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EMBEDDING RELATIONS BETWEEN WEIGHTED COMPLEMENTARY LOCAL MORREY-TYPE SPACES AND WEIGHTED LOCAL MORREY-TYPE SPACES

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Abstract. In this paper embedding relations between weighted complementary local Morrey-type spaces ${}^{\mathfrak{c}}LM_{p\theta,\omega}(\mathbb{R}^n, v)$ and weighted local Morrey-type spaces $LM_{p\theta,\omega}(\mathbb{R}^n, v)$ are characterized. In particular, two-sided estimates of the optimal constant c in the inequality

$$\left(\int_0^\infty \left(\int_{B(0,t)} f(x)^{p_2} v_2(x)\,dx\right)^{\frac{q_2}{p_2}} u_2(t)\,dt\right)^{\frac{1}{q_2}} \le c\left(\int_0^\infty \left(\int_{{}^{\mathfrak{c}}B(0,t)} f(x)^{p_1} v_1(x)\,dx\right)^{\frac{q_1}{p_1}} u_1(t)\,dt\right)^{\frac{1}{q_1}}, \quad f \ge 0$$

are obtained, where p_1, p_2, q_1, $q_2 \in (0,\infty)$, $p_2 \le q_2$ and u_1, u_2 and v_1, v_2 are weights on $(0,\infty)$ and $\mathbb{R}^n$, respectively. The proof is based on the combination of the duality techniques with estimates of optimal constants of the embedding relations between weighted local Morrey-type and complementary local Morrey-type spaces and weighted Lebesgue spaces, which allows to reduce the problem to using of the known Hardy-type inequalities.

1 Introduction

The classical Morrey spaces $\mathcal{M}_{p,\lambda} \equiv \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$, were introduced by C. Morrey in [26] in order to study regularity questions which appear in the Calculus of Variations, and defined as follows: for $0 \le \lambda \le n$ and $1 \le p \le \infty$,

$$\mathcal{M}_{p,\lambda} := \left\{ f \in L_p^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}_{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{\frac{\lambda-n}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\},$$

where $B(x,r)$ is the open ball centered at x of radius r.

Note that $\mathcal{M}_{p,0}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$ and $\mathcal{M}_{p,n}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

These spaces describe local regularity more precisely than Lebesgue spaces and appeared to be quite useful in the study of the local behavior of solutions to partial differential equations, a priori estimates and other topics in the theory of partial differential equations (cf. [16]).

The classical Morrey spaces were widely investigated during the last decades, including the study of classical operators of Harmonic and Real Analysis - maximal, singular and potential operators - in generalizations of these spaces (the so-called Morrey-type spaces). The local Morrey-type spaces and the complementary local Morrey-type spaces introduced by Guliyev in his doctoral thesis [23].

The research on local Morrey-type spaces mainly includes the study of the boundedness of classical operators in these spaces (see, for instance, [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 24]), and the investigation of the functional-analytic properties of them and relation of these spaces with other known function spaces (see, for instance, [12, 1, 27, 17, 18]). We refer the reader to the surveys [2] and [3] for a comprehensive discussion of the history of $LM_{p\theta,\omega}$ and ${}^{^\complement}\!LM_{p\theta,\omega}$.

Let A be any measurable subset of $\mathbb{R}^n$, $n \geq 1$. By $\mathfrak{M}(A)$ we denote the set of all measurable functions on A. The symbol $\mathfrak{M}^+(A)$ stands for the collection of all $f \in \mathfrak{M}(A)$ which are non-negative on A. The family of all weight functions (also called just weights) on A, that is, measurable, positive and finite a.e. on A, is given by $\mathcal{W}(A)$.

For $p \in (0,\infty]$, we define the functional $\|\cdot\|_{p,A}$ on $\mathfrak{M}(A)$ by

$$\|f\|_{p,A} := \begin{cases} \left(\int_A |f(x)|^p\,dx\right)^{1/p} & \text{if} \quad p<\infty \\ \operatorname{ess\,sup}_A |f(x)| & \text{if} \quad p=\infty \end{cases}.$$

If $w \in \mathcal{W}(A)$, then the weighted Lebesgue space $L_p(w,A)$ is given by

$$L_p(w,A) \equiv L_{p,w}(A) := \{f \in \mathfrak{M}(A) : \ \|f\|_{p,w,A} := \|fw\|_{p,A} < \infty\}.$$

When $A = \mathbb{R}^n$, we often write simply $L_{p,w}$ and $L_p(w)$ instead of $L_{p,w}(A)$ and $L_p(w,A)$, respectively.

Throughout the paper, we always denote by c and C positive constants, which are independent of main parameters but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$, ($b \gtrsim a$) we mean that $a \leq \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. We will denote by $\mathbf{1}$ the function $\mathbf{1}(x) = 1$, $x \in \mathbb{R}$.

Given two quasi-normed vector spaces X and Y, we write $X = Y$ if X and Y are equal in the algebraic and the topological sense (their quasi-norms are equivalent). The symbol $X \hookrightarrow Y$ ($Y \hookleftarrow X$) means that $X \subset Y$ and the natural embedding I of X in Y is continuous, that is, there exist a constant $c > 0$ such that $\|z\|_Y \leq c\|z\|_X$ for all $z \in X$. The best constant of the embedding $X \hookrightarrow Y$ is $\|\,\mathrm{I}\,\|_{X\to Y}$.

The weighted local Morrey-type spaces $LM_{p\theta,\omega}(\mathbb{R}^n,v)$ and weighted complementary local Morrey-type spaces $LM_{p\theta,\omega}(\mathbb{R}^n,v)$ are defined as follows: Let $0 < p,\theta \leq \infty$. Assume that $\omega \in \mathfrak{M}^+(0,\infty)$ and $v \in \mathcal{W}(\mathbb{R}^n)$.

$$LM_{p\theta,\omega}(\mathbb{R}^n,v) := \left\{ f \in L^{\mathrm{loc}}_{p,v}(\mathbb{R}^n) : \ \|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n,v)} < \infty \right\},$$

where

$$\|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n,v)} := \big\| \|f\|_{p,v,B(0,r)} \big\|_{\theta,\omega,(0,\infty)},$$

and

$${}^{^\complement}\!LM_{p\theta,\omega}(\mathbb{R}^n,v) := \left\{ f \in \bigcap\nolimits_{t>0} L_{p,v}({}^{^\complement}\!B(0,t)) : \ \|f\|_{{}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n,v)} < \infty \right\},$$

where

$$\|f\|_{{}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n,v)} := \big\| \|f\|_{p,v,{}^{\complement}B(0,r)} \big\|_{\theta,\omega,(0,\infty)}.$$

