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LARGE SEPARATED SETS OF UNIT VECTORS IN BANACH SPACES OF CONTINUOUS FUNCTIONS

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ABSTRACT. The paper is concerned with the problem whether a nonseparable $\mathcal{C}(K)$ space must contain a set of unit vectors whose cardinality equals to the density of $\mathcal{C}(K)$ such that the distances between every two distinct vectors are always greater than one. We prove that this is the case if the density is at most continuum and we prove that for several classes of $\mathcal{C}(K)$ spaces (of arbitrary density) it is even possible to find such a set which is 2-equilateral; that is, the distance between every two distinct vectors is exactly 2.

In this paper we deal with distances between unit vectors in Banach spaces. For $r \in \mathbb{R}$ a set A in a Banach space X is said to be *r-separated*, *$(r+)$-separated* and *r-equilateral* if $\|v_1 - v_2\| \geq r$, $\|v_1 - v_2\| > r$ and $\|v_1 - v_2\| = r$ for distinct $v_1, v_2 \in A$, respectively.

A natural question considered in the literature is whether, given a Banach space, there is a big equilateral set or there is a big (1+)-separated set in the unit sphere of the space. Our investigation is motivated mainly by the recent papers [11, 12, 14], where this question has been addressed also for nonseparable Banach spaces of the form $\mathcal{C}(K)$, where K is a compact space.

Let us summarize what is known. By [12], it is undecidable in ZFC whether there always exists an uncountable 2-equilateral set in the unit sphere of a nonseparable $\mathcal{C}(K)$ space. On the other hand, by [14] and [11], there always exists an uncountable (1+)-separated set in the unit sphere of a nonseparable $\mathcal{C}(K)$ space. Moreover, if K is nonmetrizable and not perfectly normal there exists an uncountable 2-equilateral set in the unit sphere [14] and if K is perfectly normal, there exists a (1+)-separated set in the unit sphere of cardinality equal to the density of $\mathcal{C}(K)$. It is mentioned in [11, page 40] that the following is a "tantalising problem" left open by the authors.

Question 1. Let K be a compact Hausdorff space with $\kappa := \operatorname{dens} \mathcal{C}(K) > \omega$. Does there exist a (1+)-separated set in the unit sphere of $\mathcal{C}(K)$ of cardinality κ?

We were not able to answer this question. However, we prove that for quite many classes of compact spaces the answer is positive.

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Let us emphasize that all compact spaces of our considerations are supposed to be infinite and Hausdorff. Recall that if K is a compact Hausdorff space then the weight of K (denoted by $w(K)$) is equal to the density of $\mathcal{C}(K)$. Our main results read as follows.

Theorem 1. *Let K be a compact space such that $w(K)$ is at most continuum. Then the unit ball of $\mathcal{C}(K)$ contains a $(1+)$-separated set of cardinality $w(K)$.*

Theorem 2. *Let K be a compact space such that at least one of the following conditions is satisfied.*

(1) *There exists a set $A \subset K$ with $\operatorname{dens} A \geq w(K)$.*
(1') *K is Valdivia or the weight of K is a strongly limit cardinal.*
(2) *K contains two disjoint homeomorphic compact spaces of the same weight as K.*
(2') *K is homogeneous, or homeomorphic to $L \times L$, or to a compact convex set in a locally convex space.*
(3) *K is dyadic (that is, a continuous image of 2^κ for some κ).*
(4) *K is a compact line (that is, a linearly ordered space with the order topology).*

Then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$.

Theorem 1 follows from a slightly more general Theorem 6, Theorem 2 summarizes the most important results from Section 3. Let us note that (1') is a consequence of (1) and (2') is a consequence of (2).

Note that separable compact spaces and first countable compact spaces are of weight at most continuum and so Theorem 1 applies. Indeed, it is a classical result, see [3] or [9, Theorem 3.3], that for a regular topological space X we have $w(X) \leq 2^{\operatorname{dens} X}$; hence, separable compact spaces have weight at most continuum. If K is a first countable compact space, then by the famous Arhangel'skii's inequality, see [1] or [9, Theorem 7.1 and 7.3], it has cardinality at most $\mathfrak{c}$. Hence, the weight is at most $\mathfrak{c}$ as well.

Let us remark that our results generalize all of the results from [11, 14] mentioned above. Indeed, if K is perfectly normal, it is first countable and we may apply Theorem 1 and if K is not perfectly normal it contains a subset of uncountable density and we may apply Theorem 7 (which is the statement from which we deduce the case (1) in Theorem 2).

Our results naturally suggest certain problems/conjectures which we summarize in the last section of this paper.

1. Preliminaries

The notation and terminology is standard, for the undefined notions see [5] for Banach spaces, [4] for topology and [13] for set theory. By $\mathfrak{c}$ we denote the cardinality of continuum. If X is a set and κ a cardinal, we denote by $[X]^\kappa$ the set of all subsets of X of cardinality κ. For a cardinal κ we denote by $\operatorname{cf}(\kappa)$ its cofinality. If X is a topological space and $A \subset X$, $\overline{A}$ stands for the closure of A, $\operatorname{dens} X$ stands for the density of X. The closed unit ball of a Banach space X is denoted by B_X. All compact spaces of our considerations are supposed to be infinite and Hausdorff. On the Banach space $\mathcal{C}(K)$ of all continuous functions on a compact space K we consider the supremum norm. In our proofs we use without mentioning the well-known fact that

$$w(K) = \operatorname{dens} \mathcal{C}(K) = \omega + \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{C}(K) \text{ separates the points of } K\}.$$

Let us mention some easy facts which we will use later. First, it is easy to see that if a compact space K is metrizable, then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$, for a proof one may for example use Theorem 7. The other easy facts are formulated as lemmas below.

