On regularity properties of solutions to the compressible Euler system

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Compressible Euler system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Periodic boundary conditions

$$\Omega = \mathbb{T}^N = \left([-1,1]|_{\{-1;1\}} \right)^N$$

Initial conditions

$$\varrho(0,\cdot) = \varrho_0, \ \ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

Weak solutions

Equation of continuity

$$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t=0}^{\tau} = \int_0^\tau \int_\Omega \left(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right) \ \mathrm{d}x\mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega)$

Balance of momentum

$$\left[\int_\Omega \varrho\mathbf{u}\cdot\varphi \ \mathrm{d}x\right]_{t=0}^{\tau}$$

$$= \int_0^\tau \int_\Omega \left(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \cdot\nabla_x\varphi + p(\varrho)\mathrm{div}_x\varphi\right) \ \mathrm{d}x \ \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega; R^N)$,

Weak solutions: Existence

Global existence for large data in 1D

The Euler system admits global-in-time weak solutions for any bounded initial data (**DiPerna, Chen et al.**). The weak solutions can be recovered as a vanishing viscosity limit of the Navier-Stokes system (**Chen and Perepelitsa**)

Global existence for large data for $N = 2, 3$

The compressible Euler system admits *infinitely many* global-in-time weak solutions for any smooth initial data (**Chiodaroli, EF** - based on the work of **DeLellis and Székelyhidi**)

Stability of 1D solutions - hypotheses

Theorem EF, Y.Sun [2015]

$$\gamma > \frac{N}{2},\ q > \max\{2, \gamma'\},\ \frac{1}{\gamma} + \frac{1}{\gamma'} = 1 \text{ if } N = 2$$

$$q > \max\left\{3, \frac{6\gamma}{5\gamma - 6}\right\} \text{ if } N = 3$$

Let $[R, V]$ be a (strong) solution of the one-dimensional *Navier-Stokes system*, with the initial data belonging to the class

$$R_0 \in W^{1,q}(0,1),\ R_0 > 0,\ V_0 \in W_0^{1,q}(0,1)$$

Let $[\varrho, \mathbf{u}]$ be a finite energy weak solution to the Navier-Stokes system in

$$(0,T) \times \Omega,\ \Omega = (0,1) \times \mathcal{T}^{N-1},$$

with the initial data

$$\varrho_0 \in L^\infty(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in L^2(\Omega; R^3).$$

Stability of 1D solutions - conclusion

Conclusion

Then

$$\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{V}|^2 + P(\varrho) - P'(R)(\varrho - R) - P(R) \right] (\tau, \cdot) \, \mathrm{d}x$$

$$\leq c(T) \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{V}_0|^2 + P(\varrho_0) - P'(R_0)(\varrho_0 - R_0) - P(R_0) \right] \, \mathrm{d}x$$

for a.a. $\tau \in (0, T)$,

$$P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma.$$

Energy conservation

Energy

$$E = \underbrace{\frac{1}{2}\varrho|\mathbf{u}|^2}_{\text{kinetic energy}} + \underbrace{P(\varrho)}_{\text{elastic energy}}, \quad P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2}\, \mathrm{d}z$$

Energy balance equation

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) + \mathrm{div}_x(p(\varrho)\mathbf{u}) = 0$$

Weak formulation

$$\left[\int_\Omega E\varphi \, \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \left(E\partial_t\varphi + E\mathbf{u}\cdot\nabla_x\varphi + p(\varrho)\mathbf{u}\cdot\nabla_x\varphi\right) \, \mathrm{d}x \, \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega)$

Admissible weak solution

Energy dissipation

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) + \mathrm{div}_x(p(\varrho)\mathbf{u}) \leq 0$$

Weak formulation

$$\left[\int_\Omega E\varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$\leq \int_0^\tau \int_\Omega (E\partial_t\varphi + E\mathbf{u}\cdot\nabla_x\varphi + p(\varrho)\mathbf{u}\cdot\nabla_x\varphi) \ \mathrm{d}x \ \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega), \ \varphi \geq 0$

Related results on Onsager's conjecture

Inhomogeneous incompressible Navier–Stokes system

T. M. Leslie and R. Shvydkoy.
The energy balance relation for weak solutions of the density-dependent Navier- Stokes equations.
arXiv:1602.08527v1, 2016.

