CHARACTERIZATION OF n-VERTEX GRAPHS WITH METRIC DIMENSION $n-3$

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Abstract. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the ordered k-vector $r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the metric representation of v with respect to W, where $d(x, y)$ is the distance between vertices x and y. A set W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W. The minimum cardinality of a resolving set for G is its metric dimension. In this paper, we characterize all graphs of order n with metric dimension $n-3$.

Keywords: resolving set; basis; metric dimension

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1. Introduction

Throughout this paper $G = (V, E)$ is a finite, simple, and connected graph of order $n(G)$. The distance between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path between u and v in G. We write it simply $d(u, v)$ when no confusion can arise. Also, the diameter of G, $\max\limits_{\{u,v\}\subseteq V(G)} d(u, v)$, is denoted by $\operatorname{diam}(G)$. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector $r(v|W) := (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the *metric representation* of v with respect to W. A set W is called a *resolving set* for G if distinct vertices have different representations. In this case we say the set W *resolves* G. A resolving set W for G with minimum cardinality is called a *basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\beta(G)$.

The concepts of the metric dimension and the resolving set were independently introduced by Slater [10] and Harary and Melter [5]. For more results related to these concepts see [1], [3], [4], [8], [12]. The concept of a resolving set has various applications in diverse areas including coin weighing problems [9], robot navigation [8], mastermind game [1], and combinatorial search and optimization [9].

It is obvious that for every graph G of order n, $1 \leqslant \beta(G) \leqslant n-1$. Khuller et al. [8] improved this bound to $\beta(G) \leqslant n - \mathrm{diam}(G)$. Chartrand et al. [2] proved that $\beta(G) = 1$ if and only if G is a path. Also, all graphs with metric dimension two were characterized by Sudhakara and Hemanth Kumar [11]. Chartrand et al. [2] proved that for $n \geqslant 2$, $\beta(G) = n-1$ if and only if G is the complete graph K_n. They also provided a characterization of graphs of order n and metric dimension $n-2$. In [6] the problem of characterization of all graphs of order n and metric dimension $n-3$ was proposed. In this paper, we answer this question and characterize these graphs. First, in the next section, we present some definitions and known results which are necessary to prove the main theorem.

2. Preliminaries

In this section, we present some definitions and known and simple results which are necessary to prove our main theorem. The symbols $\sim$ and $\nsim$ denote the adjacency and non-adjacency relations between two vertices. An edge with end vertices u and v is denoted by uv. A path of order n, P_n, and a cycle of order n, C_n, are denoted by $(v_1, v_2, \ldots, v_n)$ and $(v_1, v_2, \ldots, v_n, v_1)$, respectively.

An ordered set W *resolves* a set T of vertices in G, if the representations of vertices in T are distinct with respect to W. When $W = \{x\}$, we say a vertex x resolves T. To see whether a given set W is a resolving set for G, it is sufficient to look at the representations of vertices in $V(G) \setminus W$, because $w \in W$ is the only vertex of G with $d(w, w) = 0$. In [2] all graphs of order n with metric dimension $n-2$ are characterized as follows.

Theorem A [2]**.** *If G is a graph of order $n \geqslant 4$, then $\beta(G) = n-2$ if and only if $G = K_{s,t}$ $(s, t \geqslant 1)$, $G = K_s \vee \overline{K}_t$ $(s \geqslant 1,\ t \geqslant 2)$, or $G = K_s \vee (K_t \cup K_1)$ $(s, t \geqslant 1)$.*

Here, $\vee$ and $\cup$ are used for the *join* and the *disjoint union* of graphs, respectively. We say a set S of vertices is *homogeneous* if the subgraph induced by S in G, $G[S]$, is a complete or an empty subgraph of G. In this terminology, it is proved in [7] that each vertex of G^* is a homogeneous subset of $V(G)$.

Proposition 1 [7]**.** *If $G \neq K_1$ is a graph, then* $\mathrm{diam}(G^*) \leqslant \mathrm{diam}(G)$. *Moreover, if $u, v \in V(G)$ are not twin vertices of G, then $d_{G^*}(u^*, v^*) = d_G(u, v)$.*

For each vertex $v \in V(G)$, let $\Gamma_i(v) := \{u \in V(G);\ d(u, v) = i\}$. Two distinct vertices u, v are *twins* if $\Gamma_1(v) \setminus \{u\} = \Gamma_1(u) \setminus \{v\}$. Clearly, if vertices u, v are twins in a graph G and S resolves G, then u or v is in S. As in [7], we define $u \equiv v$ if and only if $u = v$ or u, v are twins. In [7], it is proved that the relation $\equiv$ is an equivalence relation. The equivalence class of the vertex v is denoted by v^*. The *twin graph* of G, denoted by G^*, is the graph with the vertex set $V(G^*) := \{v^*;\ v \in V(G)\}$, where $u^*v^* \in E(G^*)$ if and only if $uv \in E(G)$. It is easy to see that u, v are adjacent in G if and only if all vertices of u^* are adjacent to all vertices of v^*, hence the notion of G^* is well defined. For each subset $S \subseteq V(G)$, let S^* denote the set $\{v^* \in V(G^*);\ v^* \subseteq S\}$. By Proposition 1 $\Gamma_i(v^*) = (\Gamma_i(v))^*$, for $i \neq 0$. Furthermore, we define $R_1(v) := \{x \in \Gamma_1(v);\ \exists y \in \Gamma_2(v)\colon\ x \sim y\}$ and $R_2(v) := \Gamma_1(v) \setminus R_1(v)$.

As in [7], we say that $v^* \in V(G^*)$ is of type (1) if $|v^*| = 1$, of type (K) if $G[v^*] \cong K_r$ and $r \geqslant 2$, and of type (N) if $G[v^*] \cong \overline{K_r}$ and $r \geqslant 2$. A vertex of G^* is of type (1K) if it is of type (1) or (K), of type (1N) if it is of type (1) or (N), and of type (KN) if it is of type (K) or (N). We denote by $\alpha(G^*)$ the number of vertices of G^* of type (K) or (N). It is obvious that G is uniquely determined by G^*, and the type and cardinality of each vertex of G^*. Hernando et al. [7] characterized all graphs of order n, diameter d and metric dimension $n - d$ by the following theorem.

Theorem B [7]**.** *Let G be a graph of order n and diameter $d \geqslant 3$. If G^* is the twin graph of G, then $\beta(G) = n - d$ if and only if G^* is one of the following graphs:*

1. *$G^* \cong P_{d+1}$ and one of the following cases holds*
   (a) *$\alpha(G^*) \leqslant 1$;*
   (b) *$\alpha(G^*) = 2$, the two vertices of G^* not of type* (1) *are adjacent, and if one is a leaf of type* (K)*, then the other is also of type* (K)*;*
   (c) *$\alpha(G^*) = 2$, the two vertices of G^* not of type* (1) *are at distance* 2 *and both are of type* (N)*;*
   (d) *$\alpha(G^*) = 3$ and there is a vertex of type* (KN) *adjacent to two vertices of type* (N)*.*
2. *$G^* \cong P_{d+1,k}$ (the path $(u_0^*, u_1^*, \ldots, u_d^*)$ with one extra vertex adjacent to u_{k-1}^*) for some integer $k \in [3, d-1]$, the degree-3 vertex u_{k-1}^* of G^* is of any type, each neighbor of u_{k-1}^* is of type* (1N)*, and every other vertex is of type* (1)*.*
3. *$G^* \cong P'_{d+1,k}$ (the path $(u_0^*, u_1^*, \ldots, u_d^*)$ with one extra vertex adjacent to u_{k-1}^* and u_k^*) for some integer $k \in [2, d-1]$, the three vertices in the cycle are of type* (1K)*, and every other vertex is of type* (1)*.*

A subgraph H of a graph G is an isometric subgraph if $d_H(u, v) = d_G(u, v)$ for all pairs of vertices in H. To prove our main theorem, we need the following observations and propositions.

Observation 1. *If H is an isometric subgraph of G and $\beta(H) = n(H) - t$, then $\beta(G) \leqslant n(G) - t$.*

Corollary 1. *If H is an induced subgraph of G, where* $\operatorname{diam}(H) = 2$, *and* $\beta(H) = n(H) - t$, *then* $\beta(G) \leqslant n(G) - t$.

Corollary 2. *If H is an induced subgraph of G, and G is an induced subgraph of a graph R, where* $\operatorname{diam}(H) = \operatorname{diam}(G) = 2$, $\beta(R) = n(R) - t$, *and* $\beta(H) = n(H) - t$, *then* $\beta(G) = n(G) - t$.

Proposition 2. *If* $\beta(G^*) = n(G^*) - t$, *then* $\beta(G) \leqslant n(G) - t$.

P r o o f. Let S^* be a basis of G^* and $T^* = V(G^*) \setminus S^*$. We choose a vertex v for each $v^* \in T^*$ and let $T = \{v;\ v^* \in T^*\}$. Since S^* is a basis of G^*, for each pair of vertices $u^*, v^* \in T^*$ there exists $x^* \in S^*$ such that $d_{G^*}(x^*, u^*) \neq d_{G^*}(x^*, v^*)$. Note that neither u nor v is a twin of x, for each $x \in x^*$. Therefore, by Proposition 1, we have $d_{G^*}(x^*, u^*) = d_G(x, u)$ and $d_{G^*}(x^*, v^*) = d_G(x, v)$. Hence, $d_G(x, u) \neq d_G(x, v)$, which implies $S = \bigcup\limits_{v^* \in S^*} v^*$ resolves T. Hence, $V(G) \setminus T$ is a resolving set for G of cardinality $n(G) - t$, thus $\beta(G) \leqslant n(G) - t$. $\square$

Observation 2. *If G is a graph and G^* is the twin graph of G, then $\beta(G) \geqslant n(G) - n(G^*)$.*

3. Main results

Let G be a connected graph of order n and metric dimension $n - 3$. Since $n - 3 = \beta(G) \leqslant n - \operatorname{diam}(G)$, $\operatorname{diam}(G) \leqslant 3$. If $\operatorname{diam}(G) = 1$, then $G \cong K_n$, contrary to $\beta(G) = n - 3$. If $\operatorname{diam}(G) = 3$, then in Theorem B let $d = 3$, which yields a characterization of graphs G with $\beta(G) = n - 3$, where $\operatorname{diam}(G) = 3$ (note that in this case the interval $[3, 2]$ is empty, hence, Case 2 dose not occur). Therefore, it is enough to consider the case $\operatorname{diam}(G) = 2$. The following theorem is our main result, which is a characterization of all graphs with metric dimension $n - 3$ and diameter 2.