Remark 1. In [5] and [7] it was proved that the spaces $LM_{p\theta,\omega}(\mathbb{R}^n) := LM_{p\theta,\omega}(\mathbb{R}^n, \mathbf{1})$ and ${}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n) := {}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n, \mathbf{1})$ are non-trivial, i.e. consists not only of functions equivalent to 0 on $\mathbb{R}^n$, if and only if

$$\|\omega\|_{\theta,(t,\infty)} < \infty, \qquad \text{for some} \qquad t > 0, \tag{1.1}$$

and

$$\|\omega\|_{\theta,(0,t)} < \infty, \qquad \text{for some} \qquad t > 0, \tag{1.2}$$

respectively. The same conclusion is true for $LM_{p\theta,\omega}(\mathbb{R}^n, v)$ and ${}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n, v)$ for any $v \in \mathcal{W}(\mathbb{R}^n)$.

The proof of the following statement is straightforward.

Lemma 1.1. (i) *If $\|\omega\|_{\theta,(t_1,\infty)} = \infty$ for some $t_1 > 0$, then*

$$f \in LM_{p\theta,\omega}(\mathbb{R}^n, v) \Rightarrow f = 0 \quad \text{a.e. on} \quad B(0, t_1).$$

(ii) *If $\|\omega\|_{\theta,(0,t_2)} = \infty$ for some $t_2 > 0$, then*

$$f \in {}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n, v) \Rightarrow f = 0 \quad \text{a.e. on} \quad {}^{\complement}B(0, t_2).$$

Let $0 < \theta \le \infty$. We denote by

$$\Omega_\theta := \big\{\omega \in \mathfrak{M}^+(0,\infty) : 0 < \|\omega\|_{\theta,(t,\infty)} < \infty, \ t > 0\big\},$$
$${}^{\complement}\Omega_\theta := \big\{\omega \in \mathfrak{M}^+(0,\infty) : 0 < \|\omega\|_{\theta,(0,t)} < \infty, \ t > 0\big\}.$$

Let $v \in \mathcal{W}(\mathbb{R}^n)$. It is easy to see that $LM_{p\theta,\omega}(\mathbb{R}^n, v)$ and ${}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n, v)$ are quasi-normed vector spaces when $\omega \in \Omega_\theta$ and $\omega \in {}^{\complement}\Omega_\theta$, respectively.

The following statements are immediate consequences of Fubini's Theorem and were observed in [5] and [7], for $v = 1$, respectively.

Lemma 1.2. *Let $0 < p \le \infty$ and $v \in \mathcal{W}(\mathbb{R}^n)$. Then*
(i) $LM_{pp,\omega}(\mathbb{R}^n, v) = L_p(w)$, *where* $w(x) := v(x)\|\omega\|_{p,(|x|,\infty)}$, $x \in \mathbb{R}^n$.
(ii) ${}^{\complement}LM_{pp,\omega}(\mathbb{R}^n, v) = L_p(w)$, *where* $w(x) := v(x)\|\omega\|_{p,(0,|x|)}$, $x \in \mathbb{R}^n$.

Recall that the embedding relations between weighted local Morrey-type spaces and weighted Lebesgue spaces, that is, the embeddings

$$L_{p_1}(v_1) \hookrightarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2), \tag{1.3}$$
$$L_{p_1}(v_1) \hookrightarrow {}^{\complement}LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2), \tag{1.4}$$
$$L_{p_1}(v_1) \hookleftarrow LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2), \tag{1.5}$$
$$L_{p_1}(v_1) \hookleftarrow {}^{\complement}LM_{p_2\theta,\omega}(\mathbb{R}^n, v_2) \tag{1.6}$$

are completely characterized in [27].

Our principal goal in this paper is to investigate the embedding relations between weighted complementary local Morrey–type spaces and weighted local Morrey type spaces and vice versa, that is, the embeddings

$${}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \hookrightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2), \tag{1.7}$$
$$LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n, v_1) \hookrightarrow {}^{\complement}LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n, v_2). \tag{1.8}$$

The approach used in this paper consists of a the duality argument combined with estimates of optimal constants of embeddings (1.3) - (1.6), which allows us to reduce the problem to using the following Hardy-type inequalities

$$\big\| \|H^* f\|_{p,u,(0,\cdot)} \big\|_{q,w,(0,\infty)} \le c \, \|f\|_{\theta,v,(0,\infty)}, \; f \in \mathfrak{M}^+(0,\infty), \tag{1.9}$$

with

$$(H^* f)(t) := \int_t^\infty f(\tau)\, d\tau, \quad t > 0,$$

where u, v, w are weights on $(0,\infty)$ and $0 < p,\, q \le \infty$, $1 < \theta < \infty$. There exists different criteria for the validity of these inequalities (for more detailed information see, for instance, [19] and [20]). We will use characterizations from [21] and [22].

Note that in view of Lemma 1.2, embeddings (1.7) - (1.8) contain embeddings (1.3) - (1.6) as a special case. Moreover, by the change of variables $x = y/|y|^2$ and $t = 1/\tau$, it is easy to see that (1.8) is equivalent to the embedding

$$^{\complement}LM_{p_1\theta_1,\tilde{\omega}_1}(\mathbb{R}^n, \tilde{v}_1) \hookrightarrow LM_{p_2\theta_2,\tilde{\omega}_2}(\mathbb{R}^n, \tilde{v}_2),$$

where $\tilde{v}_i(y) = v_i(y/|y|^2)|y|^{-2n/p_i}$ and $\tilde{\omega}_i(\tau) = \tau^{-2/\theta_i}\omega_i\big(1/\tau\big)$, $i = 1, 2$. This observation allows us to concentrate our attention on characterization of (1.7). On the negative side of things we have to admit that the duality approach works only in the case when, in (1.7) - (1.8), one has $p_2 \le \theta_2$. Unfortunately, in the case when $p_2 > \theta_2$ the problem of characterization of these embeddings remains open.

