_S \text{ on} S(0)\},$$

where $\mathcal{R} \equiv \{\phi : \phi = \chi_\phi + \omega_\phi \times \mathbf{y}, \ \chi_\phi \in \mathbb{R}^3, \ \omega_\phi \in \mathbb{R}\}$, and

$$\mathcal{V} \equiv \{\phi \in \mathcal{H}; \phi_f \in H^1(\Omega_F(0)), \text{ and satisfy } \phi_f, \ \phi_S \text{ satisfy: } \phi_F|_{\partial\Omega} = 0, \ (\phi_F - \phi_S)\cdot\mathbf{N} = 0\}.$$

For $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ we define the inner product by $(\cdot,\cdot)$ by

$$(\mathbf{u}, \mathbf{v}) = \int_{\Omega_F(0)} \phi_F \mathbf{u}_F \cdot \mathbf{v}_F + \int_{S(0)} \rho_S \mathbf{u}_S \mathbf{v}_S,$$

which implies that

$$(\mathbf{u}, \mathbf{v}) = \int_{\Omega_F(0)} \rho_F \mathbf{u}_F \cdot \mathbf{v}_F + M\chi_{\mathbf{u}_S} \cdot \chi_{\mathbf{v}_S} + J\boldsymbol{\omega}_{\mathbf{u}_S} \cdot \boldsymbol{\omega}_{u_S}.$$

To solve the linear equation (3.22) we introduce the following space

$$D(A) = \{\phi \in \mathcal{H}; \ \phi_F \in H^2(\Omega_F(0)); \ \phi_F, \phi_S \text{ satisfy } \phi_F|_{\partial\Omega} = 0, \ (\phi_F - \phi_S)\cdot N|_{\partial S(0)} = 0, \\ \beta(\phi_T - \phi_S)\cdot\tau|_{\partial S(0)} = -2(\mathbb{D}(\phi_F)\cdot N)\cdot\tau|_{\partial S(0)}\}.$$

Let us define an operator $\mathcal{A}$ as

$$\mathcal{A}\mathbf{u} = \begin{cases} -\nu\Delta u \text{ in } \Omega_F(0), \\ \frac{2\nu}{m}\int_{\partial S(0)} \mathbb{D}(\mathbf{u})\mathbf{N}\, d\sigma + \left(\frac{2\nu}{\mathbb{J}}\int_{\partial S(0)} \mathbb{D}(\mathbf{u})\mathbf{n} \times \mathbf{y} d\sigma\right) \times \mathbf{y}, \quad \mathbf{y} \text{ in } S(0) \end{cases} \tag{3.24}$$

and

$$Au = \mathbb{P}\mathcal{A}\mathbf{u}, \tag{3.25}$$

where $\mathbb{P}$ is the orthogonal projector on $\mathcal{H}$ in $L^2(\Omega)$. Define

$$D(A) = \{\phi \in H_0^1(\Omega), \phi \in H^2(\Omega_F), \mathrm{div}\phi = 0 \text{ in } \Omega, \mathbb{D}(\phi) = 0 \text{ in } S\}. \tag{3.26}$$

Proposition 3.2 *The operator A defined by (3.24) - (3.26) is self-adjoint and positive. Consequently A is generator of contraction semigroup in $\mathcal{H}$. Moreover, there exists a constant $C > 0$ such that for any $\mathbf{u} \in D(A)$ we have*

$$\|\mathbf{u}\|_{H^2(\Omega)} \leq C\|A\mathbf{u}\|_{L^2(\Omega)}. \tag{3.27}$$

Proof. See [28, 31].