Theorem 1. *If G is a graph of order n and diameter 2 and G^* is the twin graph of G, then $\beta(G) = n - 3$ if and only if G^* has one of the following structures:*

$\mathbf{G_1}$. *$G^* \cong K_3$ and has at most one vertex of type* (1K)*;*

$\mathbf{G_2}$. *$G^* \cong P_3$, a leaf is of type* (K)*, the other leaf is of type* (KN) *or the degree-2 vertex is of type* (N)*;*

G_3. *G^* is a paw (a triangle with a pendant edge), one of the degree-2 vertices is of type* (N), *the other is of type* (1K), *and the leaf is of type* (1N). *Moreover, a degree-2 vertex of type* (K) *yields the leaf and the degree-3 vertex are not of type* (N);

G_4. *$G^* \cong C_5$, and each vertex is of type* (1);

G_5. *G^* is a house, the adjacent degree-2 vertices are of type* (1) *and the other vertices are of type* (1K);

G_6. *$G^* \cong P_4 \vee K_1$, all vertices except the degree-4 vertex are of type* (1K). *Moreover, two non-adjacent vertices are not of type* (K), *and two adjacent vertices are not of different types* (K) *and* (N);

G_7. *G^* is a kite with a pendant edge adjacent to a degree-3 vertex, the leaf is of type* (1), *the degree-4 and degree-3 vertices are type* (1K), *one of the degree-2 vertices is of type* (K) *and the other is of type* (1);

G_8. *G^* is a kite, one of the degree-2 vertices is of type* (K), *the other is of type* (1), *one of the degree-3 vertices is of type* (N), *and the other is of type* (1K);

G_9. *$G^* \cong C_4$, two adjacent vertices are of type* (K) *and the others are of type* (1);

G_{10}. *$G^* \cong C_4 \vee K_1$, two degree-3 adjacent vertices are of type* (K), *degree-4 vertex is of type* (1K), *and others are of type* (1).

In Figure 1 *the scheme of the* 10 *above structures is shown.*

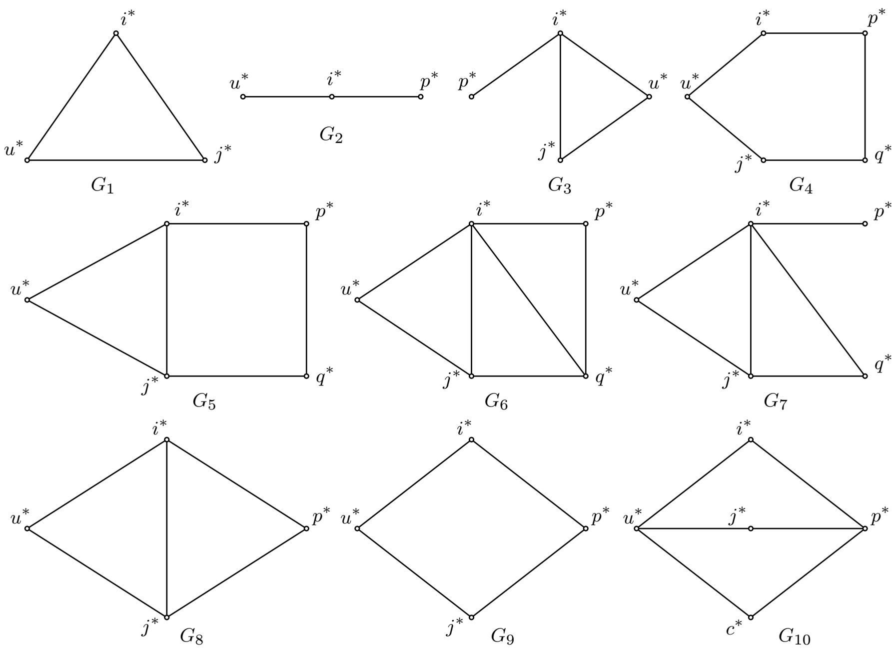


Figure 1. The twin graph of graphs with diameter 2 and metric dimension $n-3$.

3.1. Proof of sufficiency. In this section we prove that, if G is a graph of order n and $\operatorname{diam}(G) = 2$ such that G^* has one of the structures G_1 through G_{10} in Theorem 3.1, then $\beta(G) = n - 3$. In the sequel we consider each structure G_1 to G_{10}, as shown in Figure 1, separately. In each case, we assume that $i \in i^*$, $j \in j^*$, $c \in c^*$, $p \in p^*$, $q \in q^*$, and $u \in u^*$.

$\mathbf{G_1}$. Since G^* has three vertices, by Observation 2, $\beta(G) \geqslant n - 3$. On the other hand, by Theorem A, $\beta(G) \neq n - 2$, so G is not a complete graph. Therefore, $\beta(G) \leqslant n - 3$. Hence $\beta(G) = n - 3$. For structure G_2, similarly to the above, we deduce $\beta(G) = n - 3$.

$\mathbf{G_3}$. Let H, H_1, H_2, and H_3 be four families of graphs such that their twin graph is the same as G_3. Assume that in H^* the vertex j^* is of type (N) and the other vertices are of type (1), in H_1^* both i^* and u^* are of type (K), j^* is of type (N), and p^* is of type (1), in H_2^* both j^* and p^* are of type (N), i^* is of type (K), and u^* is of type (1), in H_3^*, u^* is of type (1) and the other vertices are of type (N).

Thus, there exists a graph H' in the family H which is an induced subgraph of G and there exists a graph H_t' in the family H_t such that G is an induced subgraph of H_t', for some t, $1 \leqslant t \leqslant 3$. Now we get the metric dimension of H and H_t. Since in H^*, the vertex j^* is of type (N), each resolving set for H contains at least $|j^*| - 1$ vertices of j^*. Moreover, $r(u|j^* \setminus \{j\}) = r(i|j^* \setminus \{j\})$, hence, $j^* \setminus \{j\}$ is not a resolving set for H. It is easy to see that $(j^* \setminus \{j\}) \cup \{u\}$ is a resolving set, and so a basis of H. Thus, $\beta(H) = n(H) - 3$. Since in H_1^*, vertices i^*, j^*, and u^* are not of type (1), each resolving set for H_1 contains at least $|i^*| - 1$, $|j^*| - 1$, and $|u^*| - 1$ vertices of i^*, j^*, and u^*, respectively. On the other hand, $r(u|i^* \cup j^* \cup u^* \setminus \{i, j, u\}) = r(i|i^* \cup j^* \cup u^* \setminus \{i, j, u\})$, therefore $i^* \cup j^* \cup u^* \setminus \{i, j, u\}$ does not resolve H_1. It is easy to see that $i^* \cup j^* \cup u^* \setminus \{j, u\}$ resolves H_1, hence, it is a basis of H_1, and so $\beta(H_1) = n(H_1) - 3$. By the same argument, we can see $\beta(H_t) = n(H_t) - 3$, $2 \leqslant t \leqslant 3$. Now, since $\operatorname{diam}(H) = \operatorname{diam}(G) = 2$, by Corollary 2, we have $\beta(G) = n - 3$.

$\mathbf{G_4}$. It is clear that $\beta(C_5) = 2$.

$\mathbf{G_5}$. Let H and R be two families of graphs such that their twin graph is the same as G_5, all vertices of H^* are of type (1) and in R^* both the vertices p^* and q^* are of type (1) and the other vertices are of type (K). Therefore, there exists a graph H' in the family H which is an induced subgraph of G and there exists a graph R' in the family R such that G is an induced subgraph of R'. It is clear that $\beta(H) = 2 = n(H) - 3$. Each resolving set for R contains at least $|i^*| - 1$, $|j^*| - 1$, and $|u^*| - 1$ vertices from i^*, j^*, and u^*, respectively. If $W = i^* \cup j^* \cup u^* \setminus \{i, j, u\}$, then $r(i|W) = r(j|W) = r(u|W)$. Hence, W is not a resolving set for R. Further, adding one of vertices i, j, and u to W cannot provide a resolving set for R, because $\{i, j, u\}$ is a clique in R. Since $\operatorname{diam}(R) = 2$, neither p nor q can resolve more than two vertices of $\{i, j, u\}$. Thus, $\beta(R) \geqslant |W| + 2 = n(R) - 3$. Since R is neither a complete graph nor any of the graphs in Theorem A, $\beta(R) \leqslant n(R) - 3$. Hence, $\beta(R) = n(R) - 3$. Since $\mathrm{diam}(H) = \mathrm{diam}(G) = 2$, Corollary 2 yields $\beta(G) = n - 3$.

$\mathbf{G_6}$. Let H, H_1, H_2, and H_3 be four families of graphs with twin graphs the same as G_6. Assume that all vertices of H^* are of type (1), in H_1^* the vertex i^* is of type (N) and the other vertices are of type (1), in H_2^* both p^* and q^* are of type (1) and the other vertices are of type (K), in H_3^* both p^* and u^* are of type (1) and the other vertices are of type (K). Hence, there exists a graph H' in the family H which is an induced subgraph of G and there exists a graph H_t' in the family H_t such that G is an induced subgraph of H_t' for some t, $1 \leqslant t \leqslant 3$. It is clear that $\beta(H) = 2 = n(H) - 3$. Each resolving set for H_1 contains at least $|i^*| - 1$ vertices from i^*. But $r(u|i^* \setminus \{i\}) = r(j|i^* \setminus \{i\}) = r(p|i^* \setminus \{i\}) = r(q|i^* \setminus \{i\})$ and there is no vertex of $\{u, j, p, q\}$ such that adding it to $i^* \setminus \{i\}$ provides a resolving set for H_1, hence $\beta(H_1) \geqslant |i^*| + 1 = n(H_1) - 3$. Since H_1 is not a complete graph, Theorem A gives $\beta(H_1) \leqslant n(H_1) - 3$, and so $\beta(H_1) = n(H_1) - 3$. Each resolving set for H_2 contains at least $|i^*| - 1$, $|j^*| - 1$, and $|u^*| - 1$ vertices from i^*, j^*, and u^*, respectively. Assume $W = i^* \cup j^* \cup u^* \setminus \{i, j, u\}$. It follows that $r(i|W) = r(j|W) = r(u|W)$, hence W does not resolve H_2. It is easy to see that to provide a resolving set for H_2 we need to add at least two vertices from $V(H_2) - W$ to W. Thus, $\beta(H_2) \geqslant |W| + 2 = n(H_2) - 3$. On the other hand, Theorem A implies $\beta(H_2) \leqslant n(H_2) - 3$, and so $\beta(H_2) = n(H_2) - 3$. In the same way, $\beta(H_3) = n(H_3) - 3$. Since $\mathrm{diam}(H) = \mathrm{diam}(G) = 2$, Corollary 2 implies $\beta(G) = n - 3$.

