Tekutiny v pohybu
Fluids in motion

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Fluids in motion

honey

beer

tornado

Sun

plane

Fluids in the real world

- wheather prediction
- ships, planes, cars, trains
- astrophysics, gaseous stars
- rivers, floods, oceans, tsunami waves
- human body, blood motion

Mathematical issues

- Modeling
- Analysis of models, well-posedness, stability, determinism (?)
- Numerical analysis and implementations, computations

Do we need mathematics?

Luc Tartar
[Compensation effects in partial differential equations]

What puzzles me more is the behaviour of people who have failed to become good mathematicians and advocate using the language of engineers ... as if they were not aware of the efficiency of the engineering approach that one can control processes that one does not understand at all

Mathematical modeling of fluids in motion

Molecular dynamics

Fluids understood as huge families of individual particles (atoms, molecules)

Kinetic models

Large ensembles of particles in *random* motion, description in terms of averages

Continuum fluid mechanics

Phenomenological theory based on observable quantities - mass density, temperature, velocity field

Models of turbulence

Essentially based on classical continuum mechanics but description in terms of averaged quantities

Conservation/balance laws

total amount at time t_2

$\int_B D(t_2, x)\,\mathrm{d}x$

minus

$-$

total amount at time t_1

$\int_B D(t_1, x)\,\mathrm{d}x$

conservation/balance

$=$

boundary flux

$-\int_{t_1}^{t_2} \int_{\partial B} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S_x\,\mathrm{d}t$

plus

$+$

sources

$\int_{t_1}^{t_2} \int_B s\,\mathrm{d}x\,\mathrm{d}t$

Conservation laws as PDE's

Limit processes

$$t_2 \to t_1, \; B = B_x \to x$$

Field equation

$$\frac{\partial}{\partial t} D + \mathrm{div}_x \mathbf{F} = s$$

Constitutive relations

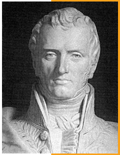
$$\mathbf{F} = \mathbf{F}(D), \; s = s(D)$$

Conclusion

The resulting equations are *partial differential equations* with *nonlinear* dependence of fields

Navier-Stokes system - Millenium Problem

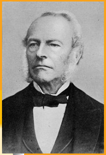
- $\mathbf{u} = \mathbf{u}(t, x)$ .... fluid velocity
- $\Pi = \Pi(t, x)$ .... pressure



"Incompressibility"

$$\mathrm{div}_x \mathbf{u} = 0$$

Claude Louis Marie Henri Navier [1785-1836]

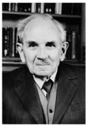


Balance of momentum

$$\partial_t \mathbf{u} + \mathrm{div}_x (\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = \Delta_x \mathbf{u}$$

George Gabriel Stokes [1819-1903]

State of the art



Jean Leray - Royal society (1991)

Jean Leray [1906-1998] Global existence of the so-called **weak** solutions for the incompressible Navier-Stokes system (3D)

Olga Aleksandrovna Ladyzhenskaya [1922-2004] Global existence of classical solutions for the incompressible 2D Navier-Stokes system

Pierre-Louis Lions[*1956] Global existence of weak solutions for the compressible barotropic Navier-Stokes system (2,3D)

and many, many others...

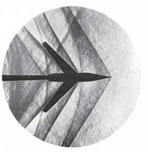
Things may go wrong

Blow-up singularities - concentrations

Solutions become large (infinite) in a finite time.
There is too much energy pumped in the system

Shock waves - oscillations

Shocks are singularities in "derivatives".
Originally smooth solutions become discontinuous in a finite time



"Bad" nonlinearities

$$\partial_t U = \boxed{U^2},\ \partial_t U + \boxed{U \partial_x U} = 0$$

Euler system (compressible inviscid)

- $\mathbf{u} = \mathbf{u}(t, x)$ .... fluid velocity
- $\varrho = \varrho(t, x)$ .... density

Leonhard Paul Euler
[1707-1783]

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Balance of momentum

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Back to integral averages

- *Pointwise* (ideal) values of functions are replaced by their *integral averages*. This idea is close to the physical concept of *measurement*