In particular, we obtain two-sided estimates of the optimal constant c in the inequality

$$\bigg(\int_0^\infty \bigg(\int_{B(0,t)} f(x)^{p_2} v_2(x)\, dx \bigg)^{\frac{q_2}{p_2}} u_2(t)\, dt \bigg)^{\frac{1}{q_2}} \le c \bigg(\int_0^\infty \bigg(\int_{^{\complement}B(0,t)} f(x)^{p_1} v_1(x)\, dx \bigg)^{\frac{q_1}{p_1}} u_1(t)\, dt \bigg)^{\frac{1}{q_1}},$$

where p_1, p_2, q_1, $q_2 \in (0,\infty)$, $p_2 \le q_2$ and u_1, u_2 and v_1, v_2 are weights on $(0,\infty)$ and $\mathbb{R}^n$, respectively.

The paper is organized as follows. We start with formulations of our main results in Section 2. The proofs of the main results are presented in Section 3.

2 Statement of the main results

We adopt the following usual conventions.

Convention 1. (i) *Throughout the paper we put* $0/0 = 0$, $0 \cdot (\pm\infty) = 0$ *and* $1/(\pm\infty) = 0$.

(ii) *We put*

$$p' := \begin{cases} \frac{p}{1-p} & \text{if} \quad 0 < p < 1, \\ \infty & \text{if} \quad p = 1, \\ \frac{p}{p-1} & \text{if} \quad 1 < p < \infty, \\ 1 & \text{if} \quad p = \infty. \end{cases}$$

(iii) *To state our results we use the notation* $p \to q$ *for* $0 < p,\, q \le \infty$ *defined by*

$$\frac{1}{p \to q} = \frac{1}{q} - \frac{1}{p} \qquad \text{if} \qquad q < p,$$

and $p \to q = \infty$ *if* $q \ge p$.

(iv) *If* $I = (a,b) \subseteq \mathbb{R}$ *and* g *is a monotone function on* I, *then by* $g(a)$ *and* $g(b)$ *we mean the limits* $\lim_{t\to a+} g(t)$ *and* $\lim_{t\to b-} g(t)$, *respectively.*

Our main results are the following theorems. Throughout the paper we will denote

$$\widetilde{V}(x) := \|v_1^{-1}v_2\|_{p_1\to p_2,B(0,x)}, \quad \text{and} \quad \mathcal{V}(t,x) := \frac{\widetilde{V}(t)}{\widetilde{V}(t)+\widetilde{V}(x)} \ (t>0,\ x>0).$$

Theorem 2.1. *Let* $0<\theta_2=p_2\le p_1=\theta_1<\infty$. *Assume that* $v_1,v_2\in\mathcal{W}(\mathbb{R}^n)$, $\omega_1\in {}^{\complement}\Omega_{\theta_1}$ *and* $\omega_2\in\Omega_{\theta_2}$. *Then*

$$\|\,\mathrm{I}\,\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \Big\|\|\omega_1\|_{p_1,(0,|\cdot|)}^{-1}\ \|\omega_2\|_{p_2,(|\cdot|,\infty)}\Big\|_{p_1\to p_2,v_1^{-1}v_2,\mathbb{R}^n}.$$

Theorem 2.2. *Let* $0<p_1,\,p_2,\,\theta_1,\,\theta_2<\infty$ *and* $\theta_2\ne p_2\le p_1=\theta_1$. *Assume that* $v_1,v_2\in\mathcal{W}(\mathbb{R}^n)$, $\omega_1\in{}^{\complement}\Omega_{\theta_1}$ *and* $\omega_2\in\Omega_{\theta_2}$.

(i) If $p_1\le\theta_2$, then

$$\|\,\mathrm{I}\,\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{t\in(0,\infty)} \Big\|\|\omega_1\|_{p_1,(0,|\cdot|)}^{-1}\Big\|_{p_1\to p_2,v_1^{-1}v_2,B(0,t)}\|\omega_2\|_{\theta_2,(t,\infty)};$$

(ii) If $\theta_2<p_1$, then

$$\|\,\mathrm{I}\,\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \bigg(\int_0^\infty \Big\|\|\omega_1\|_{p_1,(0,|\cdot|)}^{-1}\Big\|_{p_1\to p_2,v_1^{-1}v_2,B(0,t)}^{p_1\to\theta_2}\, d\Big(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1\to\theta_2}\Big)\bigg)^{\frac{1}{p_1\to\theta_2}}.$$

Theorem 2.3. *Let* $0<p_1,\,p_2,\,\theta_1,\,\theta_2<\infty$ *and* $\theta_2=p_2\le p_1\ne\theta_1$. *Assume that* $v_1,v_2\in\mathcal{W}(\mathbb{R}^n)$, $\omega_1\in{}^{\complement}\Omega_{\theta_1}$ *and* $\omega_2\in\Omega_{\theta_2}$.

(i) *If* $\theta_1\le p_2$, *then*

$$\|\,\mathrm{I}\,\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{t\in(0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-1}\Big\|\|\omega_2\|_{p_2,(|\cdot|,\infty)}\Big\|_{p_1\to p_2,v_1^{-1}v_2,B(0,t)};$$

(ii) *If* $p_2<\theta_1$, *then*

$$\begin{aligned}&\|\,\mathrm{I}\,\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}\\ &\approx \bigg(\int_0^\infty \Big\|\|\omega_2\|_{p_2,(|\cdot|,\infty)}\Big\|_{p_1\to p_2,v_1^{-1}v_2,B(0,t)}^{\theta_1\to p_2} d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1\to p_2}\Big)\bigg)^{\frac{1}{\theta_1\to p_2}}\\ &\quad + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1}\Big\|\|\omega_2\|_{p_2,(|\cdot|,\infty)}\Big\|_{p_1\to p_2,v_1^{-1}v_2,\mathbb{R}^n}.\end{aligned}$$

In view of Lemma 1.2, Theorems 2.1 - 2.3 are straightforward corrolaries of [27, Theorem 3.1] and [27, Theorem 4.2].

To state further results we need the following definitions.

Definition 1. Let U be a continuous, strictly increasing function on $[0,\infty)$ such that $U(0)=0$ and $\lim_{t\to\infty}U(t)=\infty$. Then we say that U is admissible.