$\mathbf{G_7}$. Let H and R be two families of graphs such that their twin graph is the same as G_7. Assume that in H^* the vertex u^* is of type (K) and the other vertices are of type (1), and in R^* both vertices p^* and q^* are of type (1) and the other vertices are of type (K). Therefore, there exists a graph H' in the family H which is an induced subgraph of G and there exists a graph R' in the family R such that G is an induced subgraph of R'. It is clear that each resolving set for H contains at least $|u^*| - 1$ vertices of u^*. Moreover, $r(u|u^* \setminus \{u\}) = r(i|u^* \setminus \{u\}) = r(j|u^* \setminus \{u\})$ and to provide a resolving set for H, we must add at least two vertices from $\{u, i, j, p, q\}$ to $u^* \setminus \{u\}$, hence $\beta(H) \geqslant |u^*| + 1 = n(H) - 3$. Further, by Theorem A, $\beta(H) \leqslant n(H) - 3$, thus $\beta(H) = n(H) - 3$. Since i^*, j^*, and u^* are not of type (1) in R^*, each resolving set for R contains at least $|i^*| - 1$, $|j^*| - 1$, and $|u^*| - 1$ vertices of i^*, j^*, and u^*, respectively. For $W = i^* \cup j^* \cup u^* \setminus \{i, j, u\}$ we have $r(i|W) = r(j|W) = r(u|W)$, hence W is not a resolving set for R. To provide a resolving set for R, we need to add at least two vertices from $\{u, i, j, p, q\}$ to W, and so, $\beta(R) \geqslant |W| + 2 = n(R) - 3$. Theorem A shows that $\beta(R) \leqslant n(R) - 3$, hence $\beta(R) = n(R) - 3$. Since $\mathrm{diam}(H) = \mathrm{diam}(G) = 2$, Corollary 2 yields $\beta(G) = n - 3$.

$\mathbf{G_8}$. Let H and R be two families of graphs such that their twin graph is the same as G_8, where in H^* both j^* and p^* are of type (1), u^* is of type (K) and

i^* is of type (N), and in R^* both vertices j^* and u^* are of type (K), i^* is of type (N) and p^* is of type (1). Then G is in one of the families H or R. Theorem A shows that $\beta(H) \leqslant n(H) - 3$ and $\beta(R) \leqslant n(R) - 3$. Note that each resolving set for H contains at least $|u^*| - 1$ and $|i^*| - 1$ vertices from u^* and i^*, respectively. If $S = (i^* \cup u^*) \setminus \{i, u\}$, then $r(u|S) = r(j|S)$. Therefore, $\beta(H) \geqslant |S| + 1 = n(H) - 3$, and so $\beta(H) = n(H) - 3$. It is clear that every resolving set for R contains at least $|i^*| - 1$, $|j^*| - 1$, and $|u^*| - 1$ vertices of i^*, j^*, and u^*, respectively. If $W = i^* \cup j^* \cup u^* \setminus \{i, j, u\}$, then $r(j|W) = r(u|W)$, hence W is not a resolving set for R, and so $\beta(R) \geqslant |W| + 1 = n(R) - 3$. It follows that $\beta(R) = n(R) - 3$. Consequently, $\beta(G) = n(G) - 3$.

$\mathbf{G_9}$. Let G^* be as G_9 in Figure 1, where i^* and u^* are of type (K) in G^*, and the other vertices are of type (1). Each resolving set for G contains at least $|i^*| - 1$ and $|u^*| - 1$ vertices of i^* and u^*, respectively. For $W = i^* \cup u^* \setminus \{i, u\}$, we have $r(i|W) = r(u|W)$, hence W is not a resolving set for G, and so $\beta(G) \geqslant |W| + 1 = n(G) - 3$. Theorem A implies $\beta(G) \leqslant n(G) - 3$. Consequently, $\beta(G) = n(G) - 3$.

$\mathbf{G_{10}}$. Let H and R be two families of graphs such that their twin graph is the same as G_{10}. In H^* both u^* and c^* are of type (K) and the other vertices are of type (1), and in R^* both vertices i^* and p^* are of type (1) and the other vertices are of type (K). Then G is in one of the families H or R. Theorem A shows that $\beta(H) \leqslant n(H) - 3$ and $\beta(R) \leqslant n(R) - 3$. Note that each resolving set for H contains at least $|u^*| - 1$ and $|c^*| - 1$ vertices from u^* and c^*, respectively. If $S = c^* \cup u^* \setminus \{c, u\}$, then $r(u|S) = r(j|S) = r(c|S)$, therefore S does not resolve H. To provide a resolving set for H, we need to add at least two vertices from the set $\{u, i, j, c, p\}$ to S, and so $\beta(H) \geqslant |S| + 2 = n(H) - 3$, hence $\beta(H) = n(H) - 3$. It is clear that every resolving set for R contains at least $|u^*| - 1$, $|j^*| - 1$, and $|c^*| - 1$ vertices from u^*, j^*, and c^*, respectively. Let $W = u^* \cup j^* \cup c^* \setminus \{u, j, c\}$. Hence, $r(u|W) = r(j|W) = r(c|W)$, therefore W is not a resolving set for R. Clearly, to provide a resolving set for R, we must add at least two vertices from the set $\{u, i, j, c, p\}$ to W, hence $\beta(R) \geqslant |W| + 2 = n(R) - 3$, thus $\beta(R) = n(R) - 3$. Consequently, $\beta(G) = n(G) - 3$.

This completes the proof of sufficiency of Theorem 3.1. □

3.2. Proof of necessity. Throughout this section, G is a graph of order n, diameter 2, metric dimension $n-3$, and G^* is the twin graph of G. Note that Proposition 1 implies that $\operatorname{diam}(G^*) \leqslant 2$. Through a sequence of lemmas and propositions, we will show that G^* has one of the structures G_1 through G_{10}.

Proposition 3. *If* $\operatorname{diam}(G^*) = 1$, *then* G^* *has the structure of* G_1.

P r o o f. Let u^*, v^* be two vertices of G^* of type (1K). Since G^* is a complete graph, every pair of vertices $u \in u^*$ and $v \in v^*$ of G are twins, thus $u^* = v^*$. Hence,

G^* has at most one vertex of type (1K). If vertices v_1^*, v_2^*, v_3^*, and v_4^* of G^* (except possibly v_1^*) are of type (N), then we choose an arbitrary vertex $v_i \in v_i^*$ for each i, $1 \leqslant i \leqslant 4$, and $u_i \in v_i^* \setminus \{v_i\}$ for each i, $2 \leqslant i \leqslant 4$. If $T = \{u_2, u_3, u_4\}$, then $r(v_1|T) = (1,1,1)$, $r(v_2|T) = (2,1,1)$, $r(v_3|T) = (1,2,1)$, and $r(v_4|T) = (1,1,2)$. Hence, the set $V(G) \setminus \{v_1, v_2, v_3, v_4\}$ is a resolving set for G and $\beta(G) \leqslant n-4$. This contradicts our assumption $\beta(G) = n-3$. Therefore, G^* has at most three vertices. Assume $G^* \cong K_t$ for some integer $t \in [1,3]$. If $t=1$, then $G \cong K_n$ or $G \cong \overline{K}_n$. If $t=2$, then $G \cong K_{r,s}$ or $G \cong K_r \vee \overline{K}_s$. Since $\beta(G) = n-3$, these cases are impossible. Consequently $G^* \cong K_3$, which is the desired conclusion. $\square$

The remainder of this section will be devoted to the case $\mathrm{diam}(G^*) = 2$. It is clear that in this case there exists $v \in V(G)$ such that $\Gamma_2(v^*) \neq \emptyset$ and $\Gamma_2(v) \neq \emptyset$.

Lemma 1. *If $\Gamma_2(v^*) \neq \emptyset$ and $v \in v^*$, then $\bigcup_{u^* \in \Gamma_i(v^*)} u^* \subseteq \Gamma_i(v)$, where $i \in \{1,2\}$, $\bigcup_{u^* \in (R_2(v))^*} u^* \subseteq R_2(v)$, and $\bigcup_{u^* \in (R_1(v))^*} u^* = R_1(v)$. Moreover, if v^* is of type* (1K), *then $\bigcup_{u^* \in \Gamma_2(v^*)} u^* = \Gamma_2(v)$, $R_1(v^*) = (R_1(v))^*$ and $R_2(v^*) = (R_2(v))^*$.*

P r o o f. It is clear that a vertex in $\Gamma_1(v)$ is not twin with any vertex of $\Gamma_2(v)$. Therefore, all twins of vertices in $\Gamma_1(v)$ are in the set $\Gamma_1(v) \cup \{v\}$ and all twins of vertices in $\Gamma_2(v)$ are in $\Gamma_2(v) \cup \{v\}$. This gives $\Gamma_i(v) \setminus v^* = \bigcup_{u^* \in \Gamma_i(v^*)} u^*$ for $i \in \{1,2\}$. Hence, $\bigcup_{u^* \in \Gamma_i(v^*)} u^* \subseteq \Gamma_i(v)$, when $i \in \{1,2\}$. Note that all twins of vertices in $R_1(v)$ are in $R_1(v)$, because each member of $R_1(v)$ is adjacent to v and has at least one neighbor in $\Gamma_2(v)$ while the vertices in $R_2(v) \cup \{v\}$ have not such neighbors. Thus $\bigcup_{u^* \in (R_1(v))^*} u^* = R_1(v)$. By the same argument, all twins of vertices in $R_2(v)$ are in $R_2(v) \cup \{v\}$. Consequently, $\bigcup_{u^* \in (R_2(v))^*} u^* = R_2(v) \setminus v^* \subseteq R_2(v)$.