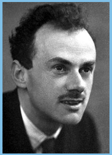
$$u \approx \left[\varphi \mapsto \int u\varphi \right]$$

- Derivatives in the equations replaced by integrals:

$$\frac{\partial u}{\partial x} \approx \left[\varphi \mapsto - \int u \partial_x \varphi \right], \ \varphi \text{ a smooth } \textit{test} \text{ function}$$

Example

Dirac distribution: $\delta_0 : \ \varphi \mapsto \varphi(0)$



Paul Adrien Maurice Dirac [1902-1984]

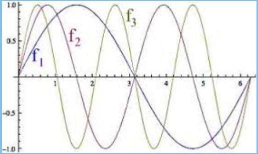
Oscillations vs. nonlinearity

Oscillatory solutions - velocity

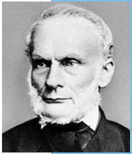
$$U(x) \approx \sin(nx),\ \ U \to 0 \text{ in the sense of avarages (weakly)}$$

Oscillatory solutions - kinetic energy

$$\frac{1}{2}|U|^2(x) \approx \frac{1}{2}\sin^2(nx) \to \frac{1}{4} \neq \frac{1}{2}0^2 \text{ in the sense of avarages (weakly)}$$



Do some solutions lose/produce energy?



Rudolph Clausius, [1822–1888]

First and Second law of thermodynamics

Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu

Mechanical energy balance for compressible fluid

classical: $$\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \ \mathrm{d}x = 0, \ P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z$$

weak: $$\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \ \mathrm{d}x \boxed{\leq} 0$$

Compressible Euler system - the state-of-art

Existence

Global-in-time solutions (in general) do not exist. Weak solutions may exist but may not be uniquely determined by the initial data.

Mechanical energy

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)$$

Admissibility criteria - mechanical energy dissipation

$$\partial_t E + \mathrm{div}_x(E\mathbf{u} + p(\varrho)\mathbf{u}) \boxed{\leq} 0$$

Wild solutions?

Charles Hermite
[1822-1901]

In a letter to Stieltjes

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives

- **Past:** What is not allowed is forbidden
- **Present:** What is not forbidden is allowed

Bad or good news for compressible Euler?

Camillo DeLellis [*1976]

Existence

Good news: There exists a global-in-time weak solution of compressible Euler system for "any" initial data

Bad news: There are infinitely many...

Admissible solutions?

Good news: Most of the "wild" solutions produce energy.

Bad news: There is a vast class of data for which there exist infinitely many admissible solutions

László Székelyhidi [*1977]

Viscosity solutions or maximal dissipation?

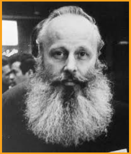
The "correct" solutions "should be" identified as limits of the viscous system

Measure-valued solutions

Young measures

$$U(t,x) \approx \nu_{t,x}[U]$$

$\nu(B), B \subset R^3$ probability that **U** belongs to the set B



Laurence Chisholm Young [1905-2000]

Siddhartha Mishra

Numerical results

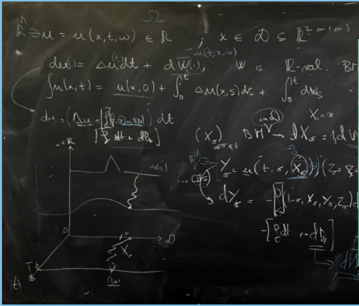
Certain numerical solutions of "inviscid" problems exhibit scheme independent oscillatory behavior

What to do?

However beautiful the strategy, you should occasionally look at the results...

Sir Winston Churchill

[1874-1965]



Some good news to finish...

Navier-Stokes system

- Wild oscillatory solutions are (sofar) not known for problems with *viscosity*, in particular, the Navier-Stokes system (compressible/incompresible)
- Most of the used *numerical schemes* is based on viscous approximation, at least implicitly
- What we compute is mostly the correct solution (??)

Synergy analysis-numerics

- Certain numerical schemes converge to *weak* solutions
- Convergence is unconditional and even error estimates are available if the limit solution is smooth
- Bounded weak solutions *are* smooth
- Bounded solutions of the numerical scheme converge (with error estimates) to the smooth solution