Let U be an admissible function. We say that a function φ is U-quasiconcave if φ is equivalent to an increasing function on $(0,\infty)$ and φ/U is equivalent to a decreasing function on $(0,\infty)$. We say that a U-quasiconcave function φ is non-degenerate if

$$\lim_{t\to 0+}\varphi(t)=\lim_{t\to\infty}\frac{1}{\varphi(t)}=\lim_{t\to\infty}\frac{\varphi(t)}{U(t)}=\lim_{t\to 0+}\frac{U(t)}{\varphi(t)}=0.$$

The family of non-degenerate U-quasiconcave functions is denoted by Q_U.

Definition 2. Let U be an admissible function, and let w be a non-negative measurable function on $(0,\infty)$. We say that the function φ, defined by

$$\varphi(t)=U(t)\int_0^\infty\frac{w(\tau)\,d\tau}{U(\tau)+U(t)},\qquad t\in(0,\infty),$$

is a fundamental function of w with respect to U. One will also say that $w(\tau)\,d\tau$ is a representation measure of φ with respect to U.

Remark 2. Let φ be the fundamental function of w with respect to U. Assume that

$$\int_0^\infty\frac{w(\tau)\,d\tau}{U(\tau)+U(t)}<\infty,\ t>0,\qquad \int_0^1\frac{w(\tau)\,d\tau}{U(\tau)}=\int_1^\infty w(\tau)\,d\tau=\infty.$$

Then $\varphi\in Q_U$.

Remark 3. Suppose that $\varphi(x)<\infty$ for all $x\in(0,\infty)$, where φ is defined by

$$\varphi(x)=\operatorname*{ess\,sup}_{t\in(0,x)}U(t)\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\frac{w(\tau)}{U(\tau)},\quad t\in(0,\infty).$$

If

$$\limsup_{t\to 0+}w(t)=\limsup_{t\to+\infty}\frac{1}{w(t)}=\limsup_{t\to 0+}\frac{U(t)}{w(t)}=\limsup_{t\to+\infty}\frac{w(t)}{U(t)}=0,$$

then $\varphi\in Q_U$.

Theorem 2.4. *Let $0<p_1,\,p_2,\,\theta_1,\,\theta_2<\infty$, $p_2<p_1$, $\theta_1\le p_2<\theta_2$. Assume that $v_1,v_2\in\mathcal{W}(\mathbb{R}^n)$, $\omega_1\in{}^{\complement}\Omega_{\theta_1}$ and $\omega_2\in\Omega_{\theta_2}$. Suppose that $\widetilde{V}$ is admissible and*

$$\varphi_1(x):=\sup_{t\in(0,\infty)}\widetilde{V}(t)\,\mathcal{V}(x,t)\ \|\omega_1\|_{\theta_1,(0,t)}^{-1}\in Q_{\widetilde{V}^{\frac{1}{p_1\to p_2}}}.$$

(i) *If $p_1\le\theta_2$, then*

$$\|\,\mathrm{I}\,\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}\approx\sup_{x\in(0,\infty)}\varphi_1(x)\sup_{t\in(0,\infty)}\mathcal{V}(t,x)\ \|\omega_2\|_{\theta_2,(t,\infty)}.$$

(ii) *If $\theta_2<p_1$, then*

$$\|\,\mathrm{I}\,\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$
$$\approx\sup_{x\in(0,\infty)}\varphi_1(x)\bigg(\int_0^\infty\mathcal{V}(t,x)^{p_1\to\theta_2}\,d\Big(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1\to\theta_2}\Big)\bigg)^{\frac{1}{p_1\to\theta_2}}.$$

Theorem 2.5. *Let $0 < p_1, p_2, \theta_1, \theta_2 < \infty$, $p_2 < p_1$ and $p_2 < \min\{\theta_1, \theta_2\}$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$, $\omega_1 \in {}^{\complement}\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Suppose that $\widetilde{V}$ is admissible and*

$$\varphi_2(x) := \bigg(\int_0^\infty [\widetilde{V}(t)\mathcal{V}(x,t)]^{\theta_1 \to p_2} \, d\Big(- \|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \to p_2} \Big) \bigg)^{\frac{1}{\theta_1 \to p_2}} \in Q_{\widetilde{V}^{\frac{1}{p_1 \to p_2}}}.$$

(i) *If $\max\{p_1, \theta_1\} \le \theta_2$, then*

$$\| \mathrm{I} \|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{x\in(0,\infty)} \varphi_2(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x) \|\omega_2\|_{\theta_2,(t,\infty)} + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t\in(0,\infty)} \widetilde{V}(t) \|\omega_2\|_{\theta_2,(t,\infty)};$$

(ii) *If $p_1 \le \theta_2 < \theta_1$, then*

$$\begin{aligned} &\| \mathrm{I} \|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \\ &\quad \approx \bigg(\int_0^\infty \varphi_2(x)^{\frac{\theta_1 \to \theta_2 \cdot \theta_1 \to p_2}{\theta_2 \to p_2}} \widetilde{V}(x)^{\theta_1 \to p_2} \Big(\sup_{t\in(0,\infty)} \mathcal{V}(t,x) \|\omega_2\|_{\theta_2,(t,\infty)} \Big)^{\theta_1 \to \theta_2} \\ &\qquad \times d\Big(- \|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \to p_2} \Big) \bigg)^{\frac{1}{\theta_1 \to \theta_2}} \\ &\qquad + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t\in(0,\infty)} \widetilde{V}(t) \|\omega_2\|_{\theta_2,(t,\infty)}; \end{aligned}$$

(iii) *If $\theta_1 \le \theta_2 < p_1$, then*

$$\begin{aligned} &\| \mathrm{I} \|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \\ &\quad \approx \sup_{x\in(0,\infty)} \varphi_2(x) \bigg(\int_0^\infty \mathcal{V}(t,x)^{p_1 \to \theta_2} d\Big(- \|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \Big) \bigg)^{\frac{1}{p_1 \to \theta_2}} \\ &\qquad + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \bigg(\int_0^\infty \widetilde{V}(t)^{p_1 \to \theta_2} d\Big(- \|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \Big) \bigg)^{\frac{1}{p_1 \to \theta_2}}; \end{aligned}$$