Now let v^* be of type (1K). Therefore, v can only be twin with vertices of $R_2(v)$. Hence, $\bigcup_{u^* \in \Gamma_2(v^*)} u^* = \Gamma_2(v)$. It is clear that $R_1(v^*) \subseteq (R_1(v))^*$. If there exists a vertex $u^* \in (R_1(v))^* \setminus R_1(v^*)$, then $u^* \in R_2(v^*)$, because $\Gamma_1(v^*) = R_1(v^*) \cup R_2(v^*)$. Since $v^* \subseteq R_2(v) \cup \{v\}$, all neighbors of each $u \in u^*$ are in the set $\Gamma_1(v) \cup \{v\}$, which contradicts the fact that $u \in R_1(v)$. Therefore, $R_1(v^*) = (R_1(v))^*$ and consequently, $R_2(v^*) = (R_2(v))^*$. $\square$

Lemma 2. *For each v, where $\Gamma_2(v) \neq \emptyset$, at least one of the sets $\Gamma_1(v)$ and $\Gamma_2(v)$ is homogeneous.*

P r o o f. Let $\Gamma_2(v) \neq \emptyset$. On the contrary, suppose that neither $\Gamma_1(v)$ nor $\Gamma_2(v)$ are homogeneous. Therefore, there exist vertices v_1, v_2, and v_3 in $\Gamma_1(v)$, and vertices

u_1, u_2, and u_3 in $\Gamma_2(v)$ such that $v_1 \sim v_2, v_2 \nsim v_3$ and $u_1 \sim u_2$, $u_2 \nsim u_3$. If $W' = \{v, v_2, u_2\}$, then $r(v_1|W') = (1, 1, *)$, $r(v_3|W') = (1, 2, *)$, $r(u_1|W') = (2, *, 1)$, and $r(u_3|W') = (2, *, 2)$, where $*$ is 1 or 2. These representations are distinct, hence, $V(G) \setminus \{v_1, v_3, u_1, u_3\}$ is a resolving set for G. Thus $\beta(G) \leqslant n-4$. This contradiction implies that at least one of the sets $\Gamma_1(v)$ or $\Gamma_2(v)$ is homogeneous. $\square$

For Lemma 2, to complete the proof of necessity, we need to consider the following two cases.

Case 1. There exists a vertex $v \in V(G)$ such that $\Gamma_2(v^*) \neq \emptyset$ and $\Gamma_1(v)$ is homogeneous.

By the assumption of Case 1, the following results are obtained.

Fact 1. $|R_1(v^*)| \leqslant 2$.

P r o o f. Since every vertex of $R_1(v^*)$ has a neighbor in $\Gamma_2(v^*)$ and $\Gamma_1(v)$ is homogeneous, for distinct vertices $x^*, y^* \in R_1(v^*)$, the sets $\Gamma_1(x^*) \bigcap \Gamma_2(v^*)$ and $\Gamma_1(y^*) \bigcap \Gamma_2(v^*)$ are distinct nonempty sets. Therefore, $\Gamma_2(v^*)$ resolves vertices of $R_1(v^*)$ in G^*. Moreover, since every vertex of $R_1(v^*)$ has a neighbor in $\Gamma_2(v^*)$ and v^* has no such neighbor, the representation of each vertex in $R_1(v^*)$ with respect to $\Gamma_2(v^*)$ has a coordinate 1 while all coordinates of $r(v^*|\Gamma_2(v^*))$ are 2. Therefore, $\Gamma_2(v^*)$ resolves $R_1(v^*) \cup \{v^*\}$ and consequently, $V(G^*) \setminus (R_1(v^*) \cup \{v^*\})$ is a resolving set for G^*. Thus $\beta(G^*) \leqslant n(G^*) - |R_1(v^*) \cup \{v^*\}|$. On the other hand, by Propositions 2, $\beta(G^*) \geqslant n(G^*) - 3$. Hence, $|R_1(v^*) \cup \{v^*\}| \leqslant 3$, which yields $|R_1(v^*)| \leqslant 2$. $\square$

Lemma 3. *If $\Gamma_2(v)$ is not homogeneous, then $R_2(v)$ and $R_2(v^*)$ are empty sets.*

P r o o f. Note that $|R_1(v^*)| \geqslant 1$, otherwise $\Gamma_2(v^*) = \emptyset$. Since $\Gamma_2(v)$ is not homogeneous, there exist vertices i, j, and k in Γ_2 such that $i \sim j$ and $j \nsim k$. If $R_2(v) \neq \emptyset$, then let $r_1 \in R_1(v) \cap \Gamma_1(j)$, $r_2 \in R_2(v)$ and $W_0 = \{v, j\}$. Thus, $r(i|W_0) = (2, 1)$, $r(k|W_0) = (2, 2)$, $r(r_1|W_0) = (1, 1)$ and $r(r_2|W_0) = (1, 2)$, and so $\beta(G) \leqslant n - 4$. This contradiction implies $R_2(v) = \emptyset$.

If $R_2(v^*) \neq \emptyset$, then $\bigcup\limits_{u^* \in R_2(v^*)} u^* \nsubseteq R_2(v)$ and hence, by Lemma 1, v^* is of type (N). Therefore, there exist two adjacent vertices $a, b \in \bigcup\limits_{u^* \in \Gamma_2(v^*)} u^*$, otherwise $\Gamma_2(v)$ is homogeneous. Since $\operatorname{diam}(G^*) = 2$, there exists $r_1 \in \bigcup\limits_{u^* \in R_1(v^*)} u^*$ such that $r_1 \sim a$. Now let $v_1, v_2 \in v^*$, $r_2 \in \bigcup\limits_{u^* \in R_2(v^*)} u^*$, and $W = \{v_1, a\}$. Thus, $r(v_2|W) = (2, 2)$, $r(r_1|W) = (1, 1)$, $r(r_2|W) = (1, 2)$, and $r(b|W) = (2, 1)$. Therefore, $V(G) \setminus \{v_2, r_1, r_2, b\}$ is a resolving set for G, which contradicts $\beta(G) = n - 3$. Consequently $R_2(v^*) = \emptyset$. $\square$

Fact 2. $|\Gamma_2(v^*)| \leqslant 3$.

P r o o f. If $\Gamma_2(v)$ is homogeneous, then since every vertex of $\Gamma_2(v^*)$ has a neighbor in $R_1(v^*)$ and $\Gamma_2(v^*)$ is homogeneous, for distinct vertices $x^*, y^* \in \Gamma_2(v^*)$ the sets $\Gamma_1(x^*) \bigcap R_1(v^*)$ and $\Gamma_1(y^*) \bigcap R_1(v^*)$ are distinct nonempty sets. Therefore, for each pair of vertices $x^*, y^* \in \Gamma_2(v^*)$ there exists $r_1^* \in R_1(v^*)$ such that r_1^* resolves x^* and y^* in G^*. Hence, $R_1(v^*)$ resolves all vertices of $\Gamma_2(v^*)$. This implies that $V(G^*) \setminus \Gamma_2(v^*)$ is a resolving set for G^*, which yields $\beta(G^*) \leqslant n(G^*) - |\Gamma_2(v^*)|$. On the other hand, by Propositions 2 we have $\beta(G^*) \geqslant n(G^*) - 3$. Thus, $|\Gamma_2(v^*)| \leqslant 3$.

If $\Gamma_2(v)$ is not homogeneous, then by Fact 1, $|R_1(v^*)| \leqslant 2$. If $|R_1(v^*)| = 1$, let $R_1(v^*) = \{r_1^*\}$, $r_1 \in r_1^*$, and for each $l \in \Gamma_2(v)$, $N_1(l) = \Gamma_1(l) \cap \Gamma_2(v)$ and $N_2(l) = \Gamma_2(l) \cap \Gamma_2(v)$. Since $\Gamma_2(v)$ is not homogeneous, there exists $x \in \Gamma_2(v)$ such that both $N_1(x)$ and $N_2(x)$ are nonempty sets. Let $a \in N_1(x)$ and $y \in N_2(x)$. Note that $\Gamma_2(v^*) = \{x^*\} \cup N_1^*(x) \cup N_2^*(x)$, and x resolves a and y. Hence, if $N_1(x)$ is not homogeneous, then there exist vertices $i, j, k \in N_1(x)$ such that $i \sim j$ and $j \nsim k$. Thus, $\{v, x, j\}$ resolves $\{i, k, y, r_1\}$, and this contradiction yields that $N_1(x)$ is homogeneous. By a similar argument $N_2(x)$ is also homogeneous. Note that the vertices outside of $N_1^*(x)$ that are adjacent to some but not all vertices in $N_1^*(x)$ are in the set $N_2^*(x)$, because $N_1(x)$ is homogeneous and its vertices share their neighbors in $\Gamma_1(v) \cup \{x\}$. Similarly, all vertices outside of $N_2^*(x)$ that are adjacent to some but not all vertices in $N_2^*(x)$ are in $N_1^*(x)$, hence, $N_1^*(x)$ and $N_2^*(x)$ resolve each other. Now let $W_1 = N_2^*(x) \cup \{x^*\}$. If each vertex of $N_1^*(x)$ has a non-neighbor vertex in $N_2^*(x)$, then the representation of each vertex of $N_1^*(x)$ with respect to $N_2^*(x)$ has the coordinate 2, all coordinates of $r(r_1^*|N_2^*(x))$ are 1, and $r(v^*|N_2^*(x))$ is again 2. Consequently, W_1 resolves $N_1^*(x) \cup \{r_1^*, v^*\}$. Thus, $\beta(G) = n - 3$ implies that $|N_1^*(x)| \leqslant 1$. Moreover, if there exists $a^* \in N_1^*(x)$ such that a^* is adjacent to all vertices of $N_2^*(x)$, then $N_1^*(x)$ has at most two vertices. Otherwise, there are two distinct vertices $b^*, c^* \in N_1^*(x)$ such that they are different from a^*, and $r(b^*|N_2^*(x))$ and $r(c^*|N_2^*(x))$ are not 1, and so W_1 resolves $\{a^*, b^*, c^*, v^*\}$, contrary to $\beta(G) = n - 3$. Hence, $|N_1^*(x)| \leqslant 2$. Furthermore, $|N_1^*(x)| = 2$ yields that there exists $a^* \in N_1^*(x)$ such that a^* is adjacent to all vertices of $N_2^*(x)$. In a similar way $|N_2^*(x)| \leqslant 2$, moreover, $|N_2^*(x)| = 2$ only if there exists $y^* \in N_2^*(x)$ such that y^* is non-adjacent to all vertices of $N_1^*(x)$. Thus, at most one of the sets $N_1^*(x)$ and $N_2^*(x)$ can have two vertices, because it is impossible that there exist a pair of vertices $a^* \in N_1^*(x)$, $y^* \in N_2^*(x)$ such that a^* is adjacent to all vertices of $N_2^*(x)$ and y^* is non-adjacent to all vertices of $N_1^*(x)$. Consequently $|\Gamma_2(v^*)| \leqslant 4$. We claim that $|\Gamma_2(v^*)| = 4$ is impossible.

If $|\Gamma_2(v^*)| = 4$, then one of the two cases below can occur.

