MONADIC $\boldsymbol{n \times m}$-VALUED ŁUKASIEWICZ-MOISIL ALGEBRAS

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Abstract. Here we initiate an investigation into the class $\boldsymbol{mLM_{n\times m}}$ of monadic $n \times m$-valued Łukasiewicz-Moisil algebras (or $mLM_{n\times m}$-algebras), namely $n \times m$-valued Łukasiewicz-Moisil algebras endowed with a unary operation. These algebras constitute a generalization of monadic n-valued Łukasiewicz-Moisil algebras. In this article, the congruences on these algebras are determined and subdirectly irreducible algebras are characterized. From this last result it is proved that $\boldsymbol{mLM_{n\times m}}$ is a discriminator variety and as a consequence, the principal congruences are characterized. Furthermore, the number of congruences of finite $mLM_{n\times m}$-algebras is computed. In addition, a topological duality for $mLM_{n\times m}$-algebras is described and a characterization of $mLM_{n\times m}$-congruences in terms of special subsets of the associated space is shown. Moreover, the subsets which correspond to principal congruences are determined. Finally, some functional representation theorems for these algebras are given and the relationship between them is pointed out.

Keywords: n-valued Łukasiewicz-Moisil algebra, monadic n-valued Łukasiewicz-Moisil algebra, congruence, subdirectly irreducible algebra, discriminator variety, Priestley space

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1. Introduction and preliminaries

In 1955, monadic Boolean algebras were introduced by P. Halmos in [11] as an algebraic counterpart of the one variable fragment of classical predicate logic. One of his well known results partially related with the present paper is that every monadic Boolean algebra is a subalgebra of a rich one. As a consequence, he proved that each monadic Boolean algebra can be embedded into a complete functional Boolean algebra. This subject caused great interest and led several authors to deepen and generalize the algebras defined by Halmos, to such an extent that research is still being conducted in this direction. For instance, the varieties of monadic MV-algebras, monadic BL-algebras, monadic Wajsberg algebras, monadic Pavelka algebras, monadic Ockham algebras, monadic distributive lattices, monadic n-valued Łukasiewicz-Moisil algebras are being studied, to mention a few. It is worth mentioning that monadic Heyting algebras constitute the first generalization of monadic Boolean algebras; they were introduced by A. Monteiro and O. Varsavsky in [15] who defined them as triples $(L, \exists, \forall)$, where L is a Heyting algebra and $\exists$, $\forall$ are unary operations verifying certain identities.

On the other hand, in 1974, L. Monteiro in his doctoral thesis [16] introduced the notion of existential quantifier on a 3-valued Łukasiewicz algebra and defined monadic 3-valued Łukasiewicz algebras. However, in 1971 Georgescu and Vraciu [10] introduced a more general class of algebras which they called monadic n-valued Łukasiewicz-Moisil algebras.

In 1975, W. Suchoń [26] introduced matrix Łukasiewicz algebras, so generalizing the notion of n-valued Łukasiewicz algebras without negation [14]. The only paper about these algebras is the one mentioned above and a brief reference to them can be found in [2]. In [24] we introduced $n \times m$-valued Łukasiewicz algebras with negation. Later, following the terminology established in [2], they were called $n \times m$-valued Łukasiewicz-Moisil algebras (or $LM_{n\times m}$-algebras for short) and since then, we have named them in this way. These algebras are both a particular case of matrix Łukasiewicz algebras and a generalization of n-valued Łukasiewicz-Moisil algebras [2]. $LM_{n\times m}$-algebras were studied in [21], [24], [25] and [8]. In particular, in [24] we provided an important example which legitimated the study of this new class of algebras. Besides, in [8] we presented a propositional calculus which has $LM_{n\times m}$-algebras as algebraic counterpart.

In the present paper, we introduce and investigate monadic $n \times m$-valued Łukasiewicz-Moisil algebras which constitute a generalization of monadic n-valued Łukasiewicz-Moisil algebras [2], [10].

The paper is organized as follows. In Section 1, we briefly summarize the main definitions and results needed throughout this article. In Section 2, we introduce monadic $n \times m$-valued Łukasiewicz-Moisil algebras (or $mLM_{n\times m}$-algebras), namely $LM_{n\times m}$-algebras endowed with a unary operation called existential quantifier. Besides, we show their most important properties which are necessary for further development. Furthermore, we determine the relationship between the existential quantifier and special subalgebras of $LM_{n\times m}$-algebras. In Section 3, we determine the $mLM_{n\times m}$-congruences and characterize subdirectly irreducible algebras. In Section 4, by applying the results obtained in the previous section, we show that this variety is a discriminator variety and as a consequence, we deduce some properties of the congruences. In Section 5, we show a topological duality for $mLM_{n\times m}$-algebras and characterize the congruences by means of special subsets of the associated space. In particular, we determine which of these subsets correspond to principal congruences. Finally, in Section 6, we describe three functional representation theorems for $mLM_{n\times m}$-algebras, pointing out the relationshhip between them.

We refer the reader to the bibliography listed here as [3], [1], [2], [12], [13], [4], [14], [10], [17], [18], [19] for specific details of the many basic notions and results of universal algebra, distributive lattices, De Morgan algebras, Boolean algebras, monadic Boolean algebras, n-valued Łukasiewicz-Moisil algebras, monadic n-valued Łukasiewicz-Moisil algebras and Priestley spaces considered in this paper.

As we mention above, from now on $n \times m$-valued Łukasiewicz algebras with negation will be called $n \times m$-valued Łukasiewicz-Moisil algebras or $LM_{n\times m}$-algebras for short.

An *$n \times m$-valued Łukasiewicz-Moisil algebra*, in which n and m are integers, $n \geqslant 2$, $m \geqslant 2$, is an algebra $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, 1\rangle$ where $(n \times m)$ is the cartesian product $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$, the reduct $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra and $\{\sigma_{ij}\}_{(i,j)\in(n\times m)}$ is a family of unary operations on L which fulfils the following conditions:

(C1) $\sigma_{ij}(x \vee y) = \sigma_{ij}x \vee \sigma_{ij}y$,

(C2) $\sigma_{ij}x \leqslant \sigma_{(i+1)j}x$,

(C3) $\sigma_{ij}x \leqslant \sigma_{i(j+1)}x$,

(C4) $\sigma_{ij}\sigma_{rs}x = \sigma_{rs}x$,

(C5) $\sigma_{ij}x = \sigma_{ij}y$ for all $(i, j) \in (n \times m)$ implies $x = y$,

(C6) $\sigma_{ij}x \vee \sim\sigma_{ij}x = 1$,

(C7) $\sigma_{ij}(\sim x) = \sim\sigma_{(n-i)(m-j)}x$.

This class of algebras will be denoted by $\boldsymbol{LM_{n\times m}}$. An algebra of this class will usually be indicated by L. The results announced here for $LM_{n\times m}$-algebras will be used throughout the paper.

(LM1) $\sigma_{ij}(L) = B(L)$ for all $(i, j) \in (n \times m)$, where $B(L)$ is the set of all Boolean elements of L [21, Proposition 2.5].

(LM2) Every $LM_{n\times 2}$-algebra is isomorphic to an n-valued Łukasiewicz-Moisil algebra. It is worth noting that $LM_{n\times m}$-algebras constitute a nontrivial generalization of the latter [24, Remark 2.1].

(LM3) Let $+$ be the binary operation on L defined as follows:

$$a + b = \bigwedge_{(i,j)\in(n\times m)} ((\sim\sigma_{ij}a \vee \sigma_{ij}b) \wedge (\sim\sigma_{ij}b \vee \sigma_{ij}a)).$$

Then $+$ satisfies the following properties:

(T1) $a + b = 1$ if and only if $a = b$,

(T2) $a + b = b + a$,

(T3) $(a + b) \wedge a = (a + b) \wedge b$, (T4) $a + 1 = \sigma_{11}a$,

(T5) $\sigma_{rs}(a + b) = a + b$ for every $(r, s) \in (n \times m)$,

(T6) $\sim(a + b)$ and $a + b$ are Boolean complements [21, Proposition 2.6].

(LM4) The class of $LM_{n\times m}$-algebras is a variety and two equational bases for it can be found in [21, Theorem 2.7] and [24, Theorem 4.6].

(LM5) Let $\rightarrow$ be the binary operation on L defined as follows:

$$x \rightarrow y = \sigma_{(n-1)(m-1)}(\sim x) \vee y.$$

Then, by defining the notion of a *deductive system* in the usual way, we infer that D is a deductive system of L if and only if D is a Stone filter of L [21, Proposition 3.3]. Besides, Stone filters were characterized as filters F of L which verify this condition: $x \in F$ implies $\sigma_{11}x \in F$ [21, Proposition 3.2]. Although these notions coincide, we will use either Stone filters or deductive systems according to the nature of the problem to solve. In what follows, we shall denote by $\mathcal{D}(L)$ and $\mathcal{F}_s(L)$ the set of all deductive systems and the set of all Stone filters of L respectively.