Proposition 3.3 *Let $T > 0$. If $\mathbf{U}_0 = (\mathbf{U}_{F,0}, \mathbf{U}_{S,0}) \in \mathcal{V}$, $\mathbf{F} \in L^2(0,T; L^2(\Omega(0,T)))$, $\mathbf{F}_j, \mathbf{F}_M \in L^2(0,T)$, then problem* (3.19) *has a unique solution on* $[0,T]$*. Moreover, we have the following estimates:*

$$\|\mathbf{U}\|_{\mathcal{U}_T(\Omega_F(0))} + \|\nabla p\|_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{\chi}\|_{H^1(0,T)} + \|\omega_S\|_{H^1(0,T)} \leq$$
$$C(\|F_M, F_y\|_{L^2(0,T)} + \|\mathbf{F}\|_{L^2(0,T;L^2(\Omega_F(0))} + \|\mathbf{U}_{S,0}\|_{H^1(S(0))} + \|\mathbf{U}_0\|_{H^1(F(0))}),$$

where C in nondecreasing function of T.

Proof. It follows after small modification from results of Takahashi and Wang, see [28, 31].

4 Nonlinear case

We know that our problem is equivalent to the following problem in the fixed domain

$$\begin{aligned}
&\rho\partial_t\mathbf{U} - \nu\mathcal{L}\mathbf{U} + \rho(\mathcal{M}\mathbf{U} + \mathcal{N}\mathbf{U}) + \mathcal{G}p = \mathbf{f},\\
&\mathrm{div}\,\mathbf{U} = 0,\\
&U_S(t,y)\cdot\mathbf{N} = (\boldsymbol{\Sigma}(t)\times\mathbf{y} + \boldsymbol{\xi}(t))\cdot\mathbf{N},\\
&m\frac{d}{dt}\boldsymbol{\xi} = -m(\boldsymbol{\Sigma}\times\boldsymbol{\xi}) - \int_{\partial S(0)}\mathcal{T}(\mathbf{U},p)\mathbf{N}d\sigma,\\
&I\frac{d}{dt}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\times(I\boldsymbol{\Sigma}) - \int_{\partial S(0)}\mathbf{y}\times\mathcal{T}(\mathbf{U},p)\mathbf{N}d\sigma,
\end{aligned}$$

with the boundary conditions

$$\begin{aligned}
&(\mathbf{U} - \mathbf{U}_S)\cdot\mathbf{N}|_{\partial S} = 0,\\
&\mathbf{U} = 0 \text{ on } \partial\Omega,\\
&\beta(\mathbf{U} - U_S)\cdot\tau|_{\partial S(0)} = -2(\mathbb{D}(\mathbf{U})\cdot\mathbf{N})\cdot\tau|_{\partial S(0)}.
\end{aligned}$$

Proof of Theorem 1 The proof is based on the fixed point argument. Let us define

$$\mathcal{F} : (\boldsymbol{V}, \pi, \boldsymbol{\xi}, \boldsymbol{\Sigma}) \to (\mathbf{U}, P, \widetilde{\boldsymbol{\xi}}, \widetilde{\boldsymbol{\Sigma}})$$

which maps

$$\mathcal{U}_T(\Omega_F) \cap L^2(0,T; H^1(\Omega_F) \times H^1(0,T) \times H^1(0,T)$$

into itself. Moreover, $(\mathbf{U}, P, \widetilde{\boldsymbol{\xi}}, \widetilde{\boldsymbol{\Sigma}})$ satisfy

$$\begin{aligned}
&\rho\partial_t \mathbf{U} - \nu\Delta\mathbf{U} + \nabla p = \boldsymbol{G},\\
&\text{div } \mathbf{U} = 0,\\
&\mathbf{U}_S(t,y) = (\widetilde{\boldsymbol{\Sigma}} \times \mathbf{y} + \widetilde{\boldsymbol{\xi}}(t)) \cdot \mathbf{N},\\
&m\frac{d}{dt}\widetilde{\boldsymbol{\xi}}(t) = -\int_{\partial S(0)} (2\mu D(\mathbf{U}) - PI) \cdot \mathbf{N} d\sigma + F_M,\\
&I\frac{d}{dt}(\widetilde{\boldsymbol{\Sigma}}(t)) = -\int_{\partial S(0)} (2\mu \mathbb{D}(\mathbf{U}) - PI) \cdot N \cdot \tau d\sigma + F_J,
\end{aligned}$$