1. $|N_1^*(x)| = 2$ and $|N_2^*(x)| = 1$. Let $N_1^*(x) = \{a^*, b^*\}$, $N_2^*(x) = \{y^*\}$, $y^* \sim a^*$, $v \in v^*, a \in a^*, b \in b^*, x \in x^*$ and $y \in y^*$. If $a^* \sim b^*$, then x^* and b^* are twins,

see Figure 2 (a). Since $x^* \sim b^*$, one of them is of type (N). Note that b^* is not of type (N), because $N_1^*(x)$ is homogeneous and $a^* \sim b^*$. Hence, b^* is of type (1K) and x^* is of type (N). Thus, $V(G) \setminus \{r_1, y, a, x\}$ is a resolving set for G, which is impossible. Therefore, $a^* \nsim b^*$, thus $V(G) \setminus \{v, r_1, x, a\}$ is a resolving set for G, which is a contradiction.

2. $|N_1^*(x)| = 1$ and $|N_2^*(x)| = 2$. Let $N_1^*(x) = \{a^*\}$, $N_2^*(x) = \{y^*, z^*\}$, $z^* \sim a^*$, $v \in v^*$, $a \in a^*$, $x \in x^*$, $y \in y^*$, $z \in z^*$ and $S = \{x, y\}$. Thus, $r(z|S) = (2, *)$, $r(a|S) = (1, 2)$, $r(r_1|S) = (1, 1)$ and $r(v|S) = (2, 2)$, where $*$ is 1 or 2. Note that $\beta(G) = n-3$ yields $* = 2$, see Figure 2 (b). Therefore, x^* and z^* are twins. Since they are non-adjacent vertices, one of them is of type (K). Clearly z^* is not of type (K), otherwise, since $N_2^*(x)$ is homogeneous, we would have $* = 1$, which is impossible. Consequently, x^* is of type (K), which implies that the set $(x^* \setminus \{x\}) \cup \{z, y\}$ resolves $\{v, r_1, a, x\}$.

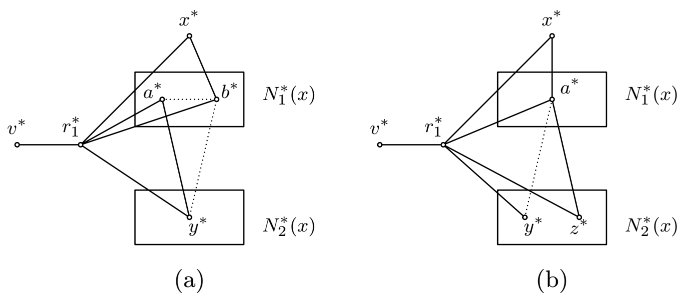


Figure 2. Different cases of $N_1^*(x)$ and $N_2^*(x)$.

These contradictions yield, $|\Gamma_2(v^*)| \leqslant 3$ when $|R_1(v^*)| = 1$.

To complete the proof we only need to consider the case $|R_1(v^*)| = 2$. In this case, since the vertices outside of $R_1(v)$ that are adjacent to some but not all vertices in $R_1(v)$ are in $\Gamma_2(v)$, $|R_1(v^*)| = 2$ implies that there exists $a \in \Gamma_2(v)$ such that $\Gamma_2(a) \cap R_1(v) \neq \emptyset$. Let $T = \Gamma_2(v) \setminus \{a\}$. Since $\Gamma_2(v)$ is not homogeneous, it has at least three vertices, hence $|T| \geqslant 2$. If T is not homogeneous, then there exists $i \in T$ such that i resolves two vertices of T. Moreover, we know that a resolves two vertices of $R_1(v)$. Hence, $\{v, i, a\}$ resolves at least four vertices of G, which is impossible. Therefore, T is homogeneous.

If a is adjacent to a vertex in T, then a is adjacent to all vertices of T, otherwise $\{a, v\}$ resolves four vertices. If a vertex $t \in T$ is not adjacent to some vertices of $R_1(v)$, then similarly to the above, it can be seen that $\Gamma_2(v) \setminus \{t\}$ is homogeneous and t is either adjacent or non-adjacent to all vertices of $\Gamma_2(v) \setminus \{t\}$. This implies that $\Gamma_2(v)$ is homogeneous, which is a contradiction to the assumption. Hence, every vertex of T is adjacent to all vertices of $R_1(v)$. Therefore, all vertices of T are twins and form a vertex b^* in G^*. Thus, $\Gamma_2(v^*)$ consists of two vertices a^* and b^*, where a^* is of type (1) and b^* is of type (KN). $\square$

By the proof of Fact 2, it is easy to see that $N_1(x)$ and $N_2(x)$ are homogeneous. This will be used later.

Proposition 4. *If $|R_1(v^*)| = 1$, then G^* has one of the structures G_2, G_3, or G_7.*

P r o o f. Let $R_1(v^*) = \{r_1^*\}$. If $\Gamma_2(v)$ is homogeneous, then every vertex of $\Gamma_2(v^*)$ is adjacent to r_1^* and all vertices of $\Gamma_2(v^*)$ are twins. Consequently, since $\Gamma_2(v)$ is homogeneous, all vertices of $\bigcup\limits_{u^* \in \Gamma_2(v^*)} u^*$ are twins. This gives $|\Gamma_2(v^*)| = 1$. If $R_2(v^*) = \emptyset$, then $G^* \cong P_3$ and $\alpha(G^*) \geqslant 2$, otherwise by Theorem B, $\beta(G) = n-2$. It is easy to check that in both the cases $\alpha(G^*) = 2$ and $\alpha(G^*) = 3$, at least one of the leaves is of type (K), otherwise by Theorem B, $\beta(G) = n-2$. Let $G^* = (x_1^*, x_2^*, x_3^*)$ and let x_1^* be of type (K). If x_2^* is of type (1K) and x_3^* is of type (1), then by Theorem B, $\beta(G) = n-2$. This contradiction implies that, if x_3^* is of type (1), then x_2^* is of type (N). This implies G^* has the structure G_2.

If $R_2(v^*) \neq \emptyset$, then all vertices of $R_2(v^*)$ have a neighbor in $R_1(v^*)$, otherwise $\operatorname{diam}(G^*) \geqslant 3$. Since $\Gamma_1(v)$ is homogeneous, every vertex of $R_2(v^*)$ is adjacent to every vertex of $R_1(v^*)$. Hence, all vertices of $R_2(v^*)$ are twins. This implies that $\Gamma_1(v)$ is a clique, all vertices of $\bigcup\limits_{u^* \in R_2(v^*)} u^*$ are twins and so $|R_2(v^*)| = 1$. In this case G^* is a paw and one of the degree-2 vertices is v^* and the other belongs to $\Gamma_1(v^*)$. Therefore, the structure of G^* is as shown in Figure 3 (a). Hence, x^* and v^* are adjacent twins. Since $x^* \neq v^*$, one of them must be of type (N). Since $\Gamma_1(v)$ is a clique, x^*, y^* are of type (1K). Thus, v^* is of type (N). If z^* is of type (K), then we choose arbitrary fixed vertices $x \in x^*, v_1, v_2 \in v^*, y \in y^*$ and $z_1, z_2 \in z^*$. Hence, $r(v_1|\{v_2, z_2\}) = (2,2)$, $r(x|\{v_2, z_2\}) = (1,2)$, $r(y|\{v_2, z_2\}) = (1,1)$, and $r(z_1|\{v_2, z_2\}) = (2,1)$. Thus, the set $V(G) \setminus \{x, y, z_1, v_1\}$ is a resolving set for G, a contradiction. Consequently, z^* is of type (1N). Similarly, if x^* is of type (K) or z^* is of type (N), then $V(G) \setminus \{x, y, z_1, v_1\}$ is a resolving set for G, which is impossible. Hence, G^* has the structure G_3.

Now let $\Gamma_2(v)$ be not homogeneous. Using the same notation as in the proof of Fact 2, we can see that vertices of $N_1(x)$ can only be twins with each other and x, while vertices of $N_2(x)$ can only be twins with each other, x, and v. By Lemma 3, we have $R_2(v^*) = \emptyset$. Hence, if $|\Gamma_2(v^*)| = 1$, then $G^* \cong P_3$, all vertices of $N_2(x)$ are twins with v, and all vertices of $N_1(x)$ are twins with x. Thus, x^* is of type (K), v^* is of type (N), and r_1^* is of any type. In this case G^* has the structure of G_2.

When $|\Gamma_2(v^*)| = 2$, there exist three cases.

1. The vertex x and all vertices of $N_1(x)$ are twins and there exist vertices in $N_2(x)$ which are not twins with v but are twins by themselves. Thus, vertices of $N_2(x) \setminus v^*$ form exactly one vertex, y^* in G^*. Hence, x^* is of type (K) and $x^* \nsim y^*$. Therefore, y^* and v^* are twins. Since $v^* \nsim y^*$, one of them is of type (K). Note that, if v^* is of type (K), then there exists $u \in V(G)$ which is adjacent to all vertices of $\Gamma_1(v) \cup \{v\}$ and is not adjacent to any vertex of $\Gamma_2(v)$. Hence, $u \in R_2(v)$, which is impossible, because by Lemma 3, $R_2(v) = \emptyset$. Thus, y^* is of type (K). If $r_1 \in r_1^*$, $y \in y^*$, $x \in x^*$ and $v \in v^*$, then $(x^* \setminus \{x\}) \cup (y^* \setminus \{y\})$ resolves $\{v, r_1, x, y\}$, a contradiction.

2. There exist vertices of $N_2(x)$ which are twins with x while the rest are twins with v, and all vertices of $N_1(x)$ are twins. Therefore, vertices of $N_1(x)$ create a vertex a^* in G^*, $a^* \sim x^*$, and x^* is of type (N). Hence, G^* is a paw, with the leaf v^*, the degree-3 vertex r_1^*, and degree-2 vertices a^* and x^*. If a^* is of type (N), then $V(G) \setminus \{v, r_1, x, a\}$ is a resolving set for G, where $r_1 \in r_1^*, v \in v^*, x \in x^*$ and $a \in a^*$. This contradiction yields a^* is of type (1K). Since $R_2(v) = \emptyset$, v^* is not of type (K). Therefore, v^* is of type (1N). By a method similar to that used before, we deduce that, if a^* is of type (K), then v^* and r_1^* cannot be of type (N). Thus, G^* has the structure of G_3.

3. All vertices of $N_2(x)$ are twins with v, and there exist vertices in $N_1(x)$ which are not twins with x. Hence, vertices of $N_1(x) \setminus x^*$ form a unique vertex a^* in G^*, $a^* \sim x^*$, and consequently G^* is a paw. Note that v^* is the leaf and its type is (N), the vertex r_1^* has degree 3, and x^*, a^* are degree-2 vertices. Since x^* and a^* are adjacent twins, one of them is of type (N). Also, since all vertices of $N_2^*(x)$ are twins with v, x^* is of type (1K), and so a^* is of type (N). Therefore, G^* has the structure of G_3.

Finally, if $|\Gamma_2(v^*)| = 3$, then we have the following three cases.

