GLOBAL BEHAVIOR OF A THIRD ORDER RATIONAL DIFFERENCE EQUATION

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Abstract. In this paper, we determine the forbidden set and give an explicit formula for the solutions of the difference equation

$$x_{n+1}=\frac{ax_nx_{n-1}}{-bx_n+cx_{n-2}},\quad n\in\mathbb{N}_0$$

where a, b, c are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers. We show that every admissible solution of that equation converges to zero if either $a<c$ or $a>c$ with $(a-c)/b<1$.

When $a>c$ with $(a-c)/b>1$, we prove that every admissible solution is unbounded. Finally, when $a=c$, we prove that every admissible solution converges to zero.

Keywords: difference equation; forbidden set; periodic solution; unbounded solution

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1. Introduction

Recently, there has been a great interest in studying properties of nonlinear and rational difference equations (see, for example [1]–[22]). Our motivation stems from some recent papers on difference equations which can be solved (see, e.g. [2], [5], [6], [9], [15], [16], [17], [18], [19], [20], [22]).

In this paper, we determine the forbidden set, give an explicit formula for the solutions and discuss the global behavior of solutions of the difference equation

$$x_{n+1}=\frac{ax_nx_{n-1}}{-bx_n+cx_{n-2}},\quad n\in\mathbb{N}_0 \tag{1.1}$$

where a,b,c are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

2. Forbidden set and solutions of equation (1.1)

In this section we derive the forbidden set and give an explicit formula for well-defined solutions of the difference equation (1.1).

Proposition 2.1. *The forbidden set F of equation* (1.1) *is*

$$F = \bigcup_{n=0}^{\infty} \Big\{(u_0, u_{-1}, u_{-2})\colon\ u_0 = u_{-2}\frac{c}{b\sum_{l=0}^{n}(a/c)^i}\Big\}$$
$$\cup \{(u_0, u_{-1}, u_{-2})\colon\ u_0 = 0\} \cup \{(u_0, u_{-1}, u_{-2})\colon\ u_{-1} = 0\}.$$

Proof. Suppose that $x_0x_{-1} = 0$. We have the following cases:

Case 1. If $x_0 = 0$ and $x_{-1} \neq 0$, then x_3 is undefined.

Case 2. If $x_{-1} = 0$ and $x_0 \neq 0$, then x_2 is undefined.

Case 3. If $x_{-2} = 0$ and $x_0x_{-1} \neq 0$, then $x_1 = -(a/b)x_{-1} \neq 0$. Therefore, we have that x_{-1}, x_0 and x_1 are different from zero. This case is reduced to the case when the initial values x_{-2}, x_{-1} and x_0 are different from zero, by shifting indices by one. The case is considered next.

Case 4. Now suppose that $x_{-i} \neq 0$ for all $i \in \{0, 1, 2\}$. From equation (1.1), using the substitution $t_n = x_{n-2}/x_n$, we obtain the linear nonhomogeneous difference equation

$$t_{n+1} = \frac{c}{a}t_n - \frac{b}{a}, \quad t_0 = \frac{x_{-2}}{x_0}. \tag{2.1}$$

We shall deduce the forbidden set of equation (1.1).

Consider the mapping $f(x) = c/ax - b/a$ and suppose that we start from an initial point (x_0, x_{-1}, x_{-2}) such that $x_{-2}/x_0 = b/c$.

Now the backward orbits $x_{n-2}/x_n = v_n$ satisfy the equation

$$v_n = f^{-1}(v_{n-1}) = \frac{a}{c}v_{n-1} + \frac{b}{c} \quad \text{with } v_0 = \frac{x_{-2}}{x_0} = \frac{b}{c},$$

hence we obtain $v_n = x_{n-2}/x_n = f^{-n}(v_0) = (b/c)\sum\limits_{i=0}^{n}(a/c)^i$. Therefore, $x_n = x_{n-2}c/b\sum\limits_{i=0}^{n}(a/c)^i$.