Inhomogeneous incompressible Euler system

R. M. Chen and C. Yu.
Onsager's energy conservation for inhomogeneous Euler equations.
arXiv:1706.08506v1, 2017.

Minimal regularity

Function spaces - Besov spaces

$$B^{\alpha}_{\infty,p}, \ \|w\|_{B^{\alpha}_{\infty,p}} \equiv \|w\|_{L^p} + \sup_{\xi \neq 0} \frac{\|w(\cdot + \xi) - w\|_{L^p}}{|\xi|^{\alpha}}$$

Mollifiers

$$w_{\varepsilon} = w * \eta_{\varepsilon}, \ \eta_p = \frac{1}{\varepsilon^N} \eta \left(\frac{x}{\varepsilon} \right)$$

Basic estimates

$$\|w_{\varepsilon} - w\|_{L^p} \lesssim \varepsilon^{\alpha} \|w\|_{B^{\alpha}_{\infty,p}}$$

$$\|\nabla w_{\varepsilon}\|_{L^p} \lesssim \varepsilon^{\alpha - 1} \|w\|_{B^{\alpha}_{\infty,p}}$$

$$B^{\alpha}_{\infty,p} \cap L^{\infty} \text{ is algebra}$$

Onsager's conjecture - incompressible fluids

Constantin, E, Titi [1994]

If a velocity field $\mathbf{u}$ solving the *incompressible* Euler system belongs to the class $B^{\alpha}_{\infty,3}$ for $\alpha > \frac{1}{3}$, then the total energy

$$\frac{1}{2} \int_{\Omega} |\mathbf{u}|^2 \, \mathrm{d}x$$

is conserved.

Energy conservation for compressible fluids

EF, Gwiazda, Swierczewska–Gwiazda, Wiedemann [2017]

Let ϱ, $\mathbf{u}$ be a weak solution of the compressible Euler system belonging to the class

$$\mathbf{u} \in B^{\alpha}_{\infty,3}((0,T)\times\mathbb{T}^N),\ \varrho, \varrho\mathbf{u} \in B^{\beta}_{\infty,3}((0,T)\times\mathbb{T}^N),\ 0 \leq \underline{\varrho} \leq \varrho \leq \overline{\varrho},$$

$$0 \leq \alpha, \beta \leq 1,\ \beta > \max\left\{1 - 2\alpha; \frac{1-\alpha}{2}\right\}.$$

Let $p \in C^2[\underline{\varrho}, \overline{\varrho}]$, $p'(0) = 0$ if $\underline{\varrho} = 0$.
Then the energy equality holds in $\mathcal{D}'((0,T)\times\Omega)$.
If, in addition, $u \in L^\infty$ and

$$\sup_{t\in(0,T)} \left[\|\varrho\|_{B^{\beta}_{\infty,3}(\mathbb{T}^N)} + \|\varrho\mathbf{u}\|_{B^{\beta}_{\infty,3}(\mathbb{T}^N)} \right] < \infty,\ \beta > 0,$$

then the energy is (weakly) continuous up to $t = 0$.

Sketch of proof

Regularization method

- Regularize the momentum equation by a space–time convolution kernel
- Use the commutator estimates
- C^2 regularity of the pressure needed

Energy conservation for BV solutions

EF, Gwiazda, Swierczewska–Gwiazda, Wiedemann [2017]

Let $p \in C^2(0,\infty) \cap C[0,\infty)$. Let ϱ, $\mathbf{u}$ be a weak solution of the compressible Euler system belonging to the class

$$\varrho,\ \mathbf{u} \in L^\infty((0,T)\times\Omega),\ \varrho(t),\ \mathbf{u}(t) \in C \cap BV(\Omega) \text{ for a.a. } t \in (0,T)$$

$$\mathrm{ess}\sup_{t\in(0,T)} \left[\|\varrho\|_{C\cap BV(\Omega)} + \|\mathbf{u}\|_{C\cap BV(\Omega)} \right] < \infty.$$

Then the energy equality holds in $\mathcal{D}'((0,T)\times\Omega)$ and the energy is (weakly) continuous up to $t = 0$.