1. Every vertex of $N_2(x)$ is a twin with v or x, and $N_1^*(x)$ has two vertices a^* and b^*. In this case a^* and b^* are twins, hence $a^* = b^*$, because $N_1(x)$ is homogeneous. Thus, $|\Gamma_2(v^*)| \leqslant 2$. Therefore this is not the case.

2. Every vertex of $N_1(x)$ is a twin with x, and $N_2^*(x)$ has two vertices y^* and z^*. Hence, y^* and z^* are twins. Since $N_2(x)$ is homogeneous, $y^* = z^*$, which is a contradiction to $|\Gamma_2(v^*)| = 3$.

3. There exist vertices in $N_1(x)$ which are not twins with x, also there exist vertices in $N_2(x)$ which are twins neither with x nor v. In this case, each one of $N_1^*(x)$ and $N_2^*(x)$ has exactly one vertex a^* and y^*, respectively. If $a^* \nsim y^*$, then v^* and y^* are non-adjacent twins. Hence one of them is of type (K). Since $R_2(v) = \emptyset$, v^* is of type (1N), and so y^* is of type (K). If $v \in v^*$, $y \in y^*$, $a \in a^*$, and $x \in x^*$, then $V(G) \setminus \{y, x, r_1, v\}$ is a resolving set for G. This contradiction yields $a^* \sim y^*$, and in consequence, G^* is a kite with a pendant edge, adjacent to a degree-3 vertex. Thus, y^* and x^* are non-adjacent twins, hence one of them is of type (K). By symmetry,

we can assume x^* is of type (K). Since $R_2(v) = \emptyset$, v^* is of type (1N). As observed before, v^*, a^*, and r_1^* are not of type (N) and y^* is not of type (KN). Therefore, G^* has the structure of G_7 and the proof is completed. $\square$

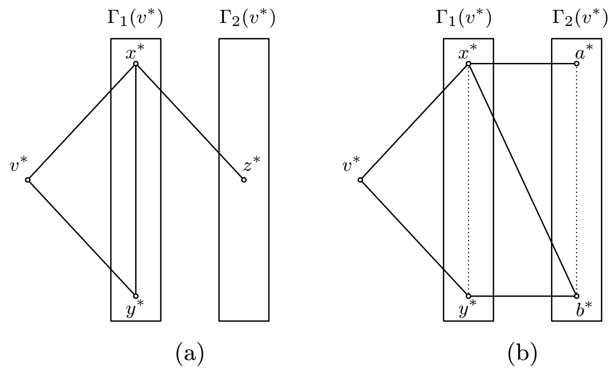


Figure 3. $|\Gamma_2(v^*)| = 1$ and $|\Gamma_2(v^*)| = 2$.

Until now, we have considered the case $|R_1(v^*)| = 1$. Now we investigate the case $|R_1(v^*)| = 2$.

Proposition 5. *If $|R_1(v^*)| = 2$ and $\Gamma_2(v)$ is not homogeneous, then G^* has the structure of G_7.*

Proof. Using the same notation as in the proof of Fact 2, let $\Gamma_2(v^*) = \{a^*, b^*\}$, where a^* is of type (1) and b^* is of type (KN). Let $R_1(v^*) = \{x^*, y^*\}$. Since $R_1(v)$ has a non-adjacent vertex to a, the vertex a^* has exactly one neighbor in $R_1(v^*)$. By the proof of Fact 2, $b^* \sim x^*$ and $b^* \sim y^*$, thus G^* is the 4-cycle, $C_4 = (v^*, x^*, b^*, y^*, v^*)$ with the pendant edge x^*a^* and possibly extra edges a^*b^* and x^*y^*, see Figure 3 (b). Because $\mathrm{diam}(G^*) = 2$, at least one of the edges a^*b^* and x^*y^* exists. If $a^* \sim b^*$, then b^* is of type (N). Let $v \in v^*$, $a \in a^*$, $b \in b^*$, $x \in x^*$, and $y \in y^*$. Consequently, the set $a^* \cup (b^* \setminus \{b\})$ resolves $\{v, x, b, y\}$, since b^* is of type (N). This contradiction yields $a^* \nsim b^*$, and so $x^* \sim y^*$, a^* is of type (1), and b^* is of type (K). Note that x^* and y^* are not of type (N), otherwise $\Gamma_1(v)$ is not homogeneous. Also, we can see easily that v^* is not of type (KN). Thus, G^* has the structure of G_7. $\square$

Now, we only need to consider the case $|R_1(v^*)| = 2$ when $\Gamma_2(v)$ is homogeneous. In this case, if $|\Gamma_2(v^*)| = 1$, then all vertices of $R_1(v)$ are twins and consequently, $|(R_1(v))^*| = 1$, which contradicts $|R_1(v^*)| = 2$. Therefore, $|\Gamma_2(v^*)| \geqslant 2$.

Lemma 4. *If $\Gamma_2(v)$ is homogeneous and $|R_1(v^*)| = 2$, then $R_2(v^*) = \emptyset$.*

P r o o f. By Fact 2 and the above argument, we have $2 \leqslant |\Gamma_2(v^*)| \leqslant 3$. Suppose on the contrary that $R_2(v^*) \neq \emptyset$. Since $\mathrm{diam}(G^*) = 2$ and $\Gamma_1(v)$ is homogeneous, every vertex of $R_2(v^*)$ is adjacent to every vertex of $R_1(v^*)$. In this way all vertices of $R_2(v^*)$ are twins, therefore all vertices of $\bigcup\limits_{u^* \in R_2(v^*)} u^*$ are twins, and so $|R_2(v^*)| = 1$. Let $R_2(v^*) = \{r_2^*\}$, $R_1(v^*) = \{x^*, y^*\}$, and $\{a^*, b^*\} \subseteq \Gamma_2(v^*)$. Therefore, r_2^* is adjacent to vertices v^*, x^* and y^*. Note that $\Gamma_1(v)$ and $\Gamma_1(v^*)$ are cliques, because $\Gamma_1(v)$ is homogeneous and r_2^* is adjacent to x^* and y^*. Thus, r_2^*, x^* and y^* are of type (1K). Since all neighbors of r_2^* and v^* are shared, r_2^* and v^* are adjacent twins. Since $r_2^* \neq v^*$, one of them is of type (N). Because r_2^* is of type (1K), v^* is of type (N) and v^* contains at least one additional vertex u beside v. This is a contradiction since $u \in \Gamma_2(v)$ is adjacent to all vertices of $R_2(v)$. Hence $R_2(v^*) = \emptyset$. $\square$

Lemma 5. *If $\Gamma_2(v)$ is homogeneous and $|\Gamma_2(v^*)| = 3$, then there is exactly one vertex $a^* \in \Gamma_2(v^*)$ such that a^* is adjacent to all vertices of $R_1(v^*)$.*

P r o o f. Let $|\Gamma_2(v^*)| = 3$. Since the vertices outside of $\Gamma_2(v^*)$ that are adjacent to some but not all vertices in $\Gamma_2(v^*)$ are in $R_1(v^*)$, $R_1(v^*)$ resolves the set $\Gamma_2(v^*)$. Suppose on the contrary that every vertex of $\Gamma_2(v^*)$ is not adjacent to all vertices of $R_1(v^*)$. Hence, at least one coordinate of the representation of each vertex in $\Gamma_2(v^*)$ with respect to $R_1(v^*)$ is 2, while every coordinate of $r(v^*|R_1(v^*))$ is 1. Therefore, $R_1(v^*)$ is a resolving set for $G^*[R_1(v^*) \cup \Gamma_2(v^*) \cup \{v^*\}]$ with cardinality at most $n(G^*) - 4$, since $|\Gamma_2(v^*) \cup \{v^*\}| = 4$. It follows that $\beta(G^*) \leqslant n(G^*) - 4$, and Proposition 2 implies that $\beta(G) \leqslant n - 4$, which is a contradiction. Therefore, there exists a vertex $a^* \in \Gamma_2(v^*)$ adjacent to all vertices of $R_1(v^*)$. If there exists another vertex $b^* \in \Gamma_2(v^*)$ adjacent to all of $R_1(v^*)$, then a^* and b^* are twins, since $\Gamma_2(v^*)$ is homogeneous. This implies that $a^* = b^*$ while $|\Gamma_2(v^*)| = 3$. Therefore, such a vertex in $\Gamma_2(v^*)$ is unique. $\square$

Lemma 6. *If $\Gamma_2(v)$ is homogeneous and $|R_1(v^*)| = 2$, then $|\Gamma_2(v^*)| \leqslant 2$.*

P r o o f. On the contrary, suppose $|\Gamma_2(v^*)| = 3$. By Lemma 5, there exists exactly one vertex $a^* \in \Gamma_2(v^*)$ such that a^* is adjacent to all vertices of $R_1(v^*)$. Let $R_1(v^*) = \{x^*, y^*\}$ and $\Gamma_2(v^*) = \{a^*, b^*, c^*\}$. Each of the vertices b^* and c^* has at least one neighbor in $R_1(v^*)$ and by Lemma 5, they have exactly one neighbor in $R_1(v^*)$. If their neighbors in $R_1(v^*)$ are the same, then they are twins, since $\Gamma_2(v^*)$ is homogeneous. This implies that every pair of vertices $b \in b^*$ and $c \in c^*$ are twins (because $\Gamma_2(v)$ is homogeneous), consequently, $b^* = c^*$, which contradicts $|\Gamma_2(v^*)| = 3$. Thus, one of them, say b^*, is (only) adjacent to x^* and the other c^*, is (only) adjacent to y^*, see Figure 4 (a) (dotted edges may exist or not). Now $r(v^*|\{b^*, c^*\}) = (2, 2)$, $r(x^*|\{b^*, c^*\}) = (1, 2)$, $r(y^*|\{b^*, c^*\}) = (2, 1)$, and $r(a^*|\{b^*, c^*\}) = (*, *)$, where $*$ is 1 or 2. If $* = 1$, then $r(a^*|\{b^*, c^*\}) = (1, 1)$, and so $V(G^*) \setminus \{a^*, v^*, x^*, y^*\}$ is a resolving set for G^*; this contradiction yields $* = 2$. Since $\Gamma_2(v)$ is homogeneous, $\Gamma_2(v)$ and $\Gamma_2(v^*)$ are independent sets.