On the other hand, we can observe that if we start from an initial point (x_0, x_{-1}, x_{-2}) such that $t_0 = x_{-2}/x_0 = (b/c)\sum\limits_{i=0}^{n_0}(a/c)^i$ for some $n_0 \in \mathbb{N}$, then according to equation (2.1) we obtain

$$t_{n_0} = \frac{x_{n_0-2}}{x_{n_0}} = \frac{b}{c}.$$

This implies that $-bx_{n_0} + cx_{n_0-2} = 0$. Therefore, x_{n_0+1} is undefined. This completes the proof. □

Theorem 2.2. *Let x_{-2}, x_{-1} and x_0 be real numbers such that $(x_0, x_{-1}, x_{-2}) \notin F$. If $a \neq c$, then the solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) is*

$$x_n = \begin{cases} x_{-1} \displaystyle\prod_{j=0}^{\frac{n-1}{2}} \frac{a-c}{\theta(c/a)^{2j+1} - b}, & n = 1, 3, 5, \ldots, \\ x_0 \displaystyle\prod_{j=0}^{\frac{n-2}{2}} \frac{a-c}{\theta(c/a)^{2j+2} - b}, & n = 2, 4, 6, \ldots \end{cases} \tag{2.2}$$

where $\theta = (a - c + b\alpha)/\alpha$ and $\alpha = x_0/x_{-2}$.

P r o o f. We can write the solution (2.2) as

$$x_{2m+i} = x_{-2+i} \prod_{j=0}^{m} \beta_i(j), \quad i = 1, 2 \text{ and } m = 0, 1, \ldots \tag{2.3}$$

where

$$\beta_i(j) = \frac{a-c}{\theta(c/a)^{2j+i} - b}, \quad i = 1, 2.$$

Hence we can see that

$$x_{-1} \frac{a-c}{(c/a)\theta - b} = x_{-1} \frac{(a-c)a\alpha}{c(a-c+b\alpha) - ba\alpha} = x_{-1} \frac{a\alpha}{c - b\alpha} = \frac{ax_0x_{-1}}{-bx_0 + cx_{-2}} = x_1$$

and

$$\begin{aligned} x_0 \frac{a-c}{(c/a)^2\theta - b} &= x_0 \frac{(a-c)a^2\alpha}{c^2(a-c+b\alpha) - ba^2\alpha} = x_0 \frac{a^2\alpha}{c^2 - b\alpha(c+a)} \\ &= \frac{a^2x_0^2}{c(cx_{-2} - bx_0) - bx_0a} = \frac{ax_0ax_0/(-bx_0 + cx_{-2})}{c - bx_0a/(-bx_0 + cx_{-2})} = \frac{ax_0x_1/x_{-1}}{c - bx_1/x_{-1}} \\ &= \frac{ax_1x_0}{-bx_1 + cx_{-1}} = x_2. \end{aligned}$$

Hence, we see that (2.2) holds for $n = 1$, $n = 2$.

Now assume that $m > 1$. Then

$$x_{2m+3} = \frac{ax_{2m+2}x_{2m+1}}{-bx_{2m+2} + cx_{2m}} = \frac{ax_0 \prod_{j=0}^{m} \beta_2(j) x_{-1} \prod_{j=0}^{m} \beta_1(j)}{-bx_0 \prod_{j=0}^{m} \beta_2(j) + cx_0 \prod_{j=0}^{m-1} \beta_2(j)}$$

$$= \frac{ax_0 \prod_{j=0}^{m} \beta_2(j) x_{-1} \prod_{j=0}^{m} \beta_1(j)}{x_0 \prod_{j=0}^{m-1} \beta_2(j)(-b\beta_2(m) + c)} = \frac{a\beta_2(m) x_{-1} \prod_{j=0}^{m} \beta_1(j)}{-b\beta_2(m) + c}$$

$$= \frac{a(a-c)/\theta(c/a)^{2m+2} - bx_{-1} \prod_{j=0}^{m} \beta_1(j)}{-b(a-c)/(\theta(c/a)^{2m+2} - b) + c} = \frac{a(a-c)x_{-1} \prod_{j=0}^{m} \beta_1(j)}{-b(a-c) + c(\theta(c/a)^{2m+2} - b)}$$