with the boundary condition

$$\begin{aligned}
&(\mathbf{U} - \mathbf{U}_S) \cdot \mathbf{N}|_{\partial S(0)} = 0,\\
&\beta(\mathbf{U} - \mathbf{U}_S) \cdot \tau|_{\partial S(0)} = -2\mathbb{D}(\mathbf{U})\mathbf{N} \cdot \tau|_{\partial S(0)},\\
&\mathbf{U} = 0 \text{ on } \partial\Omega,
\end{aligned}$$

where

$$\begin{aligned}
&G = \mu(\mathcal{L} - \Delta)\boldsymbol{V} + \rho\mathcal{M}\boldsymbol{V} + (\nabla - G)\pi - \rho\mathcal{N}\boldsymbol{V},\\
&F_M = m(\boldsymbol{\Sigma} \times \boldsymbol{\xi}), \;\; F_J = \boldsymbol{\Sigma} \times (I\boldsymbol{\Sigma}).
\end{aligned}$$

Let $T > 0$.

We define

$$\begin{aligned}
\mathcal{K} = \{(\boldsymbol{V}, \pi, \boldsymbol{\xi}, \boldsymbol{\omega}) \in \mathcal{U}_T(\Omega_F) \times L^2(0,T; H^1(\Omega_F) \times H^1(0,T) \times H^1(0,T),\\
\|\boldsymbol{V}\|_{U_T(\Omega_F)} + \|\pi\|_{L^2} + |\widetilde{\boldsymbol{\xi}}\|_{H^1(0,T)} + \|\widetilde{\boldsymbol{\Sigma}}\|_{H^1(0,T)} \leq R\}.
\end{aligned}$$

As a first step we show that $\mathcal{F}(\mathcal{K}) \subset \mathcal{K}$.

We denote

$$B_0 = B_0 = (\boldsymbol{\xi}_0, \|\mathbf{u}_{S,0}\|_{H^1(S(0))}, \|\mathbf{U}_0\| H^1(\Omega_F(0)), T, \|\mathbf{f}\|_{L^2_{\text{loc}}}(0,\infty; L^2(\Omega))).$$

From Proposition 3.3 we get

$$\begin{aligned}
&\|\mathbf{U}\|_{\mathcal{U}_T(\Omega_F)} + \|P\|_{L^2(0,T,H^1(\Omega(F)))} + \|\widetilde{\boldsymbol{\xi}}\|_{H^1(0,T)} + \|\widetilde{\boldsymbol{\Sigma}}\|_{H^1(0,T)}\\
&\leq B_0(1 + \|G\|_{L^2(0,T,L^2(\Omega_F)} + \|F_M, F_J\|_{L^2(0,T)}.
\end{aligned}$$

From [28] we have

$$\|G\|_{L^2(0,T;L^2(\Omega_F))} + \|F_M, F_J\|_{L^2(0,T)} \le K_0 T + B_0.$$

It follows that

$$\begin{aligned} &\|\underset{\sim}{\mathrm{U}}\|_{\mathcal{U}_T(\Omega_R)} + \|P\|_{L^2(0,T,H^1(\Omega_F)} \\ &+ \|\widetilde{\boldsymbol{\xi}}\|_{H^1(0,T)} + \|\widetilde{\boldsymbol{\Sigma}}_{H^1(0,T)} \le K_0 T^{1/10} + B_0. \end{aligned}$$

Now taking $B_0 \le R/4$, $T \le \left(\frac{R}{2K_0}\right)^{10}$ we get $K_0 T^{1/10} + B_0 < R$ and $\mathcal{F}(\mathcal{K}) \subset \mathcal{K}$.

Secondly, we prove that $\mathcal{F}$ is a contraction operator when T is small enough and R large enough.