Lemma 3. *Let K be a compact space and let L be a closed subset or a continuous image of K. If the unit ball of $\mathcal{C}(L)$ contains a $(1+)$-separated (resp. 2-equilateral) set of cardinality κ, then the unit ball of $\mathcal{C}(K)$ contains a $(1+)$-separated (resp. 2-equilateral) set of cardinality κ.*

Proof. If $L \subset K$ then we conclude using Tietze's extension theorem. If $\varphi : K \to L$ is continuous and surjective we realize that $\mathcal{C}(L)$ is isometric to a subspace of $\mathcal{C}(K)$ by the mapping $f \mapsto f \circ \varphi$. □

Lemma 4. *If a compact space K contains a zero-dimensional compact subspace of weight $w(K)$, then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$.*

Proof. Let L be a zero-dimensional compact subspace of K of weight $w(K)$ and let $(U_\alpha)_{\alpha<\kappa}$ be a basis of L consisting of clopen sets. Note that $\kappa \geq w(K)$. Then for every $\alpha < w(K)$ the norm-one function given by

$$f_\alpha(x) := \begin{cases} 1, & x \in U_\alpha, \\ -1, & x \in L \setminus U_\alpha, \end{cases}$$

is continuous. Clearly $\{f_\alpha : \alpha < w(K)\}$ is a 2-equilateral set. By Lemma 3 we obtain a 2-equilateral set in $\mathcal{C}(K)$ of cardinality $w(K)$. □

Lemma 5 ([14, Theorem 1])**.** *Let K be a compact space and κ be an infinite cardinal. Then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality κ if and only if it contains a $(1+\varepsilon)$-separated set of cardinality κ for some $\varepsilon > 0$.*

2. Sets separated by more than 1

Theorem 6. *Let K be a compact space. Then the unit ball of $\mathcal{C}(K)$ contains a $(1+)$-separated set of cardinality $w(K)$ or it contains a 2-equilateral set of cardinality $\mathfrak{c}$.*

Proof. We may without loss of generality assume that K is nonmetrizable, see e.g. Theorem 7. Given $f \in \mathcal{C}(K)$ and $x \in K$, we say that $t \in \mathbb{R}$ is a *local maximum of f at x* if $f(x) = t$ and there exists an open neighbourhood U of x such that $f(y) \leq t$ for every $y \in U$. Let us consider the following condition inspired by the proof of [11, Theorem 4.11]:

$$\begin{aligned} &\forall x, y \in K, x \neq y, \quad \exists f \in B_{\mathcal{C}(K)} : f(x) = 1, \\ &f(z) = -1 \text{ for every } z \text{ in some neighborhood of } y \\ &\text{and } 0 \text{ is not a local maximum of } f \text{ at any point.} \end{aligned} \tag{P1}$$

First, let us assume that the condition (P1) holds. Take a maximal (1+)-separated family $\mathcal{F}$ (with respect to inclusion) of norm-one functions such that 0 is not a local maximum of any $f \in \mathcal{F}$ at any point. We *claim* that the cardinality of $\mathcal{F}$ equals $w(K)$.

In order to get a contradiction, let us assume that $\mathcal{F}$ does not separate the points of K. Thus, for some pair of distinct points $x, y \in K$ and every $g \in \mathcal{F}$ we have $g(x) = g(y)$. Since (P1) holds, we may find a norm-one function $f \in \mathcal{C}(K)$ such that $f(x) = 1$, $f(z) = -1$ in some neighborhood U of y and 0 is not a local maximum of f at any point. Fix any $g \in \mathcal{F}$. If $g(x) = g(y) \neq 0$, then

$$\|f - g\| \geq \max\{|1 - g(x)|, |-1 - g(y)|\} > 1.$$

If $g(x) = g(y) = 0$, since 0 is not a local maximum of g at y, there is $y' \in U$ with $g(y') > 0$ and we have

$$\|f - g\| \geq |f(y') - g(y')| = |-1 - g(y')| > 1.$$

Therefore, we have $\|f - g\| > 1$ for any $g \in \mathcal{F}$ which is a contradiction with the maximality of $\mathcal{F}$.

On the other hand, let us assume that (P1) does not hold. Then there is a pair of distinct points $x, y \in K$ which witnesses the negation of (P1). Pick a function $f \in B_{\mathcal{C}(K)}$ such that $f(x) = 1$ and $f(z) = -1$ for every z in some neighborhood of y. By the choice of the pair x, y, we know that every $t \in (-1, 1)$ is a local maximum of f at some point $x_t \in K$. Indeed, if $t \in (-1, 1)$ is not a local maximum of f at any point, then we can easily modify the function f in such a way that 0 is not a local maximum at any point and this would contradict the choice of the pair x, y.

Hence, for every $t \in (-1, 1)$, there exists a neighborhood U_t of x_t with $f(z) \leq t$ for every $z \in U_t$. We have $x_s \notin U_t$ for $s > t$ and thus $x_t \notin \overline{\{x_s : s > t\}}$. Therefore, for every $t \in (-1, 1)$, we may pick a function $f_t \in B_{\mathcal{C}(K)}$ with $f_t(x_t) = 1$ and $f_t(x_s) = -1$ for every $s > t$. Then $\{f_t : t \in (-1, 1)\}$ is a 2-equilateral set of cardinality $\mathfrak{c}$. $\square$

3. Equilateral sets

Let us start with the following simple observation which already leads to interesting consequences. Recall that for a point x in a topological space X the *character* $\chi(x, X)$ is the minimal cardinality of a local basis at x.

Theorem 7. *Let K be a compact space and κ be a cardinal. Suppose at least one of the following conditions is satisfied.*

(i) There exists a set $A \subset K$ with $\operatorname{dens} A \geq \kappa$.
(ii) There exists a point $x \in K$ with $\chi(x, K) \geq \kappa$.

Then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality κ.

Proof. If there is $A \subset K$ with $\operatorname{dens} A \geq \kappa$, we inductively find points $\{x_\alpha : \alpha < \kappa\} \subset A$ such that $x_\alpha \notin \overline{\{x_\beta : \beta < \alpha\}}$ and for each $\alpha < \kappa$ we pick a norm-one function f_α such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) = -1$ for $\beta < \alpha$. Then $\{f_\alpha : \alpha < \kappa\}$ is a 2-equilateral set.

Let us assume that (ii) holds. Recall that the character of a point in a compact space equals to the pseudocharacter, see e.g. [2, page 127]. In other words, $\{x\}$ is not the intersection of less than κ open sets. We shall inductively find points x_α and open sets U_α for $\alpha < \kappa$ such that $x \in U_\alpha$ and

$$x_\alpha \in \bigcap_{\beta<\alpha} \overline{U_\beta} \setminus \overline{U_\alpha}. \tag{1}$$

Pick $x_0 \neq x$ and an open $U_0 \ni x$ with $x_0 \notin \overline{U_0}$. Having chosen x_β and U_β for every $\beta < \alpha$, we pick a point $x_\alpha \in \bigcap_{\beta<\alpha} U_\beta \setminus \{x\}$ and then we find $U_\alpha \ni x$ such that $x_\alpha \notin \overline{U_\alpha}$. In this way we have picked all the x_α's and, by (1), we have $x_\beta \notin \overline{\{x_\alpha :\ \alpha > \beta\}}$ for every $\beta < \kappa$. Hence, for each $\beta < \kappa$ we may pick a norm-one function f_β such that $f_\beta(x_\beta) = 1$ and $f_\beta(x_\alpha) = -1$ for $\alpha > \beta$. Then $\{f_\beta :\ \beta < \kappa\}$ is a 2-equilateral set. $\square$

We may apply Theorem 7 to several classes of compact spaces which include also classes studied in functional analysis. For a survey about Valdivia and Corson compacta we refer to [10], for information about Eberlein compacta to [6]. Let us recall that a cardinal κ is *strongly limit* if $2^\lambda < \kappa$ whenever $\lambda < \kappa$.

Corollary 8. *Let K be a compact space. Suppose at least one of the following conditions holds.*

- *K is Valdivia (e.g. K is metrizable, Eberlein or Corson);*
- *$w(K)$ is a strongly limit cardinal;*
- *K is a connected continuous image of a linearly ordered compact space.*

Then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$.

Proof. If K is Valdivia, then we may apply Theorem 7 since every dense Σ-subset $A \subset K$ satisfies $\operatorname{dens} A = w(K)$, see [10, Lemma 3.4].

If $w(K)$ is a strongly limit cardinal, then $\operatorname{dens} K = w(K)$. Indeed, it is a classical result, see [3] or [9, Theorem 3.3], that for a regular topological space X we have $w(X) \leq 2^{\operatorname{dens} X}$; hence, if $\operatorname{dens} K < w(K)$ we would get $w(K) < w(K)$, a contradiction.

If K is a connected continuous image of a linearly ordered compact space then, by [16], we have $\operatorname{dens} K = w(K)$ and we may apply Theorem 7. $\square$

Another class of compact spaces where an analogous statement holds is given by the following result.

Theorem 9. *Let K be a compact space. Then the unit ball of $\mathcal{C}(K \times \{0,1\})$ contains a 2-equilateral set of cardinality $w(K)$.*

Proof. By Theorem 7, we may assume that K is nonmetrizable. By Lemma 5, it is sufficient to find a $\frac{3}{2}$-separated set of cardinality $w(K)$.

For $f \in \mathcal{C}(K \times 2)$ consider the following condition:

$$\forall z \in K :\ |f(z,0)| < \tfrac{1}{2} \implies f(z,1) = -1. \tag{P2}$$

Take a maximal $\frac{3}{2}$-separated family $\mathcal{F}$ (with respect to inclusion) of norm-one functions satisfying the condition (P2). We *claim* that the cardinality of $\mathcal{F}$ equals $w(K)$. In order to get a contradiction, let us assume that $\mathcal{F}$ does not separate the points of $K \times \{0\}$. Thus, for some pair of distinct points $x, y \in K$ and every $g \in \mathcal{F}$ we have $g(x,0) = g(y,0)$. Now, consider any norm-one function $f \in \mathcal{C}(K \times 2)$ satisfying the condition (P2) such that $f(y,0) = -1$ and $f(x,0) = f(x,1) = 1$. Such a function exists because we may pick any $\tilde{f} \in B_{\mathcal{C}(K)}$ with $\tilde{f}(x) = 1 = -\tilde{f}(y)$ and take any continuous extension of a function defined on disjoint closed sets $K \times \{0\}$, $\{(x,1)\}$ and $\tilde{f}^{-1}\left([-\frac{1}{2}, \frac{1}{2}]\right) \times \{1\}$ in the obvious way, that is, $f(z,0) = \tilde{f}(z)$ for every $z \in K$, $f(x,1) = 1$ and $f(z,1) = -1$ for $z \in \tilde{f}^{-1}\left([-\frac{1}{2}, \frac{1}{2}]\right)$.

Fix any $g \in \mathcal{F}$. If $g(x,0) = g(y,0) \geq \frac{1}{2}$, then

$$\|f - g\| \geq |-1 - g(y,0)| = 1 + g(y,0) \geq \tfrac{3}{2}.$$

If $g(x,0) = g(y,0) \leq -\frac{1}{2}$, then

$$\|f - g\| \geq |1 - g(x,0)| = 1 - g(x,0) \geq \tfrac{3}{2}.$$

If $|g(x,0)| < \frac{1}{2}$, then since g satisfies (P2) we have

$$\|f - g\| \geq |f(x,1) - g(x,1)| = 1 - g(x,1) = 2.$$

Therefore, we have $\|f - g\| \geq \frac{3}{2}$ for any $g \in \mathcal{F}$ which is a contradiction with the maximality of $\mathcal{F}$. □

Corollary 10. *Let K be a compact space which contains two disjoint homeomorphic compact spaces of weight $w(K)$. Then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$.*

Proof. The statement follows immediately from Theorem 9 and Lemma 3. □

This result has interesting consequences. The first one is a strengthening of [14, Corollary 2.11 and 2.12]. To prove it, we need the following simple lemma.

Lemma 11. *Let K be a compact convex subset of a locally convex space E. Then K contains two disjoint subsets homeomorphic to itself.*

Proof. Let $x, y \in K$ be two distinct points and let $x^* \in E^*$ be such that $x^*(y - x) > 0$. It is sufficient to show that

$$(1-\lambda)x + \lambda K \quad \text{and} \quad (1-\lambda)y + \lambda K$$

are disjoint for a small enough $\lambda \in (0,1]$. Assuming the opposite for some λ, we obtain that there are $u, v \in K$ such that

$$(1-\lambda)x + \lambda u = (1-\lambda)y + \lambda v,$$

which implies

$$\sup x^*(K) - \inf x^*(K) \geq x^*(u) - x^*(v) = \frac{1-\lambda}{\lambda} x^*(y - x).$$

Therefore, any $\lambda \in (0,1]$ satisfying $\frac{1-\lambda}{\lambda} x^*(y-x) > \sup x^*(K) - \inf x^*(K)$ works. □

Corollary 12. *Let K be a compact space which is homeomorphic to either $L \times L$ for a compact space L or to a compact convex set in a locally convex space. Then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$.*

Proof. If K is homeomorphic to $L \times L$, we use Corollary 10 and the fact that $L \times L$ contains $L \times \{x\}$ and $L \times \{y\}$ for a pair of distinct points $x, y \in L$. If it is homeomorphic to a compact convex set in a locally convex space, we use Corollary 10 and Lemma 11. □

Before proving another corollary of Theorem 9, let us formulate the following easy observation.

Lemma 13. *Let K be a compact space. Then there exists a point $x \in K$ such that $w(U) = w(K)$ for every neighborhood U of x.*

Proof. In order to get a contradiction, let us assume that for every point x there exists an open neighborhood U_x of x with $w(U_x) < w(K)$. By compactness, there are points $x_1, \ldots, x_n \in K$ such that $U_{x_1} \cup \ldots \cup U_{x_n} = K$; hence, $w(K) = w(U_{x_1} \cup \ldots \cup U_{x_n}) \leq w(U_{x_1}) + \ldots + w(U_{x_n}) < w(K)$, a contradiction. □

Corollary 14. *Let K be a homogeneous compact space. Then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$.*

Proof. By Lemma 13, there exists a point $x \in K$ such that $w(U) = w(K)$ for every neighborhood of x. Pick $y \in K \setminus \{x\}$ and a homeomorphism $h : K \to K$ with $h(x) = y$.

Find open neighborhoods U and V of x and y respectively such that $\overline{U} \cap \overline{V} = \emptyset$ and $h(U) = V$. This is indeed possible since we may pick arbitrary neighborhoods U_0 and V_0 of x and y respectively such that $\overline{U_0} \cap \overline{V_0} = \emptyset$ and put $U := h^{-1}(V_0) \cap U_0$, $V := h(U)$.

Now, $\overline{U}$ and $\overline{V}$ are homeomorphic, disjoint and $w(\overline{U}) = w(K)$; hence, we may apply Corollary 10. □

The next corollary of Theorem 9 is based on a variant of the Ramsey theorem for higher cardinalities.

Definition 15. Let κ and λ be cardinals. By writing

$$\kappa \to (\lambda)^2_2$$

we mean that the following statement is true: for every set X of cardinality κ and for every $F : [X]^2 \to \{0, 1\}$ there exists a subset Y of X of cardinality λ such that $F|_{[Y]^2}$ is constant.

Corollary 16. *Let K be a compact space and let κ be the weight of K. If λ is a cardinal with $\kappa \to (\lambda)^2_2$, then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality λ.*

Proof. By Theorem 9, in the unit ball of $\mathcal{C}(K \times \{0, 1\})$ there exists a 2-equilateral set X of cardinality κ. Consider the mapping $F : [X]^2 \to \{0, 1\}$ such that $F(\{f, g\}) = 0$ if and only if there exists a point $x \in K \times \{0\}$ with $|(f - g)(x)| = 2$. If there is a set $Y \subset X$ of cardinality λ such that $F|_{[Y]^2} \equiv 0$ then $\{f|_{K \times \{0\}} : f \in Y\}$ is a 2-equilateral set in the unit ball of $\mathcal{C}(K \times \{0\})$. Otherwise, there is a set $Y \subset X$ of cardinality λ such that $F|_{[Y]^2} \equiv 1$ and $\{f|_{K \times \{1\}} : f \in Y\}$ is a 2-equilateral set. □

As a corollary, we may obtain the following result.

Corollary 17. *Let K be a compact space with $w(K) \geq (2^{<\kappa})^+$ for some cardinal κ. Then there exists a 2-equilateral set of cardinality κ in the unit ball of $\mathcal{C}(K)$.*

Proof. It follows from Corollary 16 and the classical Erdős-Rado theorem (see e.g. [8, Theorem 2.9]), which states that $(2^{<\kappa})^+ \to (\kappa)^2_2$ whenever κ is an infinite cardinal. $\square$

Recall that a compact space K is said to be *dyadic* if it is a continuous image of 2^κ for some κ. For more information about dyadic compacta we refer to [15, Section 7].

Lemma 18. *Let K be a dyadic compact space and $F, H \subset K$ disjoint closed subsets. Then there exists dyadic compact $C \subset K$ such that $F \subset C$ and $H \cap C = \emptyset$.*

Proof. Since K is dyadic, there is a continuous map f of 2^κ onto K. Clearly the sets $f^{-1}(F)$ and $f^{-1}(H)$ are disjoint and closed, hence we can find for every point $x \in f^{-1}(F)$ its basic clopen neighborhood U_x which is disjoint from $f^{-1}(H)$. By compactness of $f^{-1}(F)$ there is a finite set S such that $\{U_s : s \in S\}$ covers $f^{-1}(F)$. Every U_s is a dyadic space, since it is a basic subset of 2^κ. It is easy to see that $U := \bigcup\{U_s : s \in S\}$ is dyadic too and hence the set $C = f(U)$ is dyadic as well. It remains to note that $F \subset C$ and $H \cap C = \emptyset$. $\square$

Theorem 19. *Let K be a dyadic compact space. Then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$.*

Proof. Put $\kappa = w(K)$. First, assume that $\mathrm{cf}(\kappa) > \omega$. Then K can be mapped continuously onto $[0,1]^\kappa$, see [7] or more generally also [15, Theorem 7.21]. Hence, by Lemmas 3 and 4, it is enough to realize that $[0,1]^\kappa$ contains the zero-dimensional subspace 2^κ.

Hence, we may assume that $\mathrm{cf}(\kappa) = \omega$ and thus that there is a sequence of uncountable regular cardinals (μ_n) whose limit is κ. By Lemma 13, there exists a point $x \in K$ such that $w(U) = \kappa$ for every neighborhood U of x. By Theorem 7, we may assume that $\lambda := \chi(x, K) < \kappa$.

We claim that for every $\mu < \kappa$ and every neighborhood U of x there is a compact set $L_{\mu,U} \subset U \setminus \{x\}$ of weight at least μ. Indeed, in the opposite case there is some neighborhood U of x and some $\mu < \kappa$ such that every compact set in $U \setminus \{x\}$ is of weight less than μ. Let W be a neighborhood of x such that $\overline{W} \subseteq U$ and let $\mathcal{V}$ be a local base at x of cardinality λ formed by open subsets of K. Then

$$\overline{W} \setminus \{x\} = \bigcup\{\overline{W} \setminus \overline{V} : V \in \mathcal{V}\}.$$

Since $\lambda < \kappa$ it follows that $w(\overline{W} \setminus \{x\}) = \kappa$. On the other hand $\overline{W} \setminus \overline{V}$ is of weight at most μ and hence the weight of $\bigcup\{\overline{W} \setminus \overline{V} : V \in \mathcal{V}\}$ is at most $\lambda \cdot \mu < \kappa$. This is a contradiction.

Using the claim, we can inductively construct a sequence (L_n) of compact subspaces of $K \setminus \{x\}$ such that the weight of L_n is at least μ_n and for each L_n there is an open set $U_n \supset L_n$ which is disjoint from every L_m, $m \neq n$. By the use of Lemma 18 there are dyadic compact spaces $K_n \subset U_n$ such that $L_n \subset K_n$ for every $n \in \omega$. By the first part of the proof we are able to find for every n a 2-equilateral set $\mathcal{F}_n$ in the unit ball of $\mathcal{C}(K_n)$ of cardinality $w(K_n) \geq w(L_n) \geq \mu_n$. We can extend every function $f \in \mathcal{C}(K_n)$ to a function $f' \in \mathcal{C}(K)$ with the same norm and satisfying $f'(y) = 1$ for $y \in K \setminus U_n$. We may without loss of generality assume that, for each $n \in \omega$ and $f \in \mathcal{F}_n$, there is a point $x \in K_n$ with $f(x) = -1$ (because, for each $n \in \omega$ there is at most one function $f \in \mathcal{F}_n$ for which it does not hold). Let $\mathcal{F} := \{f' : f \in \mathcal{F}_n, n \in \omega\}$. It is easily checked that $\mathcal{F}$ is a 2-equilateral set in the unit ball of $\mathcal{C}(K)$ of cardinality κ. □

Theorem 20. *Let K be a linearly ordered compact space. Then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$.*

Proof. Put $\kappa := w(K)$. By Theorem 7 we may suppose that $\lambda := \operatorname{dens}(K) < \kappa$. Let $D \subset K$ be a dense set of cardinality λ. The cardinality of the system of open intervals $\{(a,b) : a, b \in D, a < b\}$ is λ, hence it is not a base for K. Put

$$L := \{x \in K : \exists a < x : (a,x) = \emptyset\}, \quad R := \{x \in K : \exists b > x : (x,b) = \emptyset\}.$$

We *claim* that either L or R is of cardinality $w(K)$. Indeed, assume the opposite case. For every $x \in L \setminus R$, there is $a_x < x$ with $(a_x, x) = \emptyset$ and $\mathcal{B}_x := \{(a_x, b) : b \in D, b > x\}$ is a neighborhood basis of x. Similarly, for every $x \in R \setminus L$ we find $b_x > x$ such that $\mathcal{B}_x := \{(a, b_x) : a \in D, a < x\}$ is a neighborhood basis of x. Note that the points of $L \cap R$ are isolated. Then

$$\mathcal{B} = \big\{(a,b) : a, b \in D, a < b\big\} \cup \bigcup \big\{\mathcal{B}_x : x \in L \triangle R\big\} \cup \big\{\{x\} : x \in L \cap R\big\}$$

is of cardinality less then κ. Moreover, it is easy to see that $\mathcal{B}$ is a basis of K; hence, $w(K) < \kappa$, a contradiction.

Now, assume that the cardinality of L is κ. For every $x \in L$ consider a continuous function f_x defined as

$$f_x(y) = \begin{cases} 1, & y \geq x, \\ -1, & y < x. \end{cases}$$

Then $\{f_x : x \in L\}$ is a 2-equilateral set of cardinality $w(K)$. The case when R is of cardinality κ is similar. □

It is worth mentioning that the density of a linearly ordered compact space can be less than the weight. This is witnessed e.g. by the Alexandrov double arrow space.

4. Remarks and questions

Up to our knowledge it is not known whether in the unit ball of a nonseparable Banach space X there exists a 1-separated set of cardinality equal to the density of X. However, if we consider only Banach spaces of the form $\mathcal{C}(K)$, this is easy.

Proposition 21. *Let K be a compact space. Then the unit ball of $\mathcal{C}(K)$ contains a 1-separated set of cardinality $w(K)$.*

Proof. By Theorem 7, we may assume that K is nonmetrizable. Take a maximal 1-separated family $\mathcal{F}$ (with respect to inclusion) of norm-one functions.

We *claim* that the cardinality of $\mathcal{F}$ equals $w(K)$. In order to get a contradiction, let us assume that $\mathcal{F}$ does not separate the points of K. Thus, for some pair of distinct points $x, y \in K$ and every $g \in \mathcal{F}$ we have $g(x) = g(y)$. Find a norm-one function $f \in \mathcal{C}(K)$ such that $f(x) = 1 = -f(y)$, then we have $\|f - g\| \geq 1$ for every $g \in \mathcal{F}$; hence $\mathcal{F} \cup \{f\}$ is 1-separated, which is a contradiction with the maximality of $\mathcal{F}$. □

We get easily from our results that the situation is quite simple under GCH.

Corollary 22 (GCH)**.** *Let K be a compact space.*

(1) If $w(K)$ is a limit cardinal, then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality $w(K)$.
(2) If $w(K) = \kappa^+$ for an infinite cardinal κ, then the unit ball of $\mathcal{C}(K)$ contains a 2-equilateral set of cardinality κ.

Proof. The first statement follows immediately from Corollary 8 because under GCH every limit cardinal is strongly limit.

Concerning the second statement, it follows from Corollary 17. Indeed, it is sufficient to notice that under GCH we have $2^{<\kappa} = \kappa$, which follows from the computation

$$2^{<\kappa} = \sup\{2^\lambda :\ \lambda < \kappa\} = \sup\{\lambda^+ :\ \lambda < \kappa\} = \kappa,$$

where the first equality follows e.g. from [13, Lemma I.13.17] and the second from GCH. □

Question 2. Does Corollary 22 hold in ZFC?

Moreover, we do not know if it is possible to have an analogue of Koszmider's example [12] for higher densities.

Question 3. Let $\kappa \geq \omega_1$ be a cardinal. Does there (at least consistently) exist a compact space of weight κ^+ such that the unit sphere of $\mathcal{C}(K)$ does not contain a 2-equilateral set of cardinality κ^+?

Since dyadic compacta are ccc, the positive answer to the following question would generalize Theorem 19.

Question 4. Let K be a compact space which is ccc. Does the unit sphere of $\mathcal{C}(K)$ contain a 2-equilateral (or at least (1+)-separated) set of cardinality $w(K)$?

Remark 23. P. Koszmider proved [12] that consistently there exists a nonmetrizable compact space K without an uncountable 2-equilateral set in the unit ball of $\mathcal{C}(K)$. Knowing in a detail his construction, it is quite easy to see that for Koszmider's example we have $\operatorname{ind}(K) \leq 2$, where $\operatorname{ind}(K)$ is the topological dimension (for a definition see e.g [4, Chapter 7]). Since in zero-dimensional nonmetrizable compact spaces there always exists an uncountable 2-equilateral set in the unit ball of $\mathcal{C}(K)$, it is of a certain interest to know what is the situation for compact spaces with dimension 1. Modifying Koszmider's example it is possible to obtain the following statement:

> It is relatively consistent with ZFC that there exists a nonmetrizable compact space K with $\operatorname{ind}(K) = 1$ such that there

does not exist an uncountable 2-equilateral set in the unit ball of $\mathcal{C}(K)$.

Let us include some details of the above mentioned modification. In order to shorten the notation, for a finite subset N of ω and $s \in 2^N$, put $\mathcal{N}_s := \{x \in 2^\omega : x|_N = s\}$. First, following the proof of [12, Theorem 3.3] (replacing the interval $[0,1]$ by the compact space 2^ω), one observes that it is sufficient to prove that consistently there are points $\{r_\xi : \xi < \omega_1\} \subset 2^\omega$ and a sequence of functions $(f_\xi : \xi < \omega_1)$, where $f_\xi : 2^\omega \setminus \{r_\xi\} \to [-1,1]$ are continuous, such that given

(a) $m \in \mathbb{N}$,
(b) a finite subset N of ω and pairwise different sequences $s_1, \ldots, s_m \in 2^N$,
(c) any sequence $(F_\alpha)_{\alpha<\omega_1}$ where $F_\alpha = \{\xi_1^\alpha, \ldots, \xi_m^\alpha\}$ are pairwise disjoint finite subsets of ω_1 such that $r_{\xi_i^\alpha} \in \mathcal{N}_{s_i}$ for every $1 \leq i \leq m$ and every $\alpha < \omega_1$,
(d) any m-tuple $\{q_1, \ldots, q_m\}$ of rational numbers from $[-1,1]$,

there are $\alpha < \beta < \omega_1$, a finite subset $M \supset N$ of ω and sequences $(t_i^\alpha)_{1\leq i\leq m}$, and $(t_i^\beta)_{1\leq i\leq m}$ from 2^M such that for each $1 \leq i \leq m$ we have:

(1) $\mathcal{N}_{t_i^\alpha} \cup \mathcal{N}_{t_i^\beta} \subset \mathcal{N}_{s_i}$ and $t_i^\alpha \neq t_i^\beta$,
(2) $r_{\xi_i^\alpha} \in \mathcal{N}_{t_i^\alpha}$ and $r_{\xi_i^\beta} \in \mathcal{N}_{t_i^\beta}$,
(3) $f_{\xi_i^\alpha} \restriction_{2^\omega \setminus (\mathcal{N}_{t_i^\alpha} \cup \mathcal{N}_{t_i^\beta})} = f_{\xi_i^\beta} \restriction_{2^\omega \setminus (\mathcal{N}_{t_i^\alpha} \cup \mathcal{N}_{t_i^\beta})}$,
(4) $f_{\xi_i^\alpha} \restriction_{\mathcal{N}_{t_i^\beta}} = q_i = f_{\xi_i^\beta} \restriction_{\mathcal{N}_{t_i^\alpha}}$.

Now, similarly as in [12, Section 4], by a forcing argument, we prove that consistently such points $\{r_\xi : \xi < \omega_1\} \subset 2^\omega$ and a sequence of functions $(f_\xi : \xi < \omega_1)$ exist. Fix any points $\{r_\xi : \xi < \omega_1\} \subset 2^\omega$. The forcing notion $\mathbb{P}$ consists of triples $(N_p, F_p, \mathcal{F}_p)$ such that

(1) $N_p \in [\omega]^{<\omega}$,
(2) F_p is a finite subset of ω_1 such that $\{r_\xi \restriction_{N_p} : \xi \in F_p\}$ are pairwise different sequences,
(3) $\mathcal{F}_P = \{f_p^\xi : \xi \in F_p\}$,
(4) $f_p^\xi : 2^\omega \setminus \mathcal{N}_{r_\xi \restriction N_p} \to [-1,1]$ is a *rationally piecewise constant function* for each $\xi \in F_p$ (i.e. for every $s \in 2^{N_p}$ with $s \neq r_\xi \restriction_{N_p}$ there is a rational number q_s such that $f_p^\xi(x) = q_s$ for every $x \in \mathcal{N}_s$).

We say that $q \leq p$ if and only if

(a) $N_q \supset N_p$,
(b) $F_q \supset F_p$,
(c) $f_q^\xi \supset f_p^\xi$ for every $\xi \in F_p$.

Similarly as in [12, Lemma 4.3] we prove that $\mathbb{P}$ is ccc; hence, it preserves cofinalities and cardinals [13, Theorem IV.7.9]. Finally, similarly as in [12, Proposition 4.4], we prove that $\mathbb{P}$ forces that there are functions $(f_\xi : \xi < \omega_1)$ with the properties indicated above.

It remains to see that our modification of Koszmider's example is 1-dimensional. As we have mentioned above, the modification is in replacing the interval $[0,1]$ by the compact space 2^ω in the construction from [12];

more precisely it is a resolution given by functions (f_ξ) in the sense of [17]. In order to describe some more details, let us recall the concept of a resolution (we use the concept of a resolution from [17]; constructed compact spaces are easily seen to be homeomorphic to the ones considered in [12]).

Let L be a compact space, $B \subset L$ and for every $b \in B$ let us have a continuous function $f_b : L \setminus \{b\} \to [-1,1]$. By a *resolution given by functions* $(f_b)_{b\in B}$ we understand the space $K = R(L,(f_b)_{b\in B}) = (B\times[-1,1]) \cup (L\setminus B)$ with the topology given by the following neighborhood basis. If $x \in L \setminus B$, then its neighborhood basis is the collection of all sets

$$\mathcal{U}(x,U) := \big((U \cap B) \times [-1,1]\big) \ \cup \ (U \setminus B),$$

where U is an open neighborhood of x in the space L. If $x \in B$ and $y \in [-1,1]$, then the neighborhood basis at (x,y) is the collection of all sets

$$\mathcal{U}(x,U,V) := (\{x\} \times V) \cup \big((U \cap f_x^{-1}(V) \cap B) \times [-1,1]\big) \cup (U \cap f_x^{-1}(V) \setminus B),$$

where U is an open neighborhood of x in the space L and V is an open neighborhood of y in the space $[-1,1]$.

Finally, we prove the following proposition which yields that the above described modification gives a 1-dimensional compact space.

Proposition 24. *Let L be a zero-dimensional compact space with countable character, $B \subset L$ and let $f_b : L \setminus \{b\} \to [-1,1]$ be a continuous function for every $b \in B$. Then the resolution $K = R(L,(f_b)_{b\in B})$ is a compact space with* $\operatorname{ind}(K) \le 1$.

Proof. It is well-known that K is a compact space [17, Theorem 3.1.33]. We will find a neighborhood basis at every point in K such that the boundary of each of its members is finite (in particular zero-dimensional). Let $x \in L \setminus B$ first. Then the set $\mathcal{U}(x,U)$ is a clopen neighborhood of x in K for a clopen neighborhood U of x in L. Moreover, sets of this type form a local neighborhood basis at x in K.

On the other hand suppose that $(x,y) \in B \times [-1,1]$ and let $\mathcal{U}(x,U,V)$ be a given neighborhood of (x,y) in K. We want to find a smaller neighborhood of (x,y) in K whose boundary is finite. We may assume that U is a clopen set. Let $W = [a,b]$ be a neighborhood of y in $[-1,1]$ such that $W \subset V$ and let W_n be open subsets of $[-1,1]$ such that $W_{n+1} \subset W_n$ for every $n \in \omega$, $W = \bigcap_{n\in\omega} \overline{W_n}$ and $W_0 = V$. We claim that there exists a clopen set C in $L \setminus \{x\}$ such that $f_x^{-1}(W) \subset C \subset f_x^{-1}(V)$ and $x \notin \overline{f_x^{-1}([-1,1] \setminus W_n) \cap C}$ for every $n \in \omega$. Indeed, let $\{B_n : n \in \omega\}$ be a local neighborhood basis at x formed by clopen sets in L with $B_{n+1} \subset B_n$ for each $n \in \omega$ and $B_0 = L$. One can easily find a clopen set $C_n \subset L \setminus \{x\}$ such that $f_x^{-1}(W) \subset C_n \subset f_x^{-1}(W_n)$. Without loss of generality we may suppose that $C_0 \supset C_1 \supset \dots$. Set $C = \bigcup_{n\in\omega}(C_n \cap B_n \setminus B_{n+1})$.

It follows that the set $M = ((C\cap U\cap B)\times[-1,1])\cup(C\cap U\setminus B)\cup(\{x\}\times W)$ is a neighborhood of (x,y) in K. We shall prove that its boundary is a subset of $\{(x,a),(x,b)\}$, hence it is finite.

First, every point of $L\setminus B$ as well as every point $(x',y') \in (B\setminus\{x\}\times[-1,1])$ is either an interior point of M or a point outside of the closure of M. Moreover, for $z \in W \setminus \{a,b\}$ we have that (x,z) is in the interior of M because $(x,z) \in \mathcal{U}(x,U,(a,b)) \subset M$.

Finally, for $z \in [-1,1] \setminus W$ the point (x,z) is outside of the closure of M. Indeed, there is $n \in \omega$ such that $z \notin \overline{W_n}$. Let U' be an open neighborhood of x in L such that $U' \cap \overline{f_x^{-1}([-1,1] \setminus W_n) \cap C} = \emptyset$. Let V' be an open neighborhood of z disjoint from W_n. Then $\mathcal{U}(x, U', V')$ is an open neighborhood of (x,z) disjoint from M. □

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