$$= \frac{a(a-c)x_{-1} \prod_{j=0}^{m} \beta_1(j)}{c\theta(c/a)^{2m+2} - ab} = x_{-1} \frac{a-c}{\theta(c/a)^{2m+3} - b} \prod_{j=0}^{m} \beta_1(j)$$

$$= x_{-1} \prod_{j=0}^{m+1} \beta_1(j).$$

To complete the inductive proof, we shall show that formula (2.2) also holds for x_{2m+4}. We have

$$x_{2m+4} = \frac{ax_{2m+3}x_{2m+2}}{-bx_{2m+3} + cx_{2m+1}} = \frac{ax_{-1} \prod_{j=0}^{m+1} \beta_1(j) x_0 \prod_{j=0}^{m} \beta_2(j)}{-bx_{-1} \prod_{j=0}^{m+1} \beta_1(j) + cx_{-1} \prod_{j=0}^{m} \beta_1(j)}$$

$$= \frac{ax_{-1} \prod_{j=0}^{m+1} \beta_1(j) x_0 \prod_{j=0}^{m} \beta_2(j)}{x_{-1} \prod_{j=0}^{m} \beta_1(j)(-b\beta_1(m+1) + c)} = \frac{a\beta_1(m+1) x_0 \prod_{j=0}^{m} \beta_2(j)}{-b\beta_1(m+1) + c}$$

$$= \frac{a(a-c)/(\theta(c/a)^{2m+3} - b) x_0 \prod_{j=0}^{m} \beta_2(j)}{-b(a-c)/\theta(c/a)^{2m+3} - b + c} = \frac{a(a-c)x_0 \prod_{j=0}^{m} \beta_2(j)}{-b(a-c) + c(\theta(c/a)^{2m+3} - b)}$$

$$= \frac{a(a-c)x_0 \prod_{j=0}^{m} \beta_2(j)}{c\theta(c/a)^{2m+3} - ab} = x_0 \frac{a-c}{\theta(c/a)^{2m+4} - b} \prod_{j=0}^{m} \beta_2(j) = x_0 \prod_{j=0}^{m+1} \beta_2(j).$$

This completes the inductive proof of the theorem. $\square$

3. Global behavior of equation (1.1)

In this section, we investigate the global behavior of equation (1.1) with $a \neq c$, using the explicit formula for its solution.

Theorem 3.1. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation* (1.1) *such that* $(x_0, x_{-1}, x_{-2}) \notin F$. *Then the following statements are true.*

(1) *If $a < c$, then $\{x_n\}_{n=-2}^{\infty}$ converges to* 0.

(2) *If $a > c$, then we have the following cases:*

  (a) *If $(a-c)/b < 1$, then $\{x_n\}_{n=-2}^{\infty}$ converges to* 0.

  (b) *If $(a-c)/b > 1$, then both $\{x_{2n}\}_{n=-1}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ are unbounded.*

P r o o f. (1) If $a < c$, then $\beta_i(j)$ converges to 0 as $j \to \infty$, $i = 1, 2$. It follows that there exists $j_0 \in \mathbb{N}$ such that $|\beta_i(j)| < \mu$, with some $0 < \mu < 1$ for all $j \geqslant j_0$. Therefore,

$$|x_{2m+i}| = |x_{-2+i}| \left| \prod_{j=0}^{m} \beta_i(j) \right| = |x_{-2+i}| \left| \prod_{j=0}^{j_0-1} \beta_i(j) \right| \left| \prod_{j=j_0}^{m} \beta_i(j) \right|$$
$$< |x_{-2+i}| \left| \prod_{j=0}^{j_0-1} \beta_i(j) \right| \mu^{m-j_0+1}.$$

As m tends to infinity, the solution $\{x_n\}_{n=-2}^{\infty}$ converges to 0.