Since $R_2(v^*) = \emptyset$ by Lemma 3, if v^* is of type (1N), then v^* and a^* are twins and every pair of vertices $v \in v^*$ and $a \in a^*$ are twins (because both a^* and v^* are of type (1N)), and so $v^* = a^*$, a contradiction. Therefore, v^* is of type (K). For arbitrary fixed vertices $v_1, v_2 \in v^*, x \in x^*, y \in y^*, a \in a^*, b \in b^*$ and $c \in c^*$ and $T = \{v_1, a, c\}$ we have $r(v_2|T) = (1, 2, 2)$, $r(x|T) = (1, 1, 2)$, $r(y|T) = (1, 1, 1)$, and $r(b|T) = (2, 2, 2)$. Hence, $V(G) \setminus \{v_2, x, y, b\}$ is a resolving set for G. This contradiction implies that $|\Gamma_2(v^*)| \leqslant 2$. $\square$

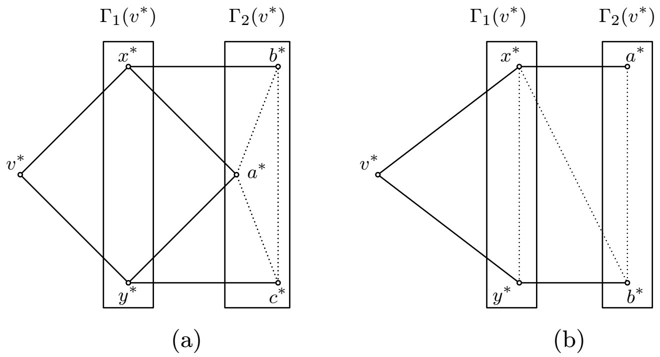


Figure 4. $|\Gamma_2(v^*)| = |\Gamma_1(v^*)| = 3$ and $|\Gamma_2(v^*)| = |\Gamma_1(v^*)| = 2$.

Corollary 3. *If $\Gamma_2(v)$ is homogeneous and $|R_1(v^*)| = 2$, then $|\Gamma_2(v^*)| = 2$.*

Proof. By Lemma 6, $|\Gamma_2(v^*)| \leqslant 2$. If $|\Gamma_2(v^*)| = 1$, then $|R_1(v^*)| = 1$, because $\Gamma_1(v)$ and $\Gamma_2(v)$ are homogeneous. This contradiction implies $|\Gamma_2(v^*)| = 2$. $\square$

On account of the above results, we only need to assume that $|\Gamma_2(v^*)| = |R_1(v^*)| = 2$ and $|R_2(v^*)| = 0$.

Proposition 6. *If $|R_1(v^*)| = 2$ and $\Gamma_2(v)$ is homogeneous, then G^* has one of the structures G_4 through G_7.*

Proof. If $R_1(v^*) = \{x^*, y^*\}$, $\Gamma_2(v^*) = \{a^*, b^*\}$, then G^* is as described in Figure 4 (b). If $a^* \nsim b^*$, then $x^* \sim y^*$ and $x^* \sim b^*$, otherwise $\mathrm{diam}(G^*) = 3$, a contradiction. Let G_0 be the path $(a^*, x^*, v^*, y^*, b^*)$. Thus, G^* must be one of the following five graphs: $H_1^* := G_0 + a^*b^*$, $H_2^* := G_0 + a^*b^* + x^*b^*$, $H_3^* := G_0 + a^*b^* + x^*y^*$, $H_4^* := G_0 + a^*b^* + x^*b^* + x^*y^*$, and $H_5^* := G_0 + x^*b^* + x^*y^*$. We fix vertices $v \in v^*$, $x \in x^*$, $y \in y^*$, $a \in a^*$ and $b \in b^*$ in each H_i^*, $1 \leqslant i \leqslant 5$. Note that $H_1^* \cong C_5$. If $G^* \cong H_1^*$, then all vertices of H_1^* are of type (1), otherwise (by a simple computation) $\beta(G) \leqslant n-4$. In this case G^* has the structure of G_4.

If $G^* \cong H_2^*$, then x^* and y^* are not of type (K), because $\Gamma_1(v)$ is homogeneous and $x^* \nsim y^*$. Similarly, a^* and b^* are not of type (N). If x^* or y^* is of type (N), then $V(G) \setminus \{v, x, y, b\}$ is a resolving set for G, with cardinality $n-4$. Further, v^* of type (N) or (K) yields $V(G) \setminus \{v, x, y, b\}$ or $V(G) \setminus \{v, x, a, b\}$ is a resolving set for G, respectively. These contradictions show that G^* has the structure of G_5.

Let $G^* \cong H_3^*$. Since $\Gamma_1(v)$ and $\Gamma_2(v)$ are homogeneous, $x^* \sim y^*$ and $a^* \sim b^*$ imply that x^*, y^*, a^*, b^* are not of type (N). If a^* or b^* is of type (K), then $V(G) \setminus \{v, x, y, a\}$ or $V(G) \setminus \{x, y, v, b\}$ is a resolving set for G. Also, v^* of type (N) yields the resolving set, $V(G) \setminus \{x, y, v, b\}$ for G. These contradictions imply that G^* has the structure of G_5.

If $G^* \cong H_4^*$ and one of the vertices v^* or y^* is of type (N), then the set $V(G) \setminus \{v, x, y, b\}$ or $V(G) \setminus \{x, y, a, b\}$, respectively, is a resolving set for G. Thus v^* and y^* are of type (1K). By symmetry, vertices a^* and b^* are of type (1K). If non-adjacent vertices v^* and b^* are of type (K), then the set $V(G) \setminus \{v, x, y, b\}$ is a resolving set of size $n-4$. Similarly, non-adjacent vertices a^* and y^* are not of type (K) simultaneously. Also, if non-adjacent vertices a^* and v^* are of type (K), then $V(G) \setminus \{v, x, y, a\}$ resolves G, which is impossible. Therefore, non-adjacent vertices are not of the same type (K). Moreover, if x^* is of type (N), and y^* or v^* is of type (K), then $V(G) \setminus \{v, x, y, a\}$ is a resolving set for G. By the same argument, if x^* is of type (N), then a^* and b^* are not of type (K). Thus, G^* has the structure of G_6.

Finally, assume that $G^* \cong H_5^*$. Since $v^* \neq b^*$ and these vertices are non-adjacent twins in H_5^*, at least one of them is of type (K). Hence, v^* is of type (K) and b^* is of type (1N), because $\Gamma_2(v)$ is homogeneous and $a^* \nsim b^*$. If b^* is of type (N), then $V(G) \setminus \{v, x, y, b\}$ resolves G, a contradiction. It follows that b^* is of type (1). In the way similar to the above, a^* is of type (1), and both x^* and y^* are of type (1K), and thus G^* has the structure of G_7. $\square$

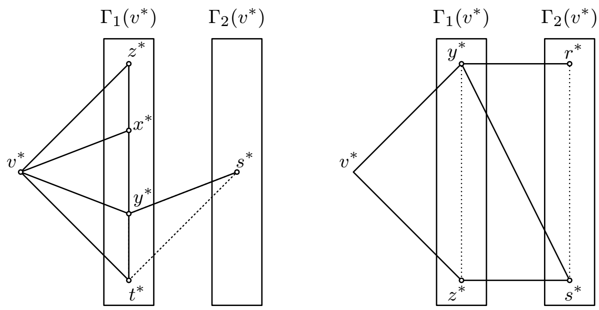
Case 2. For each vertex $v \in V(G)$ with $\Gamma_2(v^*) \neq \emptyset$, $\Gamma_1(v)$ is not homogeneous.

We choose a fixed vertex $v \in V(G)$ with $\Gamma_2(v^*) \neq \emptyset$. Lemma 2 concludes that in this case, $\Gamma_2(v)$ is homogeneous. For each $x \in \Gamma_1(v)$, let $M_1(x) := \Gamma_1(v) \cap \Gamma_1(x)$ and $M_2(x) := \Gamma_1(v) \cap \Gamma_2(x)$. Since $M_2(x) \subseteq \Gamma_2(x)$ and $\Gamma_2(x)$ is homogeneous, $M_2(x)$ is also homogeneous. If $M_1(x)$ is not homogeneous, then there exist vertices i, j, and k in $M_1(x)$ such that $i \sim j$ and $k \nsim j$. Thus, for each pair of vertices $y \in M_2(x)$ and $c \in \Gamma_2(v)$ we have $r(i|\{v, x, j\}) = (1, 1, 1)$, $r(k|\{v, x, j\}) = (1, 1, 2)$, $r(y|\{v, x, j\}) = (1, 2, *)$, $r(c|\{v, x, j\}) = (2, *_1, *_2)$, where $*$, $*_1$ and $*_2$ are 1 or 2.

However, these representations are distinct, which is a contradiction. Therefore, $M_1(x)$ is homogeneous.

Proposition 7. *If there exists $x \in R_2(v)$ with $M_2(x) \neq \emptyset$, then G^* has the structure of G_6.*

Proof. Since $x \in R_2(v)$, we have $\Gamma_1(x) = M_1(x) \cup \{v\}$. Note that v is adjacent to all vertices of $M_1(x)$. Since $M_1(x)$ is homogeneous and $\Gamma_1(x)$ is not homogeneous, we conclude $M_1(x)$ is an independent set and contains at least two vertices. Now let m_1 and m_2 be two arbitrary vertices in $M_1(x)$. Thus, m_1 resolves m_2 and v, hence m_1 cannot resolve any pair of vertices in $\Gamma_2(x)$, otherwise the set $\{x, m_1\}$ resolves at least four vertices. Therefore, m_1 is either adjacent to all vertices of $\Gamma_2(x)$ or non-adjacent to all of them. Since m_1 is an arbitrary vertex of $M_1(x)$, all vertices of $\Gamma_2(x)$ have the same neighbors in $M_1(x)$. Note that $\Gamma_2(x) = M_2(x) \cup \Gamma_2(v)$, because $x \in R_2(v)$. Moreover, all vertices of $M_2(x)$ are adjacent to v, and all vertices of $\Gamma_2(v)$ are not adjacent to v. Thus, every pair of vertices in $M_2(x)$ and also every pair of vertices of $\Gamma_2(v)$ are twins. Let $t^* = M_2(x)$ and $s^* = \Gamma_2(v)$ be the corresponding vertices in G^*. Moreover, vertices of $M_1(x)$ that are adjacent to all of $\Gamma_2(x)$ are twins and form a vertex y^* in G^*, also the remaining vertices of $M_1(x)$ are twins with each other and create a vertex z^* in G^*. Therefore, G^* has at most six vertices v^*, x^*, y^*, z^*, t^*, and s^*, where x^* is adjacent to v^*, z^*, and y^*. Also, v^* and y^* are adjacent to all vertices except s^* and z^*, respectively. There is no other edge in G^* except possibly s^*t^*, see Figure 5 (a).



(a) (b)

Figure 5. $|\Gamma_1(v^*)| = 4$ and $|\Gamma_1(v^*)| = 2$ in Case 2.

