Weak solution approach in fluid mechanics

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Generalized solutions

Why weak solutions?

- Existence for "any" data and "any" time interval
- Natural limits of numerical schemes
- The only alternative for certain class of problems (nonlinear, singularities)

What is a good weak solution?

- The most general object that "satisfies" a given system of equations with given data
- Easy to identify
- **Compatibility.** A strong (classical) solution is a weak solution. "Regular" weak solution is a classical solution.
- **Weak–strong uniqueness.** A strong (classical) solutions coincides with the weak solution corresponding to the same data

Hierarchy of solution classes

Strong vs. weak solutions

- **Strong (classical) solutions.** They possess the necessary *smoothness* required by the principles of continuum mechanics. They require strong *a priori* estimates not always available (inevitable presence of singularities for certain nonlinear problems)
- **Weak (distributional solutions.** Classical derivatives replaced by distributional ones. Considerably weaker requirements concerning *a priori* bounds. *Compactness* (strong a.a. stability) needed - no oscillatory solutions.
- **Measure–valued solutions.** Oscillations allowed, only uniform bounds on all quantities needed. The exact values of the unknowns are replaced by their *expectations* with respect to a probability measure.
- **Dissipative measure–valued solutions.** Concentrations allowed. Only uniform tightness of the probability measures required

Compressible Navier-Stokes/Euler system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I},\ \ \mu \geq 0,\ \ \eta \geq 0$$

No-flux/no-slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \ [\mathbf{u}]_{\mathrm{tan}}|_{\partial\Omega} = 0$$

Thermodynamics stability

Pressure potential

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z$$

Pressure-density state equation

$$p \in C[0,\infty) \cap C^2(0,\infty), \ \ p(0) = 0$$

$$\boxed{p'(\varrho) > 0} \text{ for } \varrho > 0, \ \liminf_{\varrho \to \infty} p'(\varrho) > 0$$

$$\liminf_{\varrho \to \infty} \frac{P(\varrho)}{p(\varrho)} > 0$$

Isentropic pressure-density state equation

$$p(\varrho) = a\varrho^\gamma, \ \ a > 0, \ \ \gamma \geq 1$$

Energy balance - conservation

Energy

$$E = \underbrace{\frac{1}{2}\varrho|\mathbf{u}|^2}_{\text{kinetic energy}} + \underbrace{P(\varrho)}_{\text{elastic energy}} \quad , \quad P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \ \mathrm{d}z$$

Energy balance equation

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) + \mathrm{div}_x(p(\varrho)\mathbf{u}) - \mathrm{div}_x\left(\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u}\right) = -\boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}}$$

Total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \boxed{\leq} 0$$

Classical (strong) solutions

Local existence

Smooth solutions exist on a maximal time interval $(0, T_{\max})$. This is true for both Navier-Stokes and Euler system

Global-in-time solutions for small data

Smooth solutions of the *Navier-Stokes system* exist globally in time provided the initial data are close to an equilibrium solution (**Matsumura and Nishida, Valli and Zajaczkowski, and others**). Solutions of the *Euler system* develop singularities in a finite time no matter how smooth and/or small the initial data are.

Global existence for the 1-D Navier-Stokes system

The Navier-Stokes system in the 1-D geometry admits global-in-time smooth solutions (**Antontsev, Kazhikhov, Shelukhin and others**)

Weak solutions

Equation of continuity

$$\left[\int_{\Omega} \varrho \varphi \, \mathrm{d}x \right]_{t=0}^{\tau} = \int_0^{\tau} \int_{\Omega} \left(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right) \, \mathrm{d}x \mathrm{d}t$$

for any $\varphi \in C_c^{\infty}([0, T) \times \overline{\Omega})$

Balance of momentum

$$\left[\int_{\Omega} \varrho \mathbf{u} \cdot \varphi \, \mathrm{d}x \right]_{t=0}^{\tau}$$

$$= \int_0^{\tau} \int_{\Omega} \left(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \cdot \nabla_x \varphi + p(\varrho) \mathrm{div}_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any $\varphi \in C_c^{\infty}([0, T) \times \overline{\Omega}; R^N)$,

$\varphi|_{\partial\Omega} = 0$ for the no-slip condition in the viscous case

Dissipative weak solutions

Energy (entropy) inequality

$$\left[\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \boxed{\leq} 0$$

for a.a. $\tau \in (0, T)$

Navier-Stokes system: Weak solutions

Global existence for large data

$$p(\varrho) \approx a\varrho^{\gamma}, \ \mu > 0$$

The Navier-Stokes system admits global-in-time weak solutions if:

- $N = 2$, $\gamma \geq 3/2$; $N = 3$, $\gamma \geq 9/5$ **P.L.Lions 1998**
- $N = 2$, $\gamma > 1$, $N = 3$, $\gamma > 3/2$ **EF et al. 2000**
- $N = 2$, $\gamma \geq 1$, $N = 3$, $\gamma \geq 3/2$ **Plotnikov and Vaigant 2014**

Dissipative weak solutions

The weak solutions are not known to be unique. The construction used in the existence theory yields *dissipative* weak solutions. Weak solutions can be obtained as a limit of certain numerical schemes (**Karper**)

Euler system: Weak solutions

Global existence for large data in 1D

The Euler system admits global-in-time weak solutions for any bounded initial data (**DiPerna, Chen et al.**). The weak solutions can be recovered as a vanishing viscosity limit of the Navier-Stokes system (**Chen and Perepelitsa**)

Global existence for large data for $N = 2, 3$

The compressible Euler system admits *infinitely many* global-in-time weak solutions for any smooth initial data (**Chiodaroli, EF** - based on the work of **DeLellis and Székelyhidi**)

Euler system: Dissipative weak solutions

Dissipative weak solutions $N = 2, 3$

- For any ϱ_0, there exists $\mathbf{u}_0$ (bounded measurable) such that the Euler system admits infinitely many dissipative weak solutions in a given time interval $(0, T)$ (**Chiodaroli, EF**)
- There is a vast class of initial data for which the Euler system admits infinitely many entropy (dissipative) weak solutions in a given time interval $(0, T)$ (**Chiodaroli, EF**, recent extension by **Luo, Xie, and Xin**)
- There exist Lipschitz (smooth) initial data for which the Euler system admits infinitely many entropy (dissipative) weak solutions in a given time interval $(0, T)$ (**Chiodaroli, DeLellis, Kreml**)
- Extension to the complete Euler system (with temperature), (**EF, Klingenberg, Kreml, Markfelder**)

Relative entropy (energy)

Relative energy functional

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right) \ \mathrm{d}x$$

Decomposition

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x - \int_\Omega \varrho\mathbf{u}\cdot\mathbf{U} \ \mathrm{d}x + \int_\Omega \frac{1}{2}\varrho|\mathbf{U}|^2 \ \mathrm{d}x$$

$$- \int_\Omega P'(r)\varrho \ \mathrm{d}x + \int_\Omega p(r) \ \mathrm{d}x$$

Dissipation inequality

Relative energy inequality

$$\left[\mathcal{E} \left(\varrho, \mathbf{u} \middle| r, \mathbf{U} \right) \right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \int_\Omega \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R} \left(\varrho, \mathbf{u} \ \middle| \ r, \mathbf{U} \right) \ \mathrm{d}t$$

Test functions

$$r > 0, \ \mathbf{U}|_{\partial\Omega} = 0 \text{ (or other relevant b.c.)}$$

Remainder

$$\int_0^{\tau} \mathcal{R}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right) \ \mathrm{d}t$$

$$\equiv \int_\Omega \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\right) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x$$

$$+ \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \ \mathrm{d}