(2) Suppose that $a > c$. Then we have the following cases:

(a) If $(a-c)/b < 1$, then $\beta_i(j)$ converges to $-(a-c)/b \in (-1, 0)$ as $j \to \infty$, $i = 1, 2$. Then there exists $j_1 \in \mathbb{N}$ such that, $\beta_i(j) \in (\mu_1, 0)$, with some $0 > \mu_1 > -1$ for all $j \geqslant j_1$ and $i = 1, 2$. Therefore, $|\beta_i(j)| < \mu_1$ for all $j \geqslant j_1$ and the solution $\{x_n\}_{n=-2}^{\infty}$ converges to 0 as in (1).

(b) If $(a-c)/b > 1$, then $\beta_i(j)$ converges to $-(a-c)/b < -1$ as $j \to \infty$, $i = 1, 2$. Then there exists $j_2 \in \mathbb{N}$ such that $\beta_i(j) < \nu < -1$ for some $\nu < -1$ for all $j \geqslant j_2$ and $i = 1, 2$.

For large values of m we have

$$|x_{2m+i}| = |x_{-2+i}| \left| \prod_{j=0}^{m} \beta_i(j) \right| = |x_{-2+i}| \left| \prod_{j=0}^{j_2-1} \beta_i(j) \right| \left| \prod_{j=j_2}^{m} \beta_i(j) \right|$$
$$> |x_{-2+i}| \left| \prod_{j=0}^{j_2-1} \beta_i(j) \right| |\nu|^{m-j_2+1}.$$

From this and since $(x_0, x_{-1}, x_{-2}) \notin F$, we have that both the subsequences $\{x_{2n}\}_{n=-1}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ are unbounded. $\square$

4. Case $a - c = b$

Using the transformation $r_n = x_n/x_{n-1}$, equation (1.1) is reduced to the equation

$$r_{n+1} = \frac{ar_{n-1}}{-br_n r_{n-1} + c}, \quad n = 0, 1, \dots . \tag{4.1}$$

Equation (4.1) has been studied in [2], [3], [4], [22].

In order to discuss equation (1.1) when $a - c = b$, we investigate the behavior of equation (4.1).

The following theorem gives the solution of equation (4.1) in terms of the parameters a, b, c.

Theorem 4.1. *Let r_{-1}, r_0 be real numbers such that $r_{-1}r_0 = \alpha \neq c/b \sum_{i=0}^{n} (a/c)^i$ for any $n \in \mathbb{N}_0$. Then the solution of equation (4.1) is*

$$r_n = \begin{cases} r_{-1} \prod\limits_{j=0}^{\frac{n-1}{2}} \dfrac{\theta(c/a)^{2j} - b}{\theta(c/a)^{2j+1} - b}, & n = 1, 3, 5, \dots, \\ r_0 \prod\limits_{j=0}^{\frac{n-2}{2}} \dfrac{\theta(c/a)^{2j+1} - b}{\theta(c/a)^{2j+2} - b}, & n = 2, 4, 6, \dots \end{cases} \tag{4.2}$$

where $\theta = (a - c + b\alpha)/\alpha$ and $\alpha = x_0/x_{-2}$.

We shall derive only some results concerning the behavior of the solutions of equation (4.1) with $a - c = b$ that we shall use.

The solution of equation (4.1) can be written as

$$r_{2m+i} = r_{-2+i} \prod_{j=0}^{m} \gamma_i(j), \quad i = 1, 2 \text{ and } m = 0, 1, \dots$$

where

$$\gamma_i(j) = \frac{\theta(c/a)^{2j+i-1} - b}{\theta(c/a)^{2j+i} - b}, \quad i = 1, 2.$$

Theorem 4.2. *Assume that $a - c = b$ and let $\{r_n\}_{n=-1}^{\infty}$ be a solution of equation (4.1) such that $r_{-1}r_0 = \alpha \neq c/b \sum_{i=0}^{n} (a/c)^i$ for any $n \in \mathbb{N}_0$. Then the necessary and sufficient condition for the solution $\{r_n\}_{n=-1}^{\infty}$ to be a period-2 solution is $\alpha = -1$.*

P r o o f. Necessity: Let $\{\dots,\varphi,\psi,\varphi,\psi,\dots\}$ be a period-2 solution of equation (4.1). Then we have that

$$\varphi=\frac{a\varphi}{-b\psi\varphi+c}\quad\text{and}\quad\psi=\frac{a\psi}{-b\varphi\psi+c}. \tag{4.3}$$

From equation (4.3) and since $a-c=b$, we get $\varphi\psi=-1$.