(iv) *If $\theta_2 < \min\{p_1, \theta_1\}$, then*

$$\begin{aligned} &\| \mathrm{I} \|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \\ &\approx \bigg(\int_0^\infty \varphi_2(x)^{\frac{\theta_1 \to \theta_2 \cdot \theta_1 \to p_2}{\theta_2 \to p_2}} \widetilde{V}(x)^{\theta_1 \to p_2} \bigg(\int_0^\infty \mathcal{V}(t,x)^{p_1 \to \theta_2} d\Big(- \|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \Big) \bigg)^{\frac{\theta_1 \to \theta_2}{p_1 \to \theta_2}} \\ &\qquad \times d\Big(- \|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \to p_2} \Big) \bigg)^{\frac{1}{\theta_1 \to \theta_2}} \\ &\quad + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \bigg(\int_0^\infty \widetilde{V}(t)^{p_1 \to \theta_2} d\Big(- \|\omega_2\|_{\theta_2,(t,\infty)}^{p_1 \to \theta_2} \Big) \bigg)^{\frac{1}{p_1 \to \theta_2}}. \end{aligned}$$

Theorem 2.6. *Let $0 < \theta_1 < p < \theta_2 < \infty$. Assume that $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $\omega_1 \in {}^{\complement}\Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$.*

$$\| \mathrm{I} \|_{{}^{\complement}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{t\in(0,\infty)} \Big\| \|\omega_1\|_{\theta_1,(0,|\cdot|)}^{-1} \Big\|_{\infty, v_1^{-1} v_2, B(0,t)} \|\omega_2\|_{\theta_2,(t,\infty)}.$$

Theorem 2.7. *Let* $0 < \theta_1, \theta_2 < \infty$ *and* $0 < p < \min\{\theta_1, \theta_2\}$. *Assume that* $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$ *such that* $v_1^{-1} v_2 \in C(\mathbb{R}^n)$. *Suppose that* $\omega_1 \in {}^{\complement}\Omega_{\theta_1}$, $\omega_2 \in \Omega_{\theta_2}$ *and*

$$0 < \|\omega_2^{-1}\|_{\theta_2 \to p,(x,\infty)} < \infty$$

holds for all $x > 0$.

(i) *If* $\theta_1 \le \theta_2$, *then*

$$\begin{aligned}
&\| \mathrm{I} \|_{{}^{\complement}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \\
&\quad \approx \sup_{x \in (0,\infty)} \Bigg(\widetilde{V}(x)^{\theta_1 \to p} \int_x^\infty d\Big(- \|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \to p} \Big) \\
&\qquad + \int_0^x \widetilde{V}(t)^{\theta_1 \to p} \, d\Big(- \|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \to p} \Big) \Bigg)^{\frac{1}{\theta_1 \to p}} \|\omega_2\|_{\theta_2,(x,\infty)} \\
&\qquad + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t \in (0,\infty)} \widetilde{V}(t) \|\omega_2\|_{\theta_2,(t,\infty)};
\end{aligned}$$

(ii) *If* $\theta_2 < \theta_1$, *then*

$$\begin{aligned}
&\| \mathrm{I} \|_{{}^{\complement}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \\
&\approx \Bigg(\int_0^\infty \bigg(\int_x^\infty d\Big(- \|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \to p} \Big) \bigg)^{\frac{\theta_1 \to \theta_2}{\theta_2 \to p}} \bigg(\sup_{0<\tau \le x} \widetilde{V}(\tau) \|\omega_2\|_{\theta_2,(\tau,\infty)} \bigg)^{\theta_1 \to \theta_2} \\
&\qquad \times d\Big(- \|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \to p} \Big) \Bigg)^{\frac{1}{\theta_1 \to \theta_2}} \\
&\quad + \Bigg(\int_0^\infty \bigg(\int_0^x \widetilde{V}(t)^{\theta_1 \to p} d\Big(- \|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1 \to p} \Big) \bigg)^{\frac{\theta_1 \to \theta_2}{\theta_2 \to p}} \widetilde{V}(x)^{\theta_1 \to p} \|\omega_2\|_{\theta_2,(t,\infty)}^{\theta_1 \to \theta_2} \\
&\qquad \times d\Big(- \|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1 \to p} \Big) \Bigg)^{\frac{1}{\theta_1 \to \theta_2}} \\
&\quad + \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \sup_{t \in (0,\infty)} \widetilde{V}(t) \|\omega_2\|_{\theta_2,(t,\infty)}.
\end{aligned}$$

3 Proofs of main results

Before proceeding to the proof of our main results we recall the following integration in polar coordinates formula.

We denote the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ in $\mathbb{R}^n$ by S^{n-1}. If $x \in \mathbb{R}^n \backslash \{0\}$, the polar coordinates of x are

$$r = |x| \in (0,\infty), \qquad x' = \frac{x}{|x|} \in S^{n-1}.$$

There is a unique Borel measure $\sigma = \sigma_{n-1}$ on S^{n-1} such that if f is Borel measurable on $\mathbb{R}^n$ and $f \ge 0$ or $f \in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_{S^{n-1}} f(rx') r^{n-1} d\sigma(x') dr$$

(see, for instance, [15, p. 78]).

It should be noted that ${}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1) \not\hookrightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)$ when $0<p_1,\,p_2,\,\theta_1,\,\theta_2\le\infty$ and $p_1<p_2$, where $v_1,v_2\in\mathcal{W}(\mathbb{R}^n)$, $\omega_1\in{}^{\complement}\Omega_{\theta_1}$ and $\omega_2\in\Omega_{\theta_2}$. To see this, assume that ${}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\hookrightarrow LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)$ holds. Then there exist $c>0$ such that

$$\|f\|_{LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}\le c\,\|f\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}$$

holds for all $f\in\mathfrak{M}^+(\mathbb{R}^n)$. Let $\tau\in(0,\infty)$ and $f\in\mathfrak{M}(\mathbb{R}^n)$: $\operatorname{supp} f\subset B(0,\tau)$. It is easy to see that

$$\begin{aligned}\|f\|_{LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} &= \big\|\|f\|_{p_2,v_2,B(0,t)}\big\|_{\theta_2,\omega_2,(0,\infty)}\\ &\ge \big\|\|f\|_{p_2,v_2,B(0,t)}\big\|_{\theta_2,\omega_2,(\tau,\infty)}\\ &\ge \|\omega_2\|_{\theta_2,(\tau,\infty)}\,\|f\|_{p_2,v_2,B(0,\tau)}\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\|f\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)} &= \big\|\|f\|_{p_1,v_1,{}^{\complement}B(0,t)}\big\|_{\theta_1,\omega_1,(0,\infty)}\\ &= \big\|\|f\|_{p_1,v_1,{}^{\complement}B(0,t)}\big\|_{\theta_1,\omega_1,(0,\tau)}\\ &\le \|\omega_1\|_{\theta_1,(0,\tau)}\,\|f\|_{p_1,v_1,B(0,\tau)}.\end{aligned}\tag{3.2}$$

Combining (3.1) with (3.2), we can assert that

$$\|\omega_2\|_{\theta_2,(\tau,\infty)}\,\|f\|_{p_2,v_2,B(0,\tau)}\le c\,\|\omega_1\|_{\theta_1,(0,\tau)}\,\|f\|_{p_1,v_1,B(0,\tau)}.$$

Since $\omega_1\in{}^{\complement}\Omega_{\theta_1}$ and $\omega_2\in\Omega_{\theta_2}$, we conclude that $L_{p_1}(B(0,\tau),v_1)\hookrightarrow L_{p_2}(B(0,\tau),v_2)$, which is a contradiction.