Sketch of proof

Regularization method

- Regularize in space and time. Use *Steklov averaging* in time

$$v_h = \int_0^T \eta_h(t-s)v(s)\,\mathrm{d}s,\ \eta_h = \frac{1}{h}1_{[-h,0]}$$

- Use the commutator estimates
- C^2 regularity of the pressure up to zero not needed here

Further observations

Remarks

- $BV \cap L^\infty \hookrightarrow B^{1/p}_{\infty,p}$ for any $p \geq 1$. In particular, the energy is conserved if

$$\varrho \in BV \cap L^\infty((0,T)\times\Omega),\ \mathbf{u} \in B^{\alpha}_{\infty,3} \cap L^\infty((0,T)\times\Omega),\ \alpha > \frac{1}{3}.$$

- Discontinuous (shock) solutions belong to the critical space and provide sharpness of the result

Less regular solutions conserving energy

Infinitely many weak solutions, Chiodaroli, EF, Luo, Xie, Xin [2016]

For any piecewise constant initial density ϱ_0, there is $\mathbf{u}_0 \in L^\infty$ such that the Euler system admits *infinitely many* weak solutions satisfying the energy equality with the energy continuous up to $t = 0$.

Lipschitz continuous data, Klingenberg, Markfelder [2017]

There exists Lipschitz continuous initial data such that the Euler system admits *infinitely many* weak solutions satisfying the energy equality with the energy (weakly) continuous up to $t = 0$.

Principle of maximal dissipation

Rewriting the energy balance

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) + \mathrm{div}_x(pu) = -\mathcal{D}$$

$$\mathcal{D} \geq 0 - \text{ dissipative defect}$$

Physical solutions

The explicit solutions of the Riemann problem have $\mathcal{D} \neq 0$.

Good and bad news

- "Most wild solutions" do not maximize the dissipation defect.
- Some "classical" solutions do not maximize the *total* dissipation defect $\int_\Omega \mathcal{D} \, \mathrm{d}x$ (**Chiodaroli, Kreml [2014]**)

Measure–valued solutions?

Equation of continuity

$$\left[\int_\Omega \langle Y_{t,x}, \varrho \rangle \varphi \ \mathrm{d}x \right]_{t=0}^{\tau}$$

$$= \int_0^\tau \int_\Omega \left(\langle Y_{t,x}; \varrho \rangle \partial_t \varphi + \langle Y_{t,x}; \varrho \mathbf{u} \rangle \cdot \nabla_x \varphi \right) \ \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x \varphi \cdot \mu_{C_1}$$

Balance of momentum

$$\left[\int_\Omega \langle Y_{t,x}; \varrho \mathbf{u} \rangle \cdot \varphi \ \mathrm{d}x \right]_{t=0}^{\tau}$$

$$= \int_0^\tau \int_\Omega \left(\langle Y_{t,x}; \varrho \mathbf{u} \rangle \cdot \partial_t \varphi + \langle Y_{t,x}; \varrho \mathbf{u} \otimes \mathbf{u} \rangle : \cdot \nabla_x \varphi \right) \ \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \langle Y_{t,x}; p(\varrho) \rangle \mathrm{div}_x \varphi \ \mathrm{d}x \ \mathrm{d}t + \int_0^\tau \int_\Omega \nabla_x \varphi : \mu_{C_2}$$

Dissipation defect

Energy inequality

$$\left[\int_\Omega \left\langle Y_{t,x}; \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right\rangle \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \mathcal{D} = 0$$

Compatibility

$$\int_0^\tau \int_\Omega [|\mu_{C^1}| + |\mu_{C_2}|] \ \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau \mathcal{D} \ \mathrm{d}t$$

MV vs weak solutions

Several reasons why to go measure–valued

- Measure valued and weak are almost "equivalent"
- Measure–valued solutions are generated by numerical schemes, they may enjoy "natural" properties
- Numerical experiments - [**Mishra**]