Sufficiency: If $\alpha=-1$, then $\theta=(a-c+b\alpha)/\alpha=0$. Therefore,

$$r_{2m+i}=r_{-2+i}\prod_{j=0}^{m}\gamma_i(j)=r_{-2+i},\quad i=1,2\ \text{and}\ m=0,1,\dots.$$

□

Theorem 4.3. *Assume that $a-c=b$ and let $\{r_n\}_{n=-1}^{\infty}$ be a solution of equation* (4.1) *such that $\alpha\neq-1$ and $r_{-1}r_0=\alpha\neq c/b\sum_{i=0}^{n}(a/c)^i$ for any $n\in\mathbb{N}_0$. Then the solution $\{r_n\}_{n=-1}^{\infty}$ converges to a period-2 solution.*

P r o o f. Let $\{r_n\}_{n=-1}^{\infty}$ be a solution of equation (4.1) such that $r_{-1}r_0=\alpha\neq c/b\sum_{i=0}^{n}(a/c)^i$ for any $n\in\mathbb{N}_0$.

The condition $\alpha\neq-1$ (where $a-c=b$) ensures that the solution $\{r_n\}_{n=-1}^{\infty}$ is not a period-2 solution.

As $\lim_{j\to\infty}\gamma_i(j)=\lim_{j\to\infty}(\theta(c/a)^{2j+i-1}-b)/(\theta(c/a)^{2j+i}-b)=1$, there exists $j_2\in\mathbb{N}$ such that $\gamma_i(j)>0$ for all $i=1,2$ and $j\geqslant j_2$.

Now for each $i\in\{1,2\}$, we have for large m

$$\begin{aligned}r_{2m+i}&=r_{-2+i}\prod_{j=0}^{m}\gamma_i(j)=r_{-2+i}\prod_{j=0}^{j_2-1}\gamma_i(j)\prod_{j=j_2}^{m}\gamma_i(j)\\&=r_{-2+i}\prod_{j=0}^{j_2-1}\gamma_i(j)\exp\Big(\sum_{j=j_2}^{m}\ln\gamma_i(j)\Big).\end{aligned}$$

Now we show the convergence of the series $\sum_{j=j_2}^{\infty}|\ln\gamma_i(j)|$.

Using the asymptotic relations $(1+x)^{-1}=1+O(x)$ and $\ln(1+x)=x+O(x^2)$, we have that

$$\begin{aligned}\ln\gamma_i(j)&=\ln\frac{\theta(c/a)^{2j+i-1}-b}{\theta(c/a)^{2j+i}-b}=\ln\Big(1+\frac{\theta}{a}\frac{(c/a)^{2j+i-1}(a-c)}{\theta(c/a)^{2j+i}-b}\Big)\\&=\ln\Big(1+\frac{\theta(c-a)}{ab}\Big(\frac{c}{a}\Big)^{2j+i-1}\Big)+o\Big(\Big(\frac{c}{a}\Big)^{2j}\Big)\\&=\frac{\theta(c-a)}{ab}\Big(\frac{c}{a}\Big)^{2j+i-1}+o\Big(\Big(\frac{c}{a}\Big)^{2j}\Big).\end{aligned}$$

From this and since $c/a < 1$, by using a known criterion for the convergence of series we get that the the series $\sum_{j=j_2}^{\infty} |\ln \gamma_i(j)|$ converges.