The following lemma is true.

Lemma 3.1. *Let $0<p_1,\,p_2,\,\theta_1,\,\theta_2<\infty$, $p_2\le p_1$ and $p_2<\theta_2$. Assume that $v_1,v_2\in\mathcal{W}(\mathbb{R}^n)$, $\omega_1\in{}^{\complement}\Omega_{\theta_1}$ and $\omega_2\in\Omega_{\theta_2}$. Then*

$$\|\mathrm{I}\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} = \left\{\sup_{g\in\mathfrak{M}^+(0,\infty)}\frac{\|\mathrm{I}\|^{p_2}_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to L_{p_2}\left(v_2(\cdot)H^*g(|\cdot|)^{\frac{1}{p_2}}\right)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}}\right\}^{\frac{1}{p_2}}.$$

Proof. By duality, interchanging suprema, we have that

$$\begin{aligned}&\|\mathrm{I}\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}\\ &\quad=\sup_{f\in\mathfrak{M}^+(\mathbb{R}^n)}\frac{\|f\|_{LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}}{\|f\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}}\\ &\quad=\sup_{f\in\mathfrak{M}^+(\mathbb{R}^n)}\frac{1}{\|f\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}}\sup_{g\in\mathfrak{M}^+(0,\infty)}\frac{\left(\int_0^\infty\left(\int_{B(0,\tau)}f(x)^{p_2}v_2(x)^{p_2}\,dx\right)g(\tau)\,d\tau\right)^{\frac{1}{p_2}}}{\|g\|^{\frac{1}{p_2}}_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}}\\ &\quad=\sup_{g\in\mathfrak{M}^+(0,\infty)}\frac{1}{\|g\|^{\frac{1}{p_2}}_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}}\sup_{f\in\mathfrak{M}^+(\mathbb{R}^n)}\frac{\left(\int_0^\infty\left(\int_{B(0,\tau)}f(x)^{p_2}v_2(x)^{p_2}\,dx\right)g(\tau)\,d\tau\right)^{\frac{1}{p_2}}}{\|f\|_{{}^{\complement}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}}.\end{aligned}$$

Applying Fubini's Theorem, we get that

$$\|\mathrm{I}\|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$
$$= \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{1}{\|g\|^{\frac{1}{p_2}}_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \sup_{f\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} f(x)^{p_2} v_2(x)^{p_2}\left(\int_{|x|}^\infty g(\tau)\,d\tau\right)dx\right)^{\frac{1}{p_2}}}{\|f\|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)}}$$
$$= \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{1}{\|g\|^{\frac{1}{p_2}}_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \|\mathrm{I}\|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to L_{p_2}\left(v_2(\cdot)H^*g(|\cdot|)^{\frac{1}{p_2}}\right)}. \tag{3.3}$$

□

Proof of Theorem 2.4. By Lemma 3.1, we have that

$$\|\mathrm{I}\|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} = \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{1}{\|g\|^{\frac{1}{p_2}}_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \|\mathrm{I}\|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to L_{p_2}\left(v_2(\cdot)H^*g(|\cdot|)^{\frac{1}{p_2}}\right)}.$$

Since $\theta_1 \le p_2$, applying [27, Theorem 4.2, (a)], we obtain that

$$\|\mathrm{I}\|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\sup_{t\in(0,\infty)} \|\omega_1\|^{-p_2}_{\theta_1,(0,t)} \|H^*g(|\cdot|)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},B(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}.$$

By using polar coordinates, we have that

$$\|H^*g(|\cdot|)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},B(0,t)} = \|H^*g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,t)}, \quad t>0,$$

where

$$\tilde{v}(r) := \int_{S^{n-1}} (v_1^{-1}v_2)(rx')^{\frac{p_1p_2}{p_1-p_2}} r^{n-1} d\sigma(x'), \quad r>0.$$

Thus, we obtain that

$$\|\mathrm{I}\|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\sup_{t\in(0,\infty)} \|\omega_1\|^{-p_2}_{\theta_1,(0,t)} \|H^*g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}.$$

Taking into account that

$$\int_0^t \tilde{v}(r)\,dr = \int_0^t \int_{S^{n-1}} (v_1^{-1}v_2)(rx')^{\frac{p_1p_2}{p_1-p_2}}\,d\sigma(x') r^{n-1} dr$$
$$= \int_{B(0,t)} (v_1^{-1}v_2)^{\frac{p_1p_2}{p_1-p_2}}(x)\,dx = \widetilde{V}(t)^{\frac{p_1p_2}{p_1-p_2}}, \tag{3.4}$$

(i) if $p_1 \le \theta_2$, then applying [21, Theorem 3.2, (i)], we arrive at

$$\| \mathrm{I} \|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{x\in(0,\infty)} \varphi_1(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|\omega_2\|_{\theta_2,(t,\infty)};$$

(ii) if $\theta_2 < p_1$, then applying [21, Theorem 3.2, (ii)], we arrive at

$$\| \mathrm{I} \|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \sup_{x\in(0,\infty)} \varphi_1(x)\bigg(\int_0^\infty \mathcal{V}(t,x)^{p_1\to\theta_2} d\Big(-\|\omega_2\|_{\theta_2,(t,\infty)}^{p_1\to\theta_2}\Big)\bigg)^{\frac{1}{p_1\to\theta_2}}.$$

□

Remark 4. In view of Remark 3, if

$$\limsup_{t\to 0+} \widetilde{V}(t)\|\omega_1\|_{\theta_1,(0,t)}^{-1} = \limsup_{t\to+\infty} \widetilde{V}(t)\|\omega_1\|_{\theta_1,(0,t)} = \limsup_{t\to 0+} \|\omega_1\|_{\theta_1,(0,t)} = \limsup_{t\to+\infty} \|\omega_1\|_{\theta_1,(0,t)}^{-1} = 0,$$

then $\varphi_1 \in Q_{\widetilde{V}^{\frac{1}{p_1\to p_2}}}$.