If all of these six vertices exist, then $d(z^*, s^*) = 3$, which contradicts $\text{diam}(G^*) = 2$. Since $s^* = \Gamma_2(v)$, the vertex z^* does not exist. It is clear that y^* is of type (N), because $M_1(x)$ is an independent set of size at least two. Let $v \in v^*$, $x \in x^*$, $y_1, y_2 \in y^*$, $s \in s^*$, and $t \in t^*$. If $s^* \nsim t^*$, then $v^* \cup x^* \cup t^* \subseteq \Gamma_2(s)$. But the set

$v^* \cup x^* \cup t^*$ is not homogeneous, and so $\Gamma_2(s)$ is not homogeneous; this contradiction yields $s^* \sim t^*$. Thus, since $s^* \cup t^* \subseteq \Gamma_2(x)$, the adjacent vertices s^* and t^* are of type (1K). Moreover, $x^* \cup v^* \subseteq \Gamma_2(s)$ and $x^* \sim v^*$ yields x^* and v^* are of type (1K). But as observed before, when y^* is of type (N) the other vertices cannot be of type (K), hence they must be of type (1); also, two non-adjacent vertices are not of the same type (K). Therefore, G^* has the structure of G_6. □

Now let for each $u \in R_2(v)$ one of the sets $M_1(u)$ and $M_2(u)$ be empty. Note that every vertex of $R_2(v)$ has a neighbor in $R_1(v)$, otherwise $\mathrm{diam}(G) \geqslant 3$. Hence, $M_1(u) \neq \emptyset$. Consequently, $M_2(u) = \emptyset$ for each $u \in R_2(v)$ and $M_1(u) = \Gamma_1(v) \setminus \{u\}$. Therefore, every vertex of $R_2(v)$ is adjacent to all vertices of $R_1(v)$, $R_2(v)$ is a clique, and $|R_2(v^*)| \leqslant 1$. We consider the cases $R_1(v)$ is homogeneous or not homogeneous separately.

Proposition 8. *If for each $u \in R_2(v)$, the set $M_2(u)$ is empty and $R_1(v)$ is homogeneous, then G^* has the structure of G_2.*

Proof. If $R_2(v) = \emptyset$, then $\Gamma_1(v) = R_1(v)$ is homogeneous, which is a contradiction. Thus, $R_2(v) \neq \emptyset$. Hence, $R_2(v)$ is a clique and all its vertices are adjacent to all vertices of $R_1(v)$. Therefore $v^* = \{v\} \cup R_2(v)$, $R_1(v) = x^*$ and $\Gamma_2(v) \neq \emptyset$. Hence G^* has a leaf of type (K). Moreover, x^* is of type (N) since $\Gamma_1(v)$ is not homogeneous. Also, since $\Gamma_2(v)$ is homogeneous, there is only one additional vertex $a^* = \Gamma_2(v)$ in G^*, which completes the proof. □

We investigate the case $R_1(v)$ is not homogeneous for two possibilities, $|\Gamma_2(v^*)| \geqslant 2$ and $|\Gamma_2(v^*)| = 1$, separately.

Proposition 9. *If for each $u \in R_2(v)$ the set $M_2(u) = \emptyset$, $R_1(v)$ is not homogeneous, and $|\Gamma_2(v^*)| \geqslant 2$, then G^* has the structure of G_6.*

Proof. Since $|\Gamma_2(v^*)| \geqslant 2$ and all neighbors of $\Gamma_2(v^*)$ are in $R_1(v^*)$ and $\Gamma_2(v)$ is homogeneous, there exist vertices $z^* \in R_1(v^*)$ and $t^* \in \Gamma_2(v^*)$ such that $z^* \nsim t^*$. If $z \in z^*$, then $z \nsim t$ for each $t \in t^*$. Since z has a neighbor $t' \in \Gamma_2(v)$, z is either adjacent to all vertices of $R_1(v) \setminus \{z\}$ or not adjacent to all these vertices, otherwise the set $\{v, z\}$ resolves four vertices of G. Moreover, if $R_1(v) \setminus \{z\}$ is not homogeneous, then there exist three vertices $i, j, k \in R_1(v) \setminus \{z\}$ such that j resolves $\{i, k\}$, and so $\{v, z, j\}$ resolves $\{i, k, t, t'\}$, which is impossible. Thus, $R_1(v) \setminus \{z\}$ is homogeneous. Therefore, $G[R_1(v)] \cong K_1 \vee \overline{K_l}$ or $K_1 \cup K_l$ for some positive integer $l \geqslant 2$, because $R_1(v)$ is not homogeneous. It follows that all vertices of $R_1(v) \setminus \{z\}$ have a neighbor and a non-neighbor vertex in $R_1(v)$. Hence, each vertex of $R_1(v) \setminus \{z\}$ is either adjacent or non-adjacent to all vertices of $\Gamma_2(v)$, since $\beta(G) = n - 3$. But by definition of $R_1(v)$, each vertex of $R_1(v)$ has a neighbor in $\Gamma_2(v)$. Consequently, each vertex of $R_1(v) \setminus \{z\}$ is adjacent to all vertices of $\Gamma_2(v)$. Thus, all vertices of $R_1(v) \setminus \{z\}$ are twins, and consequently they form a vertex y^* of type (KN) in G^*, and z^* is a vertex of type (1) in G^*. Therefore, $|R_1(v^*)| = 2$ and y^* is adjacent to all vertices of $\Gamma_2(v^*)$. Therefore, $|\Gamma_2(v^*)|$, because $\Gamma_2(v^*)$ is homogenous and z^* can devide vertices of $\Gamma_2(v^*)$ to two classes, its neighbors and non-neighbors.

Since for each $u \in R_2(v)$ the set $M_2(u)$ is empty, $v^* = \{v\} \cup R_2(v)$ is of type (1K). Let $\Gamma_2(v^*) = \{r^*, s^*\}$. Hence $G^*[V(G^*) \setminus R_2(v^*)]$ is as in Figure 5 (b). If neither the edge y^*z^* nor r^*s^* exist, then $d(r^*, z^*) = 3$, which contradicts $\operatorname{diam}(G^*) = 2$, therefore one of them exists. Let $y \in y^*$, $r \in r^*$, $s \in s^*$. If $y^* \sim z^*$ and $r^* \nsim s^*$, then y^* is of type (N), since $\Gamma_1(v)$ is not homogeneous. Also, r^* is of type (K), otherwise $\Gamma_1(r) = y^*$ is an independent set, which is impossible. Thus, $V(G) \setminus \{v, y, z, r\}$ is a resolving set for G. This contradiction shows that $r^* \sim s^*$ in G^*. If $y^* \nsim z^*$, then y^* is of type (K), since $\Gamma_1(v)$ is not homogeneous. Moreover, s^* and r^* are of type (1K), since $\Gamma_2(v)$ is homogeneous. However, $\Gamma_1(r)$ is a clique, this contradiction showing that both the edges r^*s^* and y^*z^* exist in G^*. Therefore, y^* is of type (N), since $\Gamma_1(v)$ is not homogeneous. Furthermore, r^* and s^* are of type (1K), because $\Gamma_2(v)$ is homogeneous. Also, since $\Gamma_2(v)$ is a clique, v^* is of type (1K). It is easy to see that if y^* is of type (N), then the other vertices of G^* are of type (1). Moreover, two non-adjacent vertices are not of the same type (K). Consequently, G^* has the structure of G_6. □

Proposition 10. *If for each $u \in R_2(v)$ the set $M_2(u)$ is empty, $R_1(v)$ is not homogeneous, and $|\Gamma_2(v^*)| = 1$, then G^* has one of the structures G_8 through G_{10}.*

P r o o f. Let $\Gamma_2(v^*) = \{w^*\}$. It is clear that $v^* = \{v\} \cup R_2(v)$, because for each $u \in R_2(v)$, $M_2(u) = \emptyset$. Hence v^* and w^* are twins in G^* and so at least one of them is of type (K), say v^*. Since $R_1(v)$ is not homogeneous, there exists $x \in R_1(v)$ such that $M_2(x) \neq \emptyset$. Let $y \in M_1(x) \cap R_1(v)$. Note that $M_1(x) \cap R_1(v)$ is not an independent set; otherwise $V(G) \setminus \{v, y, w, z\}$ is a resolving set for G, where $z \in M_2(x) \cap R_1(v)$. If there exist vertices $z_1, z_2 \in M_2(x) \cap R_1(v)$ such that $y \sim z_1$ and $y \nsim z_2$, then $V(G) \setminus \{v, z_1, z_2, w\}$ is a resolving set for G, which is a contradiction. Therefore each vertex of $M_1(x) \cap R_1(v)$ is either adjacent to all vertices in $M_2(x) \cap R_1(v)$ or non-adjacent to all of them. That is, all vertices of $M_2(x) \cap R_1(v)$ are twins. Now, one of the following cases can occur.

1. All vertices in $M_1(x) \cap R_1(v)$ are adjacent to all vertices in $M_2(x) \cap R_1(v)$. In this case, if $M_2(x) \cap R_1(v)$ is an independent set, then $x^* = \{x\} \cup (M_2(x) \cap R_1(v))$ is of type (N) and $y^* = M_1(x) \cap R_1(v)$ is of type (1K). Also, $\beta(G) = n - 3$ implies that w^* is of type (1). Therefore G^* has the structure of G_8. If $M_2(x) \cap R_1(v)$ is a clique of size at least 2, then $y^* = M_1(x) \cap R_1(v)$ is of type (1k) and $z^* = M_2(x) \cap R_1(v)$

is of type (K). Also, $\beta(G) = n - 3$ implies that w^* is of type (1). Hence G^* has the structure of G_{10}.

2. There is no edge between $M_1(x) \cap R_1(v)$ and $M_2(x) \cap R_1(v)$. In this case, $x^* = \{x\} \cup (M_1(x) \cap R_1(v))$ is of type (K) and $z^* = M_2(x) \cap R_1(v)$. Also, $\beta(G) = n - 3$ implies that z^* and w^* are of type (1). Therefore G^* has the structure of G_9.

3. There exists a non-empty subset $S \subset M_1(x) \cap R_1(v)$ such that all vertices in S are adjacent to all vertices in $M_2(x) \cap R_1(v)$ and there is no edge between $M_1(x) \cap R_1(v) \setminus S$ and $M_2(x) \cap R_1(v)$. In this case, $x^* = \{x\} \cup (M_1(x) \cap R_1(v) \setminus S)$ is of type (K) and $s^* = S$. Also, $\beta(G) = n - 3$ implies that z^* and w^* are of type (1). Therefore G^* has the structure of G_{10}. □

The proof of necessity is completed. □

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