Hence, there are two real numbers $\varrho_i \in \mathbb{R}$ such that

$$\lim_{m\to\infty} r_{2m+i} = \varrho_i, \quad i \in \{0, 1\}.$$

If we set $n = 2m + i - 1$, $i = 0, 1$ in equation (4.1), we get

$$r_{2m+1} = \frac{ar_{2m-1}}{-br_{2m-1}r_{2m} + c} \quad \text{and} \quad r_{2m+2} = \frac{ar_{2m}}{-br_{2m}r_{2m+1} + c}, \quad m = 0, 1, \dots .$$

By taking the limit as $m \to \infty$, we obtain

$$\varrho_1 = \frac{a\varrho_1}{-b\varrho_1\varrho_0 + c} \quad \text{and} \quad \varrho_0 = \frac{a\varrho_0}{-b\varrho_0\varrho_1 + c}. \tag{4.4}$$

If $\varrho_1 = 0$, then from the second equation in (4.4), we get $\varrho_0 = 0$. This is a contradiction, as the equilibrium point $\bar{r} = 0$ of equation (4.1) is unstable (a repeller) when $a > c$ (see [2]).

This implies that $\varrho_i \neq 0$, $i = 0, 1$ and $\varrho_0\varrho_1 = -1$. Therefore, $\{r_n\}_{n=-1}^{\infty}$ converges to the 2-periodic solution

$$\{\dots, \varrho_0, \varrho_1, \varrho_0, \varrho_1, \dots\} \quad \text{with } \varrho_0\varrho_1 = -1.$$

□

Now we are ready to formulate the main results in this section.

Theorem 4.4. *Assume that $\{x_n\}_{n=-2}^{\infty}$ is a solution of equation (1.1) such that $(x_0, x_{-1}, x_{-2}) \notin F$ and let $a - c = b$. If $\alpha = -1$, then $\{x_n\}_{n=-2}^{\infty}$ is an eventually periodic solution with period 4.*

Proof. Assume that $a - c = b$. If $\alpha = -1$, then $\theta = 0$. Therefore,

$$\begin{aligned} x_{2m+i} &= x_{-2+i} \prod_{j=0}^{m} \frac{a-c}{\theta(c/a)^{2j+i} - b} = x_{-2+i} \prod_{j=0}^{m} (-1) \\ &= x_{-2+i}(-1)^{m+1}, \quad i = 1, 2 \text{ and } m = 0, 1, \dots . \end{aligned}$$

Now if we set $m = 2n + l - 1$, $l = 0, 1$, then

$$x_{4n+2l+i-2} = x_{-2+i}(-1)^{2n+l}, \quad i = 1, 2,\ l = 0, 1 \text{ and } n = 0, 1, \dots .$$

Therefore,

$$x_{4n-1} = x_{-1}, \quad x_{4n} = x_0, \quad x_{4n+1} = -x_{-1}, \quad x_{4n+2} = -x_0.$$

□

Theorem 4.5. *Assume that* $\{x_n\}_{n=-2}^{\infty}$ *is a solution of equation* (1.1) *such that* $(x_0, x_{-1}, x_{-2}) \notin F$ *and let* $a - c = b$. *If* $\alpha \neq -1$, *then* $\{x_n\}_{n=-2}^{\infty}$ *converges to a period-*4 *solution* $\{\mu_0, \mu_1, -\mu_0, -\mu_1\}$ *such that* $\mu_1 = \mu_0|\varrho_1|$, *where* ϱ_1 *is as in Theorem* 4.3.

Proof. Suppose that $\{x_n\}_{n=-2}^{\infty}$ is a solution of equation (1.1) such that $(x_0, x_{-1}, x_{-2}) \notin F$ and let $a - c = b$. As

$$\lim_{j\to\infty} \beta_i(j) = \frac{a-c}{\theta(c/a)^{2j+i} - b} = -1, \quad i = 1, 2,$$

there exists $j_0 \in \mathbb{N}$ such that $\beta_i(j) < 0$ for all $i = 1, 2$ and $j \geqslant j_0$.