Proof of Theorem 2.5. By Lemma 3.1, applying [27, Theorem 4.2, (c)], we have that

$$\| \mathrm{I} \|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|H^*g(|\cdot|)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},\mathbb{R}^n}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}} + \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\bigg(\int_0^\infty \|H^*g(|\cdot|)\|_{\frac{p_1}{p_1-p_2},(v_1^{-1}v_2)^{p_2},B(0,t)}^{\frac{\theta_1}{\theta_1-p_2}} d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\frac{\theta_1p_2}{\theta_1-p_2}}\Big)\bigg)^{\frac{\theta_1-p_2}{\theta_1}}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}}.$$

By using polar coordinates, we have that

$$\| \mathrm{I} \|_{{}^{\mathbf{c}}LM_{p_1\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p_2\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1} \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|H^*g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,\infty)}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}} + \left\{ \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\bigg(\int_0^\infty \|H^*g\|_{\frac{p_1}{p_1-p_2},\tilde{v}^{\frac{p_1-p_2}{p_1}},(0,t)}^{\frac{\theta_1}{\theta_1-p_2}} d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\frac{\theta_1p_2}{\theta_1-p_2}}\Big)\bigg)^{\frac{\theta_1-p_2}{\theta_1}}}{\|g\|_{\frac{\theta_2}{\theta_2-p_2},\omega_2^{-p_2},(0,\infty)}} \right\}^{\frac{1}{p_2}} := C_1 + C_2.$$

Assume first that $p_1 \le \theta_2$. By using the characterization of the boundedness of the operator H^* in weighted Lebesgue spaces (see, for instance, [28, 25]), we arrive at

$$C_1 \approx \|\omega_1\|^{-1}_{\theta_1,(0,\infty)} \sup_{t\in(0,\infty)} \widetilde{V}(t)\, \|\omega_2\|_{\theta_2,(t,\infty)}.$$

(i) Let $\theta_1 \le \theta_2$. By applying [21, Theorem 3.1, (i)], we obtain that

$$C_2 \approx \sup_{x\in(0,\infty)} \varphi_2(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x)\, \|\omega_2\|_{\theta_2,(t,\infty)}.$$

Consequently, the proof is completed in this case.

(ii) Let $\theta_2 < \theta_1$. By using [21, Theorem 3.1, (ii)], we have that

$$C_2 \approx \bigg(\int_0^\infty \varphi_2(x)^{\frac{\theta_1\to\theta_2\cdot\theta_1\to p_2}{\theta_2\to p_2}} \widetilde{V}(x)^{\theta_1\to p_2} \Big(\sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|\omega_2\|_{\theta_2,(t,\infty)} \Big)^{\theta_1\to\theta_2} \times d\Big(- \|\omega_1\|^{-\theta_1\to p_2}_{\theta_1,(0,x)} \Big) \bigg)^{\frac{1}{\theta_1\to\theta_2}},$$

and the statement follows in this case.

Let us now assume that $\theta_2 < p_1$. Then, using the characterization of the boundedness of the operator H^* in weighted Lebesgue spaces, we have that

$$C_1 \approx \|\omega_1\|^{-1}_{\theta_1,(0,\infty)} \bigg(\int_0^\infty \widetilde{V}(t)^{p_1\to\theta_2}\, d\Big(- \|\omega_2\|^{p_1\to\theta_2}_{\theta_2,(t,\infty)} \Big) \bigg)^{\frac{1}{p_1\to\theta_2}}.$$

(iii) Let $\theta_1 \le \theta_2$, then [21, Theorem 3.1, (iii)] yields that

$$C_2 \approx \sup_{x\in(0,\infty)} \varphi_2(x) \bigg(\int_0^\infty \mathcal{V}(t,x)^{p_1\to\theta_2} d\Big(- \|\omega_2\|^{p_1\to\theta_2}_{\theta_2,(t,\infty)} \Big) \bigg)^{\frac{1}{p_1\to\theta_2}},$$

and these completes the proof in this case.

(iv) If $\theta_2 < \theta_1$, then on using [21, Theorem 3.1, (iv)], we arrive at

$$C_2 \approx \bigg(\int_0^\infty \varphi_2(x)^{\frac{\theta_1\to\theta_2\cdot\theta_1\to p_2}{\theta_2\to p_2}} \widetilde{V}(x)^{\theta_1\to p_2} \bigg(\int_0^\infty \mathcal{V}(t,x)^{p_1\to\theta_2} d\Big(- \|\omega_2\|^{p_1\to\theta_2}_{\theta_2,(t,\infty)} \Big) \bigg)^{\frac{\theta_1\to\theta_2}{p_1\to\theta_2}} \times d\Big(- \|\omega_1\|^{-\theta_1\to p_2}_{\theta_1,(0,x)} \Big) \bigg)^{\frac{1}{\theta_1\to\theta_2}},$$

and in this case the proof is completed. □

Remark 5. Assume that $\varphi_2(x) < \infty,\ x > 0$. In view of Remark 2, if

$$\int_0^1 \Big(\int_0^t \omega_1^{\theta_1} \Big)^{-\frac{\theta_1}{\theta_1 - p_2}} \omega_1^{\theta_1}(t)\, dt = \int_1^\infty \widetilde{V}(t)^{\frac{\theta_1 p_2}{\theta_1 - p_2}} \Big(\int_0^t \omega_1^{\theta_1} \Big)^{-\frac{\theta_1}{\theta_1 - p_2}} \omega_1^{\theta_1}(t)\, dt = \infty,$$

then $\varphi_2 \in Q_{\widetilde{V}^{\frac{1}{p_1\to p_2}}}$.