Let $\mathcal{F}(V_F^i, T_F^i \xi^i, \Sigma') = (U_F^i, P_F^i, \widetilde{\xi}^i, \widetilde{\omega}^i)$ for $i = 1, 2$ where $(V_F^i, \pi_F^i, \xi^i, \Sigma^i) \in \mathcal{K}$. We set

$$\begin{aligned} \mathbf{U} &= \mathbf{U}^1 - \mathbf{U}^2, \\ \mathbf{U}_S &= \mathbf{U}_S^1 - \mathbf{U}_S^2, \end{aligned}$$

etc. then

$$\begin{aligned} &\rho \partial_t \mathbf{U} - \nu \Delta \mathbf{U} + \nabla P = G, \quad \operatorname{div} \mathbf{U} = 0 \quad \text{in } [0,T) \times \Omega_F, \\ &\mathbf{U}_S(t, \mathbf{y}) = \widetilde{\boldsymbol{\xi}}(t) + \widetilde{\boldsymbol{\Sigma}}(t) \times \mathbf{y} \text{ in } \Omega_S, \\ &m \frac{d}{dt} \widetilde{\boldsymbol{\xi}} = -\int_{\partial S(0)} 2\mu (\mathrm{D}(\mathbf{U}) - PI) \mathbf{N} d\sigma + F_M, \\ &I \frac{d}{dt} (\widetilde{\boldsymbol{\Sigma}}(t)) = -\int_{\partial S(0)} 2\mu (\mathrm{D}(\mathbf{U}) - PI) \mathbf{N} \times (\mathbf{y}) d\sigma + F_J, \end{aligned}$$

where

$$\begin{aligned} G = {} & \nu[(\mathcal{L}^1 - \Delta)\boldsymbol{V}] + \nu \mathcal{L} \boldsymbol{V}^2 - \mathcal{M}^1 \boldsymbol{V} \\ & - \mathcal{M} \boldsymbol{V}^2 + [(\nabla - \mathcal{G}^1)\pi + \mathcal{G}\pi^2 \\ & + \mathcal{N}^1 \boldsymbol{V}^1 - \mathcal{N}^2 \boldsymbol{V}^2, \end{aligned}$$

$$F_M = m(\boldsymbol{\Sigma}_1 \times \boldsymbol{\xi}_1) - m(\boldsymbol{\Sigma}_2 \times \boldsymbol{\xi}_2),$$

$$F_J = \boldsymbol{\Sigma}_1 \times (I \boldsymbol{\Sigma}_1) - \boldsymbol{\Sigma}_2 (I \boldsymbol{\Sigma}_2).$$

Applying [28] Corollary 6.16 we get

$$\begin{aligned} &\|G\|_{L^2(0,T;L^2(\Omega_F(0))} + \|F_M\|_{L^2(0,T)} \\ &\le K_0 T^{1/10} (\|\boldsymbol{V}\|_{\mathcal{U}_T}(\Omega_F(0)) + \|\pi\|_{L^2(0,T;H^1(\Omega_F(0))} + \|\boldsymbol{\xi}\|_{H^1(0,T)} + |\boldsymbol{\omega}\|_{H^1(0,T)}. \end{aligned}$$

Applying Proposition 3.3 we have

$$\begin{aligned} &\|\mathbf{U}\|_{\mathcal{U}_T(\Omega_F(0))} + \|P\|_{L^2(0,T;H^1(\Omega_F(0))} + \|\widetilde{\boldsymbol{\xi}}\|_{H^1(0,T)} + |\widetilde{\boldsymbol{\Sigma}}\|_{H^1(0,T)} \\ &\le K_0 T^{1/10} (\|\boldsymbol{V}\|_{\mathcal{U}_T(\Omega_F(0))} + \|\Pi\|_{L^2(0,T;H^1(\Omega_F(0))} + \|\boldsymbol{\xi}\|_{H^1(0,T)} + \|\boldsymbol{\Sigma}\|_{H^1(0,T)}). \end{aligned}$$

Thus, when T is small enough, $\mathcal{F}$ is a contraction operator.

Acknowledgements:

The work of H. Al Baba and Š. Nečasová was supported by Grant No. 16-03230S of GAČ in the framework of RVO 67985840. The work of B. Muha was supported by by Croatian Science Foundation grant number 9477. The work of N.V. Chemetov was supported by PEst-OE/MAT/UI0209/2013, financiado pela Fundação para a Ciência e a Tecnologia.

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