Hence

$$\begin{aligned} |x_{2m+i}| &= |x_{-2+i}|\Big|\prod_{j=0}^{m} \beta_i(j)\Big| = |x_{-2+i}|\Big|\prod_{j=0}^{j_0-1} \beta_i(j)\Big| \prod_{j=j_0}^{m} |\beta_i(j)| \\ &= |x_{-2+i}|\Big|\prod_{j=0}^{j_0-1} \beta_i(j)\Big| \exp\Big(\sum_{j=j_0}^{m} \ln|\beta_i(j)|\Big). \end{aligned}$$

Now we show the convergence of the series $\sum\limits_{j=j_0}^{\infty} |\ln(-\beta_i(j))|$. Using the asymptotic relations $(1+x)^{-1} = 1 + x + O(x^2)$ and $\ln(1+x) = x + O(x^2)$, we have that

$$\ln|\beta_i(j)| = \ln\Big(1 + \frac{\theta}{b}\Big(\frac{c}{a}\Big)^{2j+i} + O\Big(\Big(\frac{c}{a}\Big)^{4j}\Big)\Big).$$

As $c/a < 1$, we get that the series $\sum\limits_{j=j_0}^{\infty} \ln|\beta_i(j)|$ is convergent.

This ensures that there are two positive real numbers μ_0, μ_1 such that

$$\lim_{m\to\infty} |x_{2m+i}| = \mu_i, \quad i \in \{0, 1\}. \tag{4.5}$$

Now set

$$\lim_{m\to\infty} x_{4m+l} = L_l, \quad l \in \{0, 1, 2, 3\}.$$

As

$$r_{4m+1}r_{4m+2} = \frac{x_{4m+2}}{x_{4m}} \quad \text{and} \quad r_{4m+2}r_{4m+3} = \frac{x_{4m+3}}{x_{4m+1}},$$

using Theorem (4.3) we obtain $L_2 = -L_0$ and $L_3 = -L_1$.

On the other hand, from (4.5) we get

$$|L_2| = |-L_0| = |L_0| = \mu_0 \quad \text{and} \quad |L_3| = |-L_1| = |L_1| = \mu_1.$$

Then

$$L_0 = \mu_0 \quad \text{or} \quad L_0 = -\mu_0 \qquad \text{and} \qquad L_1 = \mu_1 \quad \text{or} \quad L_1 = -\mu_1.$$

Without loss of generality, we take $L_0 = \mu_0$ and $L_1 = \mu_1$. Then the solution $\{x_n\}_{n=-2}^{\infty}$ converges to the period-4 solution

$$\{\ldots, \mu_0, \mu_1, -\mu_0, -\mu_1, \mu_0, \mu_1, -\mu_0, -\mu_1, \ldots\}.$$

Moreover, as $|x_{2m+1}| = |x_{2m} r_{2m+1}|$, we have $\mu_1 = \mu_0 |\varrho_1|$ where

$$\varrho_1 = r_{-1} \prod_{j=0}^{\infty} \frac{\theta(c/a)^{2j} - b}{\theta(c/a)^{2j+1} - b} \quad \text{and} \quad \mu_0 = |x_0| \prod_{j=1}^{\infty} \frac{b}{|\theta(c/a)^{2j} - b|}.$$

□

5. Case $a = c$

In this section, we study the case when $a = c$.

Proposition 5.1. *Assume that $a = c$. Then the forbidden set G of equation* (1.1) *is*

$$G = \bigcup_{n=0}^{\infty} \Big\{(u_0, u_{-1}, u_{-2}) \colon\ u_0 = u_{-2} \frac{a}{b(n+1)}\Big\}$$
$$\cup \{(u_0, u_{-1}, u_{-2}) \colon\ u_0 = 0\} \cup \{(u_0, u_{-1}, u_{-2}) \colon\ u_{-1} = 0\}.$$

Let x_{-2}, x_{-1} and x_0 be real numbers such that $(x_0, x_{-1}, x_{-2}) \notin G$. If $a = c$, then the solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) is

$$x_n = \begin{cases} x_{-1} \displaystyle\prod_{j=0}^{\frac{n-1}{2}} \frac{a\alpha}{a - b\alpha(2j+1)}, & n = 1, 3, 5, \ldots, \\ x_0 \displaystyle\prod_{j=0}^{\frac{n-2}{2}} \frac{a\alpha}{a - b\alpha(2j+2)}, & n = 2, 4, 6, \ldots \end{cases} \tag{5.1}$$

where $\alpha = x_0 / x_{-2}$.