Proof of Theorem 2.6. By Lemma 3.1, applying [27, Theorem 4.2, (b)], we get that

$$\| \mathrm{I} \|_{{}^{\mathfrak{c}}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)} = \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\sup_{t\in(0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p} \|H^* g(|\cdot|)\|_{\infty,(v_1^{-1}v_2)^p,B(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}}.$$

Recall that, whenever F, G are non-negative measurable functions on $(0,\infty)$ and F is non-increasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{\tau\in(0,t)} G(\tau). \tag{3.5}$$

Observe that

$$\|H^* g(|\cdot|)\|_{\infty,(v_1^{-1}v_2)^p,B(0,t)} = \sup_{\tau\in(0,t)} \sup_{|y|=\tau} \big(v_1^{-1}(y)v_2(y)\big)^p H^* g(|y|) = \|H^* g\|_{\infty,\tilde{\tilde{v}},(0,t)} \tag{3.6}$$

holds for all $t>0$, where $\tilde{\tilde{v}}(\tau) := \big(\sup_{|y|=\tau} v_1^{-1}(y)v_2(y)\big)^p$, $\tau>0$.

By using (3.5), we get that

$$\begin{aligned} \| \mathrm{I} \|_{{}^{\mathfrak{c}}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)} &= \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\sup_{t\in(0,\infty)} \|\omega_1\|_{\theta_1,(0,t)}^{-p} \|H^* g\|_{\infty,\tilde{\tilde{v}},(0,t)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}} \\ &= \left\{ \sup_{g \in \mathfrak{M}^+(0,\infty)} \frac{\|H^* g\|_{\infty,\|\omega_1\|_{\theta_1,(0,\cdot)}^{-p}\tilde{\tilde{v}}(\cdot),(0,\infty)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}} \right\}^{\frac{1}{p}}. \end{aligned}$$

By using the characterization of the boundedness of H^* in weighted Lebesgue spaces, we obtain that

$$\begin{aligned} &\| \mathrm{I} \|_{{}^{\mathfrak{c}}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1) \to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)} \\ &\qquad \approx \sup_{t\in(0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left(\sup_{s\in(0,t)} \|\omega_1\|_{\theta_1,(0,s)}^{-1} \tilde{\tilde{v}}(s)^{\frac{1}{p}} \right) \\ &\qquad = \sup_{t\in(0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left(\sup_{s\in(0,t)} \sup_{|y|=s} \|\omega_1\|_{\theta_1,(0,|y|)}^{-1} v_1^{-1}(y)v_2(y) \right) \\ &\qquad = \sup_{t\in(0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \left(\sup_{x\in B(0,t)} \|\omega_1\|_{\theta_1,(0,|x|)}^{-1} v_1^{-1}(x)v_2(x) \right) \\ &\qquad = \sup_{t\in(0,\infty)} \|\omega_2\|_{\theta_2,(t,\infty)} \Big\| \|\omega_1\|_{\theta_1,(0,|\cdot|)}^{-1} \Big\|_{\infty,v_1^{-1}v_2,B(0,t)}. \end{aligned}$$

□

Proof of Theorem 2.7. By Lemma 3.1, applying [27, Theorem 4.2, (d)], and using (3.6), we get that

$$\|\,\mathrm{I}\,\|_{{}^{\mathbf{c}}LM_{p\theta_1,\omega_1}(\mathbb{R}^n,v_1)\to LM_{p\theta_2,\omega_2}(\mathbb{R}^n,v_2)}$$
$$\approx \|\omega_1\|_{\theta_1,(0,\infty)}^{-1}\left\{\sup_{g\in\mathfrak{M}^+(0,\infty)}\frac{\|H^*g\|_{\infty,\tilde{\tilde{v}},(0,\infty)}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}}\right\}^{\frac{1}{p}}$$
$$+\left\{\sup_{g\in\mathfrak{M}^+(0,\infty)}\frac{\left(\int_0^\infty\|H^*g\|_{\infty,\tilde{\tilde{v}},(0,t)}^{\frac{\theta_1}{\theta_1-p}}d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\frac{\theta_1 p}{\theta_1-p}}\Big)\right)^{\frac{\theta_1-p}{\theta_1}}}{\|g\|_{\frac{\theta_2}{\theta_2-p},\omega_2^{-p},(0,\infty)}}\right\}^{\frac{1}{p}}$$
$$:= C_3+C_4.$$

Again, by using the characterization of the boundedness of H^* in weighted Lebesgue spaces, we obtain that

$$C_3\approx\|\omega_1\|_{\theta_1,(0,\infty)}^{-1}\sup_{t\in(0,\infty)}\widetilde{V}(t)\|\omega_2\|_{\theta_2,(t,\infty)}.$$

(i) Let $\theta_1\le\theta_2$, then by [22, Theorem 4.1], we have that

$$C_4\approx\sup_{x\in(0,\infty)}\left(\widetilde{V}(x)^{\theta_1\to p}\int_x^\infty d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1\to p}\Big)\right.$$
$$\left.+\int_0^x\widetilde{V}(t)^{\theta_1\to p}\,d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1\to p}\Big)\right)^{\frac{1}{\theta_1\to p}}\|\omega_2\|_{\theta_2,(x,\infty)},$$

and the statement follows in this case.

(ii) Let $\theta_2<\theta_1$, then [22, Theorem 4.4] yields that

$$C_4\approx\left(\int_0^\infty\left(\int_x^\infty d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1\to p}\Big)\right)^{\frac{\theta_1\to\theta_2}{\theta_2\to p}}\left(\sup_{0<\tau\le x}\widetilde{V}(\tau)\|\omega_2\|_{\theta_2,(\tau,\infty)}\right)^{\theta_1\to\theta_2}\right.$$
$$\left.\times d\Big(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1\to p}\Big)\right)^{\frac{1}{\theta_1\to\theta_2}}$$
$$+\left(\int_0^\infty\left(\int_0^x\widetilde{V}(t)^{\theta_1\to p}d\Big(-\|\omega_1\|_{\theta_1,(0,t)}^{-\theta_1\to p}\Big)\right)^{\frac{\theta_1\to\theta_2}{\theta_2\to p}}\widetilde{V}(x)^{\theta_1\to p}\|\omega_2\|_{\theta_2,(t,\infty)}^{\theta_1\to\theta_2}\right.$$
$$\left.\times d\Big(-\|\omega_1\|_{\theta_1,(0,x)}^{-\theta_1\to p}\Big)\right)^{\frac{1}{\theta_1\to\theta_2}},$$

and the proof is completed in this case. □

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