(LM6) For each $X \subseteq L$, $D(X) = F(\sigma_{11}X)$ [22, Proposition 2.3.3], where $D(X)$ and $F(X)$ denote the deductive system and the filter generated by X on L respectively. In particular, if $X = \{a\}$ we shall write $F(a)$ instead of $F(\{a\})$.

(LM7) Let L be an $LM_{n\times m}$-algebra with more than one element and let $\mathrm{Con}(L)$ be the lattice of all congruences on L. Then $\mathrm{Con}(L) = \{R(F)\colon F \in \mathcal{F}_s(L)\}$, where $R(F) = \{(x, y) \in L \times L\colon$ there exists $f \in F$ such that $x \wedge f = y \wedge f\}$. Besides, the lattices $\mathrm{Con}(L)$ and $\mathcal{F}_s(L)$ are isomorphic considering the mappings $\theta \mapsto [1]_\theta$ and $F \mapsto R(F)$ which are mutually inverse, where $[x]_\theta$ stands for the equivalence class of x modulo θ [21, Theorem 3.6].

(LM8) An *$n \times m$-valued Łukasiewicz-Moisil space* (or *$lm_{n\times m}$-space*) is a triple $(X, g, \{f_{ij}\}_{(i,j)\in(n\times m)})$ which verifies the following conditions:

(E1) (X, g) is a De Morgan space (or m-space) [6],

(E2) $f_{ij}\colon X \rightarrow X$ is an increasing and continuous function,

(E3) $f_{ij}(x) \leqslant f_{(i+1)j}(x)$,

(E4) $f_{ij}(x) \leqslant f_{i(j+1)}(x)$,

(E5) $f_{ij} \circ f_{rs} = f_{ij}$,

(E6) $f_{ij} \circ g = f_{ij}$,

(E7) $g \circ f_{ij} = f_{(n-i)(m-j)}$,

(E8) if for every U, $V \in IC(X)$ it is verified that $f_{ij}^{-1}(U) = f_{ij}^{-1}(V)$ for all $(i, j) \in (n \times m)$, then $U = V$, where $IC(X)$ denotes the lattice of all increasing clopen subsets of X.

Besides, if $(X, g, \{f_{ij}\}_{(i,j)\in(n\times m)})$ and $(X', g', \{f'_{ij}\}_{(i,j)\in(n\times m)})$ are $lm_{n\times m}$-spaces, an $lm_{n\times m}$*-function* is an isotone continuous function $f\colon X \to X'$ such that $f \circ g = g' \circ f$ and $f'_{ij} \circ f = f \circ f_{ij}$ for all $(i,j) \in (n \times m)$.

Then, taking into account the topological duality given by W. Cornish and P. Fowler for De Morgan algebras [6], we proved that the category of $lm_{n\times m}$-spaces and $lm_{n\times m}$-functions is naturally equivalent to the dual of the category of $LM_{n\times m}$-algebras and their corresponding homomorphisms [25, Theorem 2.1].

(LM9) Let X be a nonempty set and let L^X be the set of all functions from X into L. Then L^X is an $LM_{n\times m}$-algebra where the operations are defined componentwise.

(LM10) Let $B(L){\uparrow}^{(n\times m)} = \{f\colon (n \times m) \to B(L)$ such that for arbitrary i, j if $r \leqslant s$, then $f(r,j) \leqslant f(s,j)$ and $f(i,r) \leqslant f(i,s)\}$. Then

$$\langle B(L){\uparrow}^{(n\times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, 0, 1\rangle$$

is an $LM_{n\times m}$-algebra where for all $f \in B(L){\uparrow}^{(n\times m)}$ and $(i,j) \in (n \times m)$ the operations $\sim$ and σ_{ij} are defined as follows: $({\sim}f)(i,j) = (f(n-i, m-j))'$, where x' denotes the Boolean complement of x, $(\sigma_{ij} f)(r,s) = f(i,j)$ for all $(r,s) \in (n \times m)$, and the remaining operations are defined componentwise [24, Proposition 3.2]. It is worth noting that this result can be generalized by replacing $B(L)$ by any Boolean algebra B. Furthermore, if B is a complete Boolean algebra, it is simple to check that $B{\uparrow}^{(n\times m)}$ is also a complete $LM_{n\times m}$-algebra.

(LM11) Every $LM_{n\times m}$-algebra L can be embedded into $B(L){\uparrow}^{(n\times m)}$ [24, Theorem 3.1]. Besides, L is isomorphic to $B(L){\uparrow}^{(n\times m)}$ if and only if L is centred [24, Corollary 3.1] where L is *centred* if for each $(i,j) \in (n \times m)$ there exists $c_{ij} \in L$ such that

$$\sigma_{rs} c_{ij} = \begin{cases} 0 & \text{if } i > r \text{ or } j > s, \\ 1 & \text{if } i \leqslant r \text{ and } j \leqslant s. \end{cases}$$

2. Monadic $n \times m$-valued Łukasiewicz-Moisil algebras

The class of algebras which is of our concern now, arises from $n \times m$-valued Łukasiewicz-Moisil algebras endowed with a unary operation.

Definition 2.1. Let $L \in \boldsymbol{LM_{n\times m}}$. An existential quantifier on L is a mapping $\exists\colon L \to L$ which verifies the identities

(e1) $\exists 0 = 0$,

(e2) $x \wedge \exists x = x$,

(e3) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,

(e4) $\exists \sigma_{ij} x = \sigma_{ij} \exists x$ for all $(i,j) \in (n \times m)$.

Definition 2.2. Let $L \in \boldsymbol{LM_{n\times m}}$. A universal quantifier on L is a mapping $\forall\colon L \to L$ verifying the conditions

(u1) $\forall 1 = 1$,

(u2) $x \wedge \forall x = \forall x$,

(u3) $\forall(x \vee \forall y) = \forall x \vee \forall y$,

(u4) $\forall \sigma_{ij} x = \sigma_{ij} \forall x$ for all $(i,j) \in (n \times m)$.

Remark 2.1. Let $\exists$ be an existential quantifier on L. By defining $\forall x = {\sim}\exists{\sim}x$ for all $x \in L$, we have that $\forall$ is a universal quantifier on L. Conversely, let $\forall$ be a universal quantifier on L. Then the operator $\exists$ defined by $\exists x = {\sim}\forall{\sim}x$ for all $x \in L$ is an existential quantifier on L.

Definition 2.3. Let $L \in \boldsymbol{LM_{n\times m}}$. A monadic $n \times m$-valued Łukasiewicz-Moisil algebra (or $mLM_{n\times m}$-algebra) is a pair $(L, \exists)$, where $\exists$ is an existential quantifier on L or equivalently, it is a pair $(L, \forall)$, where $\forall$ is a universal quantifier on L.

In what follows we will denote by $\boldsymbol{mLM_{n\times m}}$ the class of $mLM_{n\times m}$-algebras.

Some of the results on $mLM_{n\times m}$-algebras given in this paper were communicated at the meetings indicated in [7], [20] and [23].

Remark 2.2. (i) From Definition 2.3 and (LM4) we infer that $\boldsymbol{mLM_{n\times m}}$ is a variety and two equational bases for it can be obtained.

(ii) If $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$, then from (e4) we have that $(B(L), \exists)$ is a monadic Boolean algebra.

(iii) Taking into account (LM2), we infer that every $mLM_{n\times 2}$-algebra is isomorphic to a monadic n-valued Łukasiewicz-Moisil algebra.

The next Lemma 2.1 summarizes the most important properties of both the existential and universal quantifiers which are necessary for further development. Its proof is an easy excercise.

Lemma 2.1. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$. Then the following conditions are satisfied:*

(e5) $\exists 1 = 1$,	(u5) $\forall 0 = 0$,
(e6) $\exists\exists x = \exists x$,	(u6) $\forall\forall x = \forall x$,

(e7) *$x \in \exists L$ if and only if $\exists x = x$,*
(e8) *$x \leqslant y$ implies $\exists x \leqslant \exists y$,*
(e9) *$\exists(x \vee y) = \exists x \vee \exists y$,*
(e10) *$\exists{\sim}\sigma_{ij}\exists x = {\sim}\sigma_{ij}\exists x$,*
(e11) *$\exists\forall x = \forall x$,*
(e12) *$\exists{\sim}\sigma_{ij}\forall x = {\sim}\sigma_{ij}\forall x$,*
(e13) *$x = \exists x$ if and only if $x = \forall x$.*

(u7) *$x \in B(L)$ implies $\forall x \in B(L)$,*
(u8) *$x \leqslant y$ implies $\forall x \leqslant \forall y$,*
(u9) *$\forall(x \wedge y) = \forall x \wedge \forall y$,*
(u10) *$\forall{\sim}\sigma_{ij}\forall x = {\sim}\sigma_{ij}\forall x$,*
(u11) *$\forall\exists x = \exists x$,*
(u12) *$\forall{\sim}\sigma_{ij}\exists x = {\sim}\sigma_{ij}\exists x$,*

Propositions 2.1 and 2.2 determine the relationship between the existential quantifier and special subalgebras of $mLM_{n\times m}$-algebras.

Proposition 2.1. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$. Then*

(i) *$\exists(L)$ is a Moore family of L and $\exists x = \bigwedge\{z \in \exists(L)\colon\ x \leqslant z\}$, where $\bigwedge X$ denotes the infimum of all elements of the set X,*
(ii) *$\exists(L)$ is a subalgebra of L,*
(iii) *for each $x \in L$, $\sigma_{ij}(\bigwedge\{z \in \exists(L)\colon\ x \leqslant z\}) = \bigwedge\{z \in \exists(L)\colon\ \sigma_{ij}x \leqslant z\}$ for all $(i, j) \in (n \times m)$,*
(iv) *if $x, y \in \exists(L)$ and there exists $x \Rightarrow y$ in L, then $x \Rightarrow y \in \exists(L)$, where $a \Rightarrow b$ stands for the relative pseudocomplement of a with respect to b.*

Proof. From (e2), (e6) and (e9) we have that $\exists$ is an additive closure operator and taking into account the well-known relationship between closure operators and Moore families [1] we conclude that (i) holds. On the other hand, it is straightforward to prove that (ii) is verified. Besides, (iii) follows from (e4). In order to prove (iv), suppose that $x, y \in \exists(L)$. Then $\exists((x \Rightarrow y)\wedge\exists x) = \exists(x\wedge(x \Rightarrow y)) = \exists(x\wedge y) \leqslant \exists y = y$. This inequality and (e3) imply that $x \wedge \exists(x \Rightarrow y) \leqslant y$ and so, $\exists(x \Rightarrow y) \leqslant x \Rightarrow y$. Therefore, from (e2) we conclude that $\exists(x \Rightarrow y) = x \Rightarrow y$ and the proof is complete. □

The following is a partial converse of Proposition 2.1.

Proposition 2.2. *Let $L \in \boldsymbol{LM_{n\times m}}$ and let M be a subset of L which verifies the following conditions:*

(i) *M is a Moore family of L,*
(ii) *M is a subalgebra of L,*
(iii) *for each $x \in L$, $\sigma_{ij}(\bigwedge\{z \in M\colon\ x \leqslant z\}) = \bigwedge\{z \in M\colon\ \sigma_{ij}x \leqslant z\}$ for all $(i, j) \in (n \times m)$,*
(iv) *for each $x, y \in M$, there exists $x \Rightarrow y$ in L and $x \Rightarrow y \in M$.*

For each $x \in L$ we define $\exists x = \bigwedge\{z \in M\colon\ x \leqslant z\}$. Then $\exists$ is an existential quantifier on L and $M = \exists(L)$.

P r o o f. From the hypothesis, it is simple to verify that conditions (e1), (e2) and (e4) hold and that $M = \exists(L)$. Besides, from (e2) we have that $x \wedge \exists y \leqslant \exists x \wedge \exists y$. By virtue of (ii), it results that $\exists x \wedge \exists y \in M$. Hence, $\exists(x \wedge \exists y) \leqslant \exists x \wedge \exists y$. On the other hand, if $k \in M$ verifies that $x \wedge \exists y \leqslant k$, then $x \leqslant \exists y \Rightarrow k$. Furthermore, from (iv) we infer that $\exists y \Rightarrow k \in M$. Therefore, $\exists x \leqslant \exists y \Rightarrow k$ and so, $\exists x \wedge \exists y \leqslant k$. Thus, $\exists x \wedge \exists y \leqslant \exists(x \wedge \exists y)$ and consequently, we conclude that (e3) holds. $\square$

It is worth noting that the dual of Propositions 2.1 and 2.2 are also true.

3. Congruences and subdirectly irreducible $\boldsymbol{mLM_{n\times m}}$-algebras

Now, we will describe the congruence lattices of $mLM_{n\times m}$-algebras.

Definition 3.1. Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$. A monadic congruence on $(L, \exists)$ is an $LM_{n\times m}$-congruence ε on L which for all $x, y \in L$ verifies the additional condition $(x, y) \in \varepsilon$ implies $(\exists x, \exists y) \in \varepsilon$.

From now on, we will denote by $\mathrm{Con}_m(L)$ the congruence lattice of $(L, \exists)$.

Definition 3.2. Let $(L, \forall)$ be an $mLM_{n\times m}$-algebra. A Stone filter (deductive system) F of L is monadic if it verifies the condition $x \in F$ implies $\forall x \in F$.

We will denote by $\mathcal{F}_{ms}(L)$ and $\mathcal{D}_m(L)$ the set of all monadic Stone filters and monadic deductive systems of $(L, \forall)$ respectively. Hence, we have that $\mathcal{F}_{ms}(L) = \mathcal{D}_m(L)$.

Theorem 3.1. *Let $(L, \exists)$ be an $mLM_{n\times m}$-algebra with more than one element. Then*

(i) $\mathrm{Con}_m(L) = \{R(F)\colon F \in \mathcal{F}_{ms}(L)\}$, *where $R(F)$ is the relation defined in* (LM7),

(ii) *the lattices $\mathrm{Con}_m(L)$ and $\mathcal{F}_{ms}(L)$ are isomorphic considering the mappings $\theta \mapsto [1]_\theta$ and $F \mapsto R(F)$, which are mutually inverse.*

P r o o f. Taking into account (LM5) and (LM7), it remains to prove that if $(x, y) \in R(F)$, then $(\exists x, \exists y) \in R(F)$ which is a direct consequence of (u2), (e11), (e3) and the fact that F is a monadic Stone filter. $\square$

In what follows, for each $F \in \mathcal{F}_{ms}(L)$ we will denote by L/F the quotient algebra of $(L, \exists)$ by $R(F)$.

R e m a r k 3.1. If $\varepsilon \in \mathrm{Con}_m(L)$ and $(x, y) \in \varepsilon$, then it is simple to check that $(x + z, y + z) \in \varepsilon$ for all $z \in L$.

Next, our attention is focused on characterizing the subdirectly irreducible $mLM_{n\times m}$-algebras. Lemma 3.1 will be fundamental for this purpose.

Lemma 3.1. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$ and let $H \subseteq L$. If $D_m(H)$ is the monadic deductive system generated by H, then $D_m(H) = F(\sigma_{11}\forall H) = D(\forall H)$.*

P r o o f. By (LM6), it only remains to prove that $D_m(H) = F(\sigma_{11}\forall H)$. Taking into account that $\sigma_{11}x \leqslant x$ holds for every $x \in L$ and (u2), we infer that $H \subseteq F(\sigma_{11}\forall H)$. Besides, it is simple to check that $F(\sigma_{11}\forall H)$ is a Stone filter and therefore, due to (LM5) it is a deductive system. Moreover, if $x \in F(\sigma_{11}\forall H)$ then by virtue of (u8), (u9), (u4) and (u6) we have that $\forall x \in F(\sigma_{11}\forall H)$. On the other hand, if T is a monadic deductive system of $(L, \exists)$ such that $H \subseteq T$, then $F(\sigma_{11}\forall H) \subseteq T$. Indeed, let $x \in F(\sigma_{11}\forall H)$. Hence, there exist $\sigma_{11}\forall h_1, \ldots, \sigma_{11}\forall h_r \in \sigma_{11}\forall H$ such that $\sigma_{11}\forall h_1 \wedge \ldots \wedge \sigma_{11}\forall h_r \leqslant x$. So, we conclude that $x \in T$. $\square$

On the other hand, for each $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$, let us consider the $LM_{n\times m}$-algebra $\exists(L)$, the monadic Boolean algebra $(B(L), \exists)$ and the Boolean algebra $B(\exists(L))$. Then by defining mappings

$$\begin{gathered}
\gamma_1\colon \mathcal{D}_m(L) \to \mathcal{D}(\exists(L)),\ \gamma_1(D) = D \cap \exists(L),\\
\gamma_2\colon \mathcal{D}_m(L) \to \mathcal{F}_m(B(L)),\ \gamma_2(D) = D \cap B(L),\\
\gamma_3\colon \mathcal{D}(\exists(L)) \to \mathcal{F}(B(\exists(L))),\ \gamma_3(D') = D' \cap B(\exists(L)),\\
\gamma_4\colon \mathcal{F}_m(B(L)) \to \mathcal{F}(B(\exists(L))),\ \gamma_4(F) = F \cap B(\exists(L)),
\end{gathered}$$

where $\mathcal{F}_m(B(L))$ and $\mathcal{F}(B(\exists(L)))$ are the set of monadic filters of $(B(L), \exists)$ and the set of filters of $B(\exists(L))$ respectively, we infer

Theorem 3.2. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$. Then the mappings γ_1, γ_2, γ_3 and γ_4 are order isomorphisms where $\mathcal{D}_m(L)$, $\mathcal{D}(\exists(L))$, $\mathcal{F}_m(B(L))$ and $\mathcal{F}(B(\exists(L)))$ are ordered by set inclusion. Besides, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{D}_m(L) \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"'] & \mathcal{D}(\exists(L)) \arrow[d, "\gamma_3"] \\
\mathcal{F}_m(B(L)) \arrow[r, "\gamma_4"'] & \mathcal{F}(B(\exists(L)))
\end{tikzcd}$$

P r o o f. Following a reasoning analogous to that given in [4] for n-valued Łukasiewicz-Moisil algebras and using well-known results of the theory of monadic Boolean algebras we have that γ_3 and γ_4 are isomorphisms. On the other hand, from Lemma 3.1 and by applying standard techniques we infer that γ_1 and γ_2 are isomorphisms. Finally, it is straightforward to prove that $\gamma_3 \circ \gamma_1 = \gamma_4 \circ \gamma_2$. $\square$

By virtue of Theorems 3.2 and 3.1 we are ready to characterize the subdirectly irreducible $mLM_{n\times m}$-algebras.

Theorem 3.3. *Let* $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$. *Then the following conditions are equivalent:*

(i) $(L, \exists)$ *is simple,*
(ii) $\exists(L)$ *is a simple* $LM_{n\times m}$*-algebra,*
(iii) $B(\exists(L))$ *is a simple Boolean algebra,*
(iv) $(L, \exists)$ *is subdirectly irreducible.*

As a consequence of Theorem 3.3 and by well-known results of universal algebra we conclude

Corollary 3.1. $\boldsymbol{mLM_{n\times m}}$ *is semisimple.*

4. The discriminator variety $\boldsymbol{mLM_{n\times m}}$

In this section, we will apply the results we developed so far to show that $\boldsymbol{mLM_{n\times m}}$ is a discriminator variety. Furthermore, we will determine the principal congruences. In what follows, for each $a, b \in L$ we will denote by $\theta(a, b)$ the principal congruence generated by (a, b).

Recall that the *ternary discriminator function* t on a set A is defined by the conditions

$$t(x, y, z) = \begin{cases} z & \text{if } x = y, \\ x & \text{otherwise.} \end{cases}$$

A variety $\mathcal{V}$ is a *discriminator variety*, if it has a polynomial p that coincides with the ternary discriminator function on each subdirectly irreducible member of $\mathcal{V}$; such a polynomial is called a *ternary discriminator polynomial for* $\mathcal{V}$.

Theorem 4.1. *The variety* $\boldsymbol{mLM_{n\times m}}$ *is a discriminator variety.*

Proof. Let $p(x, y, z) = (\forall(x + y) \wedge z) \vee ({\sim}\forall(x + y) \wedge x)$. In view of (T1) and (u1) we have that $p(x, x, z) = z$. If $x \neq y$, then from (T1) and (u2) we infer that $\forall(x+y) \neq 1$. Moreover, from (T6), (u7) and (e11) it results that $\forall(x+y) \in B(\exists(L))$. So, by Theorem 3.3 we conclude that $\forall(x + y) = 0$. Hence, $p(x, y, z) = x$. □

Corollary 4.1. *The variety of monadic* n*-valued Łukasiewicz-Moisil algebras is a discriminator variety.*

Proof. This follows from Theorem 4.1 and (iii) in Remark 2.2. □

Corollary 4.2.

(i) $\boldsymbol{mLM_{n\times m}}$ *is arithmetic,*

(ii) *for each* $(L,\exists)\in \boldsymbol{mLM_{n\times m}}$ *and* $a,b,c,d\in L$ *we have that* $(c,d)\in\theta(a,b)$ *if and only if* $p(a,b,c)=p(a,b,d)$, *i.e.* $\boldsymbol{mLM_{n\times m}}$ *has equationally definable principal congruences,*

(iii) *every principal congruence on* $(L,\exists)\in \boldsymbol{mLM_{n\times m}}$ *is a factor congruence,*

(iv) *the principal congruences on* $(L,\exists)\in \boldsymbol{mLM_{n\times m}}$ *form a sublattice of the lattice* $\mathrm{Con}_m(L)$,

(v) *each compact congruence on* $(L,\exists)\in \boldsymbol{mLM_{n\times m}}$ *is a principal congruence,*

(vi) *the congruences on each* $(L,\exists)\in \boldsymbol{mLM_{n\times m}}$ *are regular, normal and filtral,*

(vii) $\boldsymbol{mLM_{n\times m}}$ *has the congruence extension property.*

P r o o f. This is a direct consequence of Theorem 4.1 and the results established in [27]. □

Lemma 4.1 will allow us to give a new description of the principal congruences on $mLM_{n\times m}$-algebras simpler than the one obtained from (ii) in Corollary 4.2.

Lemma 4.1. *Let* $(L,\exists)\in \boldsymbol{mLM_{n\times m}}$. *Then*

(i) $\theta(a,b)=\theta(\forall(a+b),1)$,

(ii) $[1]_{\theta(a,b)}=F(\forall(a+b))$.

P r o o f. From Remark 3.1 we can assert that $(a+b,b+b)\in\theta(a,b)$. Taking into account (T1) and (u1) we infer that $(\forall(a+b),1)\in\theta(a,b)$. On the other hand, from (u2) we have $(a+b,1)\in\theta(\forall(a+b),1)$ and so, from (T3) it results that $(a,b)\in\theta(\forall(a+b),1)$. Consequently, (i) holds. On the other hand, let $x\in[1]_{\theta(a,b)}$. By virtue of (i) and item (ii) in Corollary 4.2, we have that $p(\forall(a+b),1,x)=p(\forall(a+b),1,1)$ and so, $(\forall(\forall(a+b)+1)\wedge x)\vee(\sim\forall(\forall(a+b)+1)\wedge\forall(a+b))=\forall(\forall(a+b)+1)\vee(\sim\forall(\forall(a+b)+1)\wedge\forall(a+b))$. Besides, from (T4), (T6), (LM1) and (u6) we get that $\forall(\forall(a+b)+1)=\forall(a+b)$. Hence, we infer that $\forall(a+b)\leqslant x$ and therefore, $[1]_{\theta(a,b)}\subseteq F(\forall(a+b))$. The other inclusion is immediate. □

Theorem 4.2. *Let* $(L,\exists)\in \boldsymbol{mLM_{n\times m}}$. *Then*

$$\theta(a,b)=\{(x,y)\in L\times L\colon\ x\wedge\forall(a+b)=y\wedge\forall(a+b)\}.$$

P r o o f. From Theorem 3.1 and (ii) in Lemma 4.1 we have that

$$\theta(a,b)=R([1]_{\theta(a,b)})=R(F(\forall(a+b)))=\{(x,y)\in L\times L\colon\ x\wedge\forall(a+b)=y\wedge\forall(a+b)\}.$$

□

R e m a r k 4.1. Taking into account (iii) in Remark 2.2, the principal congruences of monadic n-valued Łukasiewicz-Moisil algebras can be characterized as in Theorem 4.2 by means of the $+$ operation defined in (LM3). By identifying the set $\{1,\ldots,n-1\}\times\{1\}$ with $\{1,\ldots,n-1\}$ and σ_{i1} with σ_i, $1\leqslant i\leqslant n-1$, the $+$ operation is provided by the formula

$$a+b=\bigwedge_{i=1}^{n-1}((\sim\sigma_i a\vee\sigma_i b)\wedge(\sim\sigma_i b\vee\sigma_i a)).$$

The following lemmas will allow us to compute the number of congruences of a finite $mLM_{n\times m}$-algebra.

Lemma 4.2. *G is a principal monadic Stone filter of an $mLM_{n\times m}$-algebra $(L,\exists)$ if and only if $G=F(\forall(\sigma_{11}a))$ for some $a\in L$.*

P r o o f. It is routine. □

Lemma 4.3. *Let $(L,\exists)\in\boldsymbol{mLM_{n\times m}}$ and let $\mathrm{Con}^P_m(L)$ be the lattice of all principal monadic congruences on $(L,\exists)$. Then $\mathrm{Con}^P_m(L)$ is a Boolean lattice, where $\mathrm{id}_L=\theta(1,1)$, $L\times L=\theta(0,1)$, $\theta(a,b)\wedge\theta(c,d)=\theta((a+b)\vee\forall(c+d),1)$ and $\theta(\sim\forall(a+b),1)$ is the Boolean complement of $\theta(a,b)$.*

P r o o f. Let $\mathcal{F}^P_{ms}(L)$ be the set of all principal monadic Stone filters of $(L,\exists)$. Taking into account (ii) in Theorem 3.1, Lemma 4.1 and (T4) we obtain an isomorphism between $\mathcal{F}^P_{ms}(L)$ and $\mathrm{Con}^P_m(L)$ by means of the correspondences $\theta(a,b)\mapsto F(\forall(a{+}b))$ and $F(\forall\sigma_{11}(a))\mapsto\theta(a,1)$, which are mutually inverse. On the other hand, it is simple to verify that $\{1\}=F(\forall\sigma_{11}(1))$, $L=F(\forall\sigma_{11}0)$, $F(\forall\sigma_{11}a)\wedge F(\forall\sigma_{11}b)=F(\forall\sigma_{11}(a\vee\forall b))$ and $F(\forall\sigma_{11}\sim\forall\sigma_{11}a)$ is the Boolean complement of $F(\forall\sigma_{11}a)$. Hence, the above correspondences allow us to complete the proof. □

Lemma 4.4. *Let $(L,\exists)$ be a finite $mLM_{n\times m}$-algebra and let $a\in L$. Then the following conditions are equivalent:*

(i) *$F(\forall\sigma_{11}(a))$ is a maximal monadic Stone filter of $(L,\exists)$,*
(ii) *$\forall\sigma_{11}(a)$ is an atom of $B(\exists(L))$.*

P r o o f. It is routine. □

Let $|X|$ stand for the number of elements in a finite set X. As a direct consequence of Lemmas 4.3 and 4.4 we obtain

Proposition 4.1. *Let $(L, \exists)$ be a finite $mLM_{n\times m}$-algebra and let $\Pi(B(\exists(L)))$ be the set of all atoms of $B(\exists(L))$. Then $|\operatorname{Con}_m(L)| = 2^{|\Pi(B(\exists(L)))|}$.*

5. A TOPOLOGICAL DUALITY FOR $mLM_{n\times m}$-ALGEBRAS

Next, we will show a topological duality for $mLM_{n\times m}$-algebras taking into account both, the one indicated in (LM8) and the one given in [5] for Q-distributive lattices.

Definition 5.1. $(X, g, \{f_{ij}\}_{(i,j)\in(n\times m)}, E)$ is a monadic $n\times m$-valued Łukasiewicz space (or $mlm_{n\times m}$-space) if $(X, g, \{f_{ij}\}_{(i,j)\in(n\times m)})$ is an $lm_{n\times m}$-space and E is an equivalence relation on X which verifies the following conditions:

(ml1) $\nabla_E U \in IC(X)$ for each $U \in IC(X)$, where $\nabla_E U$ stands for the union of all the equivalence classes that contain an element of U,

(ml2) the equivalence classes modulo E are closed in X,

(ml3) for each $U \in IC(X)$, $f_{ij}^{-1}(\nabla_E U) = \nabla_E f_{ij}^{-1}(U)$ for all $(i,j) \in (n\times m)$.

Let $(X, g, \{f_{ij}\}_{(i,j)\in(n\times m)}, E)$ and $(X', g', \{f'_{ij}\}_{(i,j)\in(n\times m)}, E')$ be $mlm_{n\times m}$-spaces. An $mlm_{n\times m}$-function is an $lm_{n\times m}$-function $f\colon X \to X'$ satisfying $\nabla_E(f^{-1}(U)) = f^{-1}(\nabla'_E(U))$ for each $U \in IC(X)$.

Proposition 5.1. *Let $(X, g, \{f_{ij}\}_{(i,j)\in(n\times m)}, E)$ be an $mlm_{n\times m}$-space. Then $m\mathcal{LM}_{n\times m}(X) = \langle IC(X), \cap, \cup, \sim, \{\sigma_{ij}^X\}_{(i,j)\in(n\times m)}, \exists_E, \emptyset, X\rangle$ is an $mLM_{n\times m}$-algebra, where for each $U \in IC(X)$, $\sim U = X \setminus g^{-1}(U)$, $\sigma_{ij}^X(U) = f_{ij}^{-1}(U)$ and $\exists_E U$ is the union of all equivalence classes that contain an element of U.*

P r o o f. By virtue of [25, Proposition 2.1] and the results obtained in [5], it only remains to prove (e4), which is a direct consequence of (ml3). ☐

Proposition 5.2. *Let $\langle L, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, \exists, 0, 1\rangle$ be an $mLM_{n\times m}$-algebra and let $X(L)$ be the Priestley space associated with L. Then $\mathfrak{mlm}_{n\times m}(L) = (X(L), g_L, \{f_{ij}^L\}_{(i,j)\in(n\times m)}, E_\exists)$ is an $mlm_{n\times m}$-space, where for each $P \in X(L)$, $g_L(P) = L \setminus \{\sim x\colon x \in P\}$, $f_{ij}^L(P) = \sigma_{ij}^{-1}(P)$ and $E_\exists = \{(P,Q) \in X(L)\times X(L)\colon P \cap \exists(L) = Q \cap \exists(L)\}$. Moreover, $\sigma_L\colon L \to IC(X(L))$ defined by $\sigma_L(a) = \{P \in X(L)\colon a \in P\}$ is an $mLM_{n\times m}$-isomorphism.*

P r o o f. Conditions (ml1) and (ml2) are direct consequences of [5]. Besides, from [25, Proposition 2.2] we have that $(X(L), g_L, \{f_{ij}^L\}_{(i,j)\in(n\times m)})$ is an $lm_{n\times m}$-space. On the other hand, let $U \in IC(X(L))$. Then there exists $a \in L$ such that $U = \sigma_L(a)$. Therefore, in order to prove (ml3), we must show that $\exists_{E_\exists} {f_{ij}^L}^{-1} \sigma_L(a) = {f_{ij}^L}^{-1} \exists_{E_\exists} \sigma_L(a)$ for all $(i,j) \in (n\times m)$. Indeed, using the hypothesis (e4) and taking into account that σ_L is an isomorphism in $\boldsymbol{LM_{n\times m}}$ and belongs to the variety of all Q-distributive lattices [5], we infer that

$$\begin{aligned}\exists_{E_\exists} {f_{ij}^{L}}^{-1}\sigma_L(a) &= \exists_{E_\exists}\sigma_L\sigma_{ij}(a) = \sigma_L\exists\sigma_{ij}(a) = \sigma_{ij}^{X(L)}\sigma_L\exists(a)\\ &= \exists_{E_\exists} {f_{ij}^{L}}^{-1}\sigma_L(a) = {f_{ij}^{L}}^{-1}\exists_{E_\exists}\sigma_L(a).\end{aligned}$$

□

Let $\boldsymbol{mlm_{n\times m}}$ be the category of $mlm_{n\times m}$-spaces and $mlm_{n\times m}$-functions, and $\boldsymbol{m\mathcal{LM}_{n\times m}}$ the category of $mLM_{n\times m}$-algebras and their corresponding homomorphisms. Then, applying Propositions 5.1, 5.2 and following standard techniques we conclude

Theorem 5.1. *The category $\boldsymbol{m\mathcal{LM}_{n\times m}}$ is naturally equivalent to the dual of the category $\boldsymbol{mlm_{n\times m}}$.*

Now, taking into account the topological duality described above, we will characterize the lattice of all $mLM_{n\times m}$-congruences. In order to do this, we introduce

Definition 5.2. Let $(X, g, \{f_{ij}\}_{(i,j)\in(n\times m)}, E)$ be an $mlm_{n\times m}$-space. A subset Y of X is semimodal if $Y \subseteq f_{ij}^{-1}(Y)$ for every $(i,j) \in (n\times m)$.

On the other hand, recall that a subset Y of a De Morgan space (X, g) is involutive if $g(Y) = Y$.

From now on, for each $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$, we will denote by $\mathcal{C}_S(\mathfrak{mlm}_{n\times m}(L))$ the set of all closed, involutive and semimodal subsets Y of $\mathfrak{mlm}_{n\times m}(L)$ such that $\exists_E(Y) = Y$.

Proposition 5.3. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$ and $Y \in \mathcal{C}_S(\mathfrak{mlm}_{n\times m}(L))$. Then $\Theta(Y) = \{(x,y) \in L\times L\colon\ \sigma_L(x)\cap Y = \sigma_L(y)\cap Y\}$ is a monadic congruence on $(L, \exists)$.*

Proof. It is a consequence of [25, Proposition 2.3] and the results established in [6] and [5]. □

Proposition 5.4. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$, $\delta \in \mathrm{Con}_m(L)$ and $Y = \{P \in X(L)\colon [1]_\delta \subseteq P\}$. Then $Y \in \mathcal{C}_S(\mathfrak{mlm}_{n\times m}(L))$ and $\Theta(Y) = \delta$.*

Proof. Taking into account [25, Proposition 2.4] we will only prove that $\exists_E(Y) = Y$. Let $P \in Y$, then $[1]_\delta \subseteq P$. Besides, let $Q \in [P]_{E_\exists}$ and $t \in [1]_\delta$. Then $\forall t \in [1]_\delta$ and therefore, $\forall t \in P \cap \forall(L) = Q \cap \forall(L)$, from which we infer that $t \in Q$. Hence, $[1]_\delta \subseteq Q$ which implies that $Q \in Y$. So, $\exists_E(Y) \subseteq Y$. The other inclusion is immediate. □

Theorem 5.2. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$. Then the lattice $\mathcal{C}_S(\mathfrak{mlm}_{n\times m}(L))$ is isomorphic to the dual lattice* $\mathrm{Con}_m(L)$ *and the isomorphism is $\gamma\colon \mathcal{C}_S(\mathfrak{mlm}_{n\times m}(L)) \to \mathrm{Con}_m(L)$ defined by $\gamma(Y) = \Theta(Y)$.*

Proof. It follows from Propositions 5.3 and 5.4 and the results established in [6]. □

Now, we will determine the elements of $\mathcal{C}_S(\mathfrak{mlm}_{n\times m}(L))$ which correspond to principal congruences on L. Since $\theta(a,b) = \theta(a \wedge b, a \vee b)$ there is no loss of generality in assuming $a \leqslant b$.

Lemma 5.1. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$ and let $(X(L), g_L, \{f_{ij}^L\}_{(i,j)\in(n\times m)}, E_\exists)$ be its associated space. Then for all $a, b \in L$ such that $a \leqslant b$ the following identity holds:*

$$\sigma_L(\forall(a+b)) = X(L) \setminus \bigcup_{(i,j)\in(n\times m)} f_{ij}^{-1}(\exists_{E_\exists}(\sigma_L(b) \setminus \sigma_L(a))).$$

Proof. Taking into account [25, Lemma 2.2] we have that $\sigma_L(a+b) = X(L) \setminus \bigcup_{(i,j)\in(n\times m)} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))$. Then (E6) and (e9) yield

$$\begin{aligned}
\sigma_L(\forall(a+b)) &= {\sim}\exists_{E_\exists}{\sim}\Big(X(L) \setminus \bigcup_{(i,j)\in(n\times m)} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))\Big)\\
&= {\sim}\exists_{E_\exists} g_L^{-1}\Big(\bigcup_{(i,j)\in(n\times m)} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))\Big)\\
&= {\sim}\exists_{E_\exists} \bigcup_{(i,j)\in(n\times m)} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))\\
&= {\sim} \bigcup_{(i,j)\in(n\times m)} \exists_{E_\exists} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))\\
&= X(L) \setminus g_L^{-1}\Big(\bigcup_{(i,j)\in(n\times m)} \exists_{E_\exists} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a))\Big)\\
&= X(L) \setminus \bigcup_{(i,j)\in(n\times m)} g_L^{-1}(\exists_{E_\exists} f_{ij}^{-1}(\sigma_L(b) \setminus \sigma_L(a)))\\
&= X(L) \setminus \bigcup_{(i,j)\in(n\times m)} (g_L^{-1} \circ f_{ij}^{-1})(\exists_{E_\exists}(\sigma_L(b) \setminus \sigma_L(a)))\\
&= X(L) \setminus \bigcup_{(i,j)\in(n\times m)} f_{ij}^{-1}(\exists_{E_\exists}(\sigma_L(b) \setminus \sigma_L(a))).
\end{aligned}$$

□

Theorem 5.3. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$ and let $(X(L), g_L, \{f_{ij}^L\}_{(i,j)\in(n\times m)}, E_\exists)$ be its associated space. Then the following conditions are equivalent for all $a, b \in L$, $a \leqslant b$:*

(i) $Y = \sigma_L(\forall(a+b))$,

(ii) $Y \in \mathcal{C}_S(\mathfrak{mlm}_{n\times m}(L))$ *and* $\Theta(Y) = \theta(a, b)$.

Proof. Let $P \in \sigma_L(\forall(a+b))$. Then from Lemma 5.1 we have that $f_{ij}(P) \notin \exists_{E_\exists}(\sigma_L(b) \setminus \sigma_L(a))$ for all $(i,j) \in (n \times m)$ and so, taking into account (E5) we infer that $f_{rs}(P) \notin \bigcup\limits_{(i,j)\in(n\times m)} f_{ij}^{-1}(\exists_{E_\exists}(\sigma_L(b) \setminus \sigma_L(a))$ for all $(r,s) \in (n \times m)$. Hence, $P \in f_{rs}^{-1}(Y)$ for all $(r,s) \in (n \times m)$. Therefore, Y is semimodal. On the other hand, let $P \in g_L(\sigma_L(\forall(a+b)))$. Then there exists $Q \in \sigma_L(\forall(a+b))$ such that $P = g_L(Q)$. From this last assertion and (E6) we get that $f_{ij}(P) = f_{ij}(Q)$ for all $(i,j) \in (n \times m)$. Furthermore, from Lemma 5.1 we infer that $f_{ij}(Q) \notin \exists_{E_\exists}(\sigma_L(b) \setminus \sigma_L(a))$ for all $(i,j) \in (n \times m)$. Hence, we obtain that $P \in \sigma_L(\forall(a+b))$. Consequently, $g_L(\sigma_L(\forall(a+b))) \subseteq \sigma_L(\forall(a+b))$. The other inclusion follows if we bear in mind that g_L is involutive. So, Y is involutive. Besides, taking into account that σ_L is an isomorphism and (e11) it results that $\exists_{E_\exists}(Y) = Y$. Since Y is a closed subset of $X(L)$ we conclude that $Y \in \mathcal{C}_S(\mathfrak{mlm}_{n\times m}(L))$. From Theorem 4.2 and the fact that σ_L is one-to-one we conclude that $\Theta(\sigma_L(\forall(a+b))) = \theta(a,b)$. The converse is immediate. $\square$

Remark 5.1. In the particular case of monadic n-valued Łukasiewicz-Moisil algebras, Theorem 5.3 and (iii) in Remark 2.2 allow us to determine the semimodal, involutive and closed subsets of the space associated with them which correspond to principal congruences.

6. Functional representation theorems for $\boldsymbol{mLM_{n\times m}}$-algebras

In the present section, we will generalize the results obtained in [9] for monadic n-valued Łukasiewicz-Moisil algebras. More precisely, we will describe three functional representation theorems for $mLM_{n\times m}$-algebras pointing out the relationship between them. To this end, the following assertions on monadic Boolean algebras will be necessary.

(H1) A constant of a monadic Boolean algebra $(A, \exists)$ is a Boolean endomorphism c on A such that (c1) $c \circ \exists = \exists$ and (c2) $\exists \circ c = c$.

This mapping has the following properties: (c3) $c \circ c = c$ and (c4) $c(x) \leqslant \exists x$ for all $x \in A$.

In particular, a constant c is a witness to an element z of $(A, \exists)$ if $\exists z = c(z)$, and we will denote it by c_z. Furthermore, a monadic Boolean algebra $(A, \exists)$ is rich if for any $x \in A$ there exists a witness to x.

(H2) Every monadic Boolean algebra is a subalgebra of a rich one [12, Theorem 11].

(H3) If $(A, \exists)$ is a monadic Boolean algebra, then there exists a set X and a Boolean algebra B such that

(i) A is isomorphic to a subalgebra S of the functional Boolean algebra B^X,

(ii) for each $f \in S$ there exists $x \in X$ such that $\exists f(x) = f(x)$ [12, Theorem 12].

Proposition 6.1. *Let* $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$. *Then* $\langle B(L)\uparrow^{(n\times m)}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, \exists, 0, 1\rangle$ *is an* $mLM_{n\times m}$*-algebra where* $\exists$ *is defined componentwise and the remaining operations are the ones defined in* (LM10).

P r o o f. The statement follows from (LM10) and the definition of $\exists$. ☐

Theorem 6.1. *Every* $mLM_{n\times m}$*-algebra* $(L, \exists)$ *can be embedded into the algbra* $(B(L)\uparrow^{(n\times m)}, \exists)$.

P r o o f. Taking into account [24, Theorem 3.1], the mapping $\tau\colon L \to B(L)\uparrow^{(n\times m)}$ defined by the prescription $\tau(x)(i,j) = \sigma_{ij}x$ for each $x \in L$ and $(i,j) \in (n \times m)$ is a one-to-one $LM_{n\times m}$-homomorphism. Besides, due to (e4) it is simple to check that $\tau(\exists x) = \exists\tau(x)$ for all $x \in L$. ☐

Corollary 6.1. *Every* $mLM_{n\times m}$*-algebra can be embedded into a complete one.*

P r o o f. This assertion follows from a well-known result on Boolean algebras, (LM10) and Theorem 6.1. ☐

By defining the notion of *centred* $mLM_{n\times m}$*-algebras* in a similar way to the one given for $LM_{n\times m}$-algebras and as a direct consequence of (LM11), we conclude

Corollary 6.2. *Let* $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$. *Then the following conditions are equivalent:*

(i) $(L, \exists)$ *is centred,*

(ii) $(L, \exists)$ *is isomorphic to* $(B(L)\uparrow^{(n\times m)}, \exists)$.

Proposition 6.2. *Let B be a complete Boolean algebra and let X be a nonempty set. Then $\langle B{\uparrow}^{(n\times m)^X}, \wedge, \vee, \sim, \{\sigma_{ij}\}_{(i,j)\in(n\times m)}, \exists, 0, 1\rangle$ is a complete $mLM_{n\times m}$-algebra where for all $x \in X$, $(\exists f)(x) = \bigvee f(X)$ for each $f \in B{\uparrow}^{(n\times m)}$ with $\bigvee f(X)$ being the supremum of $f(X) = \{f(y)\colon\ y \in X\}$ and the remaining operations are defined componentwise.*

Proof. From (LM9) and (LM10) we have that $B{\uparrow}^{(n\times m)^X}$ is a complete $LM_{n\times m}$-algebra and that $\exists$ is well defined on $B{\uparrow}^{(n\times m)^X}$. Besides, it is simple to check that identities (e1), (e2), (e3) and (e4) hold. □

For the proof of the next functional representation theorem we will apply the results given by P. Halmos for monadic Boolean algebras mentioned at the beginning of this section.

Theorem 6.2. *Let $(L, \exists)$ be an $mLM_{n\times m}$-algebra. Then there exists a nonempty set X and a Boolean algebra B such that $(L, \exists)$ can be embedded into $((\exists(B)){\uparrow}^{(n\times m)^X}, \exists)$ and $B(L)$ is a subalgebra of B.*

Proof. From (ii) in Remark 2.2 we have that $(B(L), \exists)$ is a monadic Boolean algebra and so, by (H2) we can assert that $(B(L), \exists)$ is a subalgebra of a rich monadic Boolean algebra B. Let X be a set of constants of B containing at least one witness to x for each $x \in B$. Let $\Phi\colon\ B \to (\exists(B))^X$ be the mapping defined by $\Phi(z)(c) = c(z)$ for all $c \in X$. Then Φ is a one-to-one monadic Boolean homomorphism [12, Theorem 12]. On the other hand, from (LM10) and (LM9) it results that $(\exists(B)){\uparrow}^{(n\times m)^X}$ is an $LM_{n\times m}$-algebra. Let us consider now the mapping $\Psi\colon\ L \to (\exists(B)){\uparrow}^{(n\times m)^X}$ defined by $(\Psi(x)(c))(i,j) = \Phi(\sigma_{ij}x)(c)$ for each $x \in L$, $c \in X$ and $(i,j) \in (n \times m)$. Taking into account the definition of Φ, (C1), (C2), (C3) and (C5) it is simple to verify that Ψ is a one-to-one homomorphism of bounded lattices.

On the other hand, since Φ is a Boolean homomorphism, from (C7) we have that

$$\begin{aligned}((\Psi(\sim x))(c))(i,j) &= (\Phi(\sigma_{ij}(\sim x)))(c) = (\Phi(\sim\sigma_{(n-i)(m-j)}(x)))(c)\\ &= (\Phi((\sigma_{(n-i)(m-j)}(x))'))(c) = (\Phi(\sigma_{(n-i)(m-j)}(x)))'(c)\\ &= ((\Phi(\sigma_{(n-i)(m-j)}(x)))(c))' = ((\Psi(x)(c))(n-i, m-j))'\\ &= (\sim(\Psi(x)(c)))(i,j) = ((\sim\Psi(x))(c))(i,j)\end{aligned}$$

for all $c \in X$ and $(i,j) \in (n \times m)$ and therefore, $\Psi(\sim x) = \sim\Psi(x)$. Furthermore, taking into account (C4), we get that

$$\begin{aligned}((\Psi(\sigma_{rs}x))(c))(i,j) &= (\Phi(\sigma_{ij}\sigma_{rs}x))(c) = (\Phi(\sigma_{rs}x))(c) = ((\Psi(x))(c))(r,s)\\ &= (\sigma_{rs}((\Psi(x))(c)))(i,j) = ((\sigma_{rs}(\Psi(x)))(c))(i,j)\end{aligned}$$

for all $c \in X$ and $(i,j) \in (n \times m)$. Hence, $\Psi(\sigma_{rs}x) = \sigma_{rs}\Psi(x)$.

If f belongs to the image of Ψ, then $\Psi(l) = f$ for some $l \in L$. Let g_l be the mapping defined by $g_l(i,j) = \exists\sigma_{ij} l$ for all $(i,j) \in (n \times m)$. It is simple to verify that $g_l \in (\exists(B))\uparrow^{(n\times m)}$. Besides, from (c4) we have that $f(c)(i,j) = \Phi(\sigma_{ij} l)(c) = c(\sigma_{ij} l) \leqslant \exists\sigma_{ij} l = g_l(i,j)$ for all $(i,j) \in (n \times m)$ and so, $f(c) \leqslant g_l$ for all $c \in X$. On the other hand, let $h \in (\exists(B))\uparrow^{(n\times m)}$ be such that $f(c) \leqslant h$ for all $c \in X$. Therefore, $f(c)(i,j) \leqslant h(i,j)$ for each $(i,j) \in (n \times m)$. In particular, $f(c_{\sigma_{ij} l})(i,j) \leqslant h(i,j)$ and so, $g_l \leqslant h$. From this last assertion and the fact that g_l is an upper bound of $\{f(c)\colon c \in X\}$, we infer that $g_l = \bigvee\{f(c)\colon c \in X\}$. Hence, for each $f \in \Psi(L)$ we define $(\exists f)(c) = \bigvee\{f(c)\colon c \in X\}$ for all $c \in X$. Moreover, from (c1) and (e4) we have that

$$\begin{aligned}((\exists(\Psi(x)))(c))(i,j) &= \exists\sigma_{ij} x = c(\sigma_{ij}\exists x) = \Phi(\sigma_{ij}\exists x)(c)\\ &= ((\Psi(\exists x))(c))(i,j)\end{aligned}$$

for all $x \in L$, $c \in X$ and $(i,j) \in (n \times m)$. So, Ψ conmmutes with $\exists$. □

Remark 6.1. (i) Let $(L, \exists)$ be an $mLM_{n\times m}$-algebra and let X be a nonempty set. There is no loss of generality in asuming that the Boolean algebra $\exists(B(L))$ is complete. Then from Proposition 6.2 we have that $((\exists(B(L)))\uparrow^{(n\times m)^X}, \exists)$ is a complete $mLM_{n\times m}$-algebra. Hence, from Theorem 6.2, $(L, \exists)$ can be embedded into a complete $mLM_{n\times m}$-algebra. It is worth noting that the latter is different from that obtained in Corollary 6.1. (ii) If all elements of an $mLM_{n\times m}$-algebra $(L, \exists)$ are Boolean ones, that is to say if $L = B(L)$, then Theorem 6.2 coincides with Halmos's functional representation theorem indicated in (H3).

With the purpose of obtaining the third functional representation theorem, we extend the notion of constant indicated in (H1) to $mLM_{n\times m}$-algebras as follows:

A *constant* of an $mLM_{n\times m}$-algebra $(L, \exists)$ is an $mLM_{n\times m}$-endomorphism c on L such that $c \circ \exists = \exists$ and $\exists \circ c = c$.

From this definition, it results that $c(L) = \exists(L)$ and therefore, $c\colon L \to \exists(L)$ is an $LM_{n\times m}$-epimorphism such that c is the identity on $\exists(L)$. The notions of *witness* and *rich $mLM_{n\times m}$-algebras* are similar to those given for monadic Boolean algebras.

Lemma 6.1. *Let $(L, \exists)$ be a rich $mLM_{n\times m}$-algebra and let X be a set of constants of L containing at least one witness to x for each $x \in L$. Then the following conditions are equivalent:*

(i) $c(x) = 1$ *for all* $c \in X$,

(ii) $x = 1$.

Proof. It is routine. □

Theorem 6.3. *Let $(L, \exists)$ be a rich $mLM_{n\times m}$-algebra. Then there exists a nonempty set X such that $(L, \exists)$ can be embedded into $((\exists(L))^X, \exists)$.*

P r o o f. Let X be a set of constants of L containing at least one witness to x for each $x \in L$. Then X is a nonempty set and from (LM9) we have that $(\exists(L))^X$ is an $LM_{n\times m}$-algebra. Let $\Omega\colon L \to (\exists(L))^X$ be the mapping defined by $\Omega(x)(c) = c(x)$ for all $c \in X$. It is straightforward to prove that Ω is an $LM_{n\times m}$-homomorphism. Furthermore, Ω is one-to-one. Indeed, let $x, y \in L$ be such that $\Omega(x) = \Omega(y)$. Then $c(x) = c(y)$ for each $c \in X$. Thus, $c(x + y) = 1$ for each $c \in X$ and so, from Lemma 6.1 and (T1) we infer that $x = y$. On the other hand, let $f = \Omega(x)$ for some $x \in L$. Hence, $f(c) = c(x) \leqslant \exists x$ for each $c \in X$. Besides, if $k \in \exists(L)$ verifies that $f(c) \leqslant k$ for all $c \in X$, then $f(c_x) \leqslant k$. From this assertion we have that $\exists x \leqslant k$ and so, $\bigvee\{f(c)\colon c \in X\} = \exists x$. For each $f \in \Omega(L)$, we define $(\exists f)(c) = \exists x$ for all $c \in X$, where $f = \Omega(x)$. It is simple to check that $\Omega(\exists y) = \exists(\Omega y)$ for all $y \in L$. Therefore, $(L, \exists)$ is isomorphic to the $mLM_{n\times m}$-algebra $(\Omega(L), \exists)$. $\square$

Since every $mLM_{n\times m}$-algebra is a monadic Boolean algebra whenever $\sigma_{ij}x = x$ for all $(i, j) \in (n \times m)$, then bearing in mind well-known results on monadic Boolean algebras we can assert that there exist $mLM_{n\times m}$-algebras which are not rich. The next theorem will characterize rich algebras in $\boldsymbol{mLM_{n\times m}}$.

Theorem 6.4. *Let $(L, \exists) \in \boldsymbol{mLM_{n\times m}}$. Then the following conditions are equivalent:*

(i) *$(L, \exists)$ is rich,*

(ii) *for each $x \in L$ there exists a Stone filter F_x such that the natural map $q_x\colon L \to L/F_x$ restricted to $\exists(L)$ is an $LM_{n\times m}$-isomorphism and $q_x(\exists x) = q_x(x)$.*

P r o o f. (i) $\Rightarrow$ (ii): Let $x \in L$ and let c_x be a witness to x. Then $F_x = c_x^{-1}(1)$ is a Stone filter of L. Hence, the natural map q_x restricted to $\exists(L)$, which we will denote by $q_x|_{\exists(L)}$, is bijective. Indeed, let $a \in L$, then $c_x(a) = y$ for some $y \in \exists(L)$ and so, $c_x(a) = c_x(y)$. Therefore, $[a]_{R(F_x)} = [y]_{R(F_x)}$. Furthermore, if $z \in \exists(L)$ verifies that $[y]_{R(F_x)} = [z]_{R(F_x)}$, then $c_x(y) = c_x(z)$ from which we infer that $y = z$. Thus, each equivalence class has only one element belonging to $\exists(L)$. This assertion allows us to infer that $q_x|_{\exists(L)}$ is an $LM_{n\times m}$-isomorphism. On the other hand, since $c_x(\exists x) = c_x(x)$, we conclude that $q_x(\exists x) = q_x(x)$.

(ii) $\Rightarrow$ (i): Let $x \in L$. Then there is a Stone filter F_x of L such that $q_x|_{\exists(L)}$, is an $LM_{n\times m}$-isomorphism and $q_x(\exists x) = q_x(x)$. Let $c = q_x|_{\exists(L)}^{-1} \circ q_x$. Then $c(\exists a) = q_x|_{\exists(L)}^{-1}(q_x(\exists a)) = \exists a$ for all $a \in L$. Besides, c is an $LM_{n\times m}$-epimorphism from L onto $\exists(L)$, from which we have that $\exists c(a) = c(a)$ for all $a \in L$. Finally, since $c(x) = q_x|_{\exists(L)}^{-1}(q_x(x)) = q_x|_{\exists(L)}^{-1}(q_x(\exists x)) = \exists x$, we conclude that c is a witness to x. $\square$

R e m a r k 6.2. Let L be an $LM_{n\times m}$-algebra and let $[0,b]=\{x\in L\colon x\leqslant b\}$, where $b\in B(L)$. Then it is easy to verify that $\langle[0,b],\wedge,\vee,-,\{\sigma_{ij}\}_{(i,j)\in(n\times m)},0,b\rangle$ is an $LM_{n\times m}$-algebra, where $-x={\sim}x\wedge b$ for each $x\in[0,b]$. Moreover, the mapping $h_b\colon L\to[0,b]$ defined by $h_b(x)=x\wedge b$ is an $LM_{n\times m}$-epimorphism whose kernel is $F(b)$.

Corollary 6.3. *Let $(L,\exists)$ be a finite $mLM_{n\times m}$-algebra. Then the following conditions are equivalent:*

(i) *$(L,\exists)$ is rich,*

(ii) *for each $x\in L$ there exists $b_x\in B(L)$ such that h_{b_x} restricted to $\exists(L)$ is an $LM_{n\times m}$-isomorphism and $h_{b_x}(\exists x)=h_{b_x}(x)$.*

P r o o f. (i) $\Rightarrow$ (ii): Let $x\in L$. By Theorem 6.4 there exists a Stone filter F_x such that $q_x\colon\exists(L)\to L/F_x$ is an $LM_{n\times m}$-isomorphism and $q_x(\exists x)=q_x(x)$. Since L is a finite algebra, $F_x=F(b_x)$ for some $b_x\in B(L)$. Then by Remark 6.2, the mapping $\gamma\colon L/F(b_x)\to[0,b_x]$ defined by $\gamma([y]_{R(F(b_x))})=y\wedge b_x$ is an $LM_{n\times m}$-isomorphism and $h_{b_x}=\gamma\circ q_x$, which allows us to conclude (ii).

(ii) $\Rightarrow$ (i): Let $x\in L$. Then due to the hypothesis there exists $b_x\in B(L)$ from which by Remark 6.2, the mapping $\beta\colon L/F(b_x)\to[0,b_x]$ such that $\beta\circ q_{b_x}=h_{b_x}$ is an $LM_{n\times m}$-isomorphism, where q_{b_x} is the natural $LM_{n\times m}$-homomorphism. Hence, $q_{b_x}=\beta^{-1}\circ h_{b_x}$. From this assertion and (ii), it is straightforward to prove that $q_{b_x}|_{\exists(L)}$ is one-to-one and so, it is an $LM_{n\times m}$-isomorphism. Besides, from (ii) we infer that $q_{b_x}(\exists x)=q_{b_x}(x)$. Therefore, Theorem 6.4 allows us to conclude that L is rich. $\square$

Finally, we will show the relationship between the functional representations obtained in Theorems 6.1, 6.2 and 6.3 in the particular case of rich $mLM_{n\times m}$-algebras.

Let X_L be a set of constants of a rich $mLM_{n\times m}$-algebra $(L,\exists)$ containing at least one witness to x for each $x\in L$ and let $X_{B(L)}=\{c^*=c|_{B(L)}\colon c\in X_L$ and c is a witness to at least one b for each $b\in B(L)\}$. From Theorem 6.3 we have that $\Omega(x)=(c(x))_{c\in X_L}$ for each $x\in L$. On the other hand, let $\tau^*\colon L^{X_L}\to(B(L){\uparrow}^{(n\times m)})^{X_L}$ and $\varepsilon:((\exists B(L)){\uparrow}^{(n\times m)})^{X_L}\to((\exists(B(L))){\uparrow}^{(n\times m)})^{X_{B(L)}}$ be the mappings defined by $\tau^*((a_c)_{c\in X_L})=(\tau(a_c))_{c\in X_L}$ and $\varepsilon((f(c))_{c\in X_L})=(f(c^*))_{c^*\in X_{B(L)}}$, where τ is the mapping introduced in Theorem 6.1. Then for each $x\in L$ we have that

$$\begin{aligned}(\varepsilon\circ\tau^*\circ\Omega)(x)&=(\varepsilon\circ\tau^*)((c(x))_{c\in X_L})=\varepsilon((\tau(c(x)))_{c\in X_L})\\&=\varepsilon(((\sigma_{ij}(c(x)))_{(i,j)\in(n\times m)})_{c\in X_L}\\&=\varepsilon(((c(\sigma_{ij}(x)))_{(i,j)\in(n\times m)})_{c\in X_L}\\&=((c^*(\sigma_{ij}(x)))_{(i,j)\in(n\times m)})_{c^*\in X_{B(L)}}=\Psi(x),\end{aligned}$$

where Ψ is the mapping given in Theorem 6.2. Then the following diagram commutes:

$$\begin{tikzcd}
L \arrow[r, "\Omega"] \arrow[ddr, "\Psi"'] & (\exists(L))^{X_L} \arrow[d, "\tau^*|_{\exists(L)}"] \\
& ((\exists(B(L)))\uparrow^{(n\times m)})^{X_L} \arrow[d, "\varepsilon"] \\
& ((\exists(B(L)))\uparrow^{(n\times m)})^{X_{B(L)}}
\end{tikzcd}$$

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