Proof. We can write the solution (5.1) as

$$x_{2m+i} = x_{-2+i} \prod_{j=0}^{m} \eta_i(j), \quad i = 1, 2 \ \text{and} \ m = 0, 1, \ldots \tag{5.2}$$

where

$$\eta_i(j) = \frac{a\alpha}{a - b\alpha(2j+i)}, \quad i = 1, 2.$$

By direct calculation, we can get the values of x_1 and x_2 as desired.

Now assume that $m > 1$. Then

$$
\begin{aligned}
x_{2m+3} &= \frac{ax_{2m+2}x_{2m+1}}{-bx_{2m+2} + ax_{2m}} = \frac{ax_0 \prod_{j=0}^{m} \eta_2(j) x_{-1} \prod_{j=0}^{m} \eta_1(j)}{-bx_0 \prod_{j=0}^{m} \eta_2(j) + ax_0 \prod_{j=0}^{m-1} \eta_2(j)} \\
&= \frac{ax_0 \prod_{j=0}^{m} \eta_2(j) x_{-1} \prod_{j=0}^{m} \eta_1(j)}{x_0 \prod_{j=0}^{m-1} \eta_2(j)(-b\eta_2(m) + a)} = \frac{a\eta_2(m) x_{-1} \prod_{j=0}^{m} \eta_1(j)}{-b\eta_2(m) + a} \\
&= \frac{a(a\alpha/(a - b\alpha(2m+2)))x_{-1} \prod_{j=0}^{m} \eta_1(j)}{-ba\alpha/(a - b\alpha(2m+2)) + a} = \frac{a(a\alpha)x_{-1} \prod_{j=0}^{m} \eta_1(j)}{-ba\alpha + a(a - b\alpha(2m+2))} \\
&= \frac{a\alpha}{a - b\alpha(2m+3)} x_{-1} \prod_{j=0}^{m} \eta_1(j) = \eta_1(m+1) x_{-1} \prod_{j=0}^{m+1} \eta_1(j) \\
&= x_{-1} \prod_{j=0}^{m+1} \eta_1(j).
\end{aligned}
$$

To complete the inductive proof, we shall show that formula (2.2) also holds for x_{2m+4}. We have

$$
\begin{aligned}
x_{2m+4} &= \frac{ax_{2m+3}x_{2m+2}}{-bx_{2m+3} + ax_{2m+1}} = \frac{ax_{-1} \prod_{j=0}^{m+1} \eta_1(j) x_0 \prod_{j=0}^{m} \eta_2(j)}{-bx_{-1} \prod_{j=0}^{m+1} \eta_1(j) + ax_{-1} \prod_{j=0}^{m} \eta_1(j)} \\
&= \frac{ax_{-1} \prod_{j=0}^{m+1} \eta_1(j) x_0 \prod_{j=0}^{m} \eta_2(j)}{x_{-1} \prod_{j=0}^{m} \eta_1(j)(-b\eta_2(m+1) + a)} = \frac{a\eta_1(m+1) x_0 \prod_{j=0}^{m} \eta_2(j)}{-b\eta_1(m+1) + a} \\
&= \frac{a(a\alpha/(a - b\alpha(2m+3)))x_0 \prod_{j=0}^{m} \eta_2(j)}{-ba\alpha/(a - b\alpha(2m+3)) + a} = \frac{a(a\alpha)x_0 \prod_{j=0}^{m} \eta_2(j)}{-ba\alpha + a(a - b\alpha(2m+3))} \\
&= \frac{a\alpha}{a - b\alpha(2m+4)} x_0 \prod_{j=0}^{m} \eta_2(j) = \eta_2(m+1) x_0 \prod_{j=0}^{m+1} \eta_2(j) \\
&= x_0 \prod_{j=0}^{m+1} \eta_2(j).
\end{aligned}
$$

This completes the inductive proof of the theorem. $\square$

Theorem 5.2. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation* (1.1) *such that* $(x_0, x_{-1}, x_{-2}) \notin G$. *If* $a = c$, *then* $\{x_n\}_{n=-2}^{\infty}$ *converges to* 0.

P r o o f. It is sufficient to see that $\eta_i(j) \to 0$ as $j \to \infty$, $i = 1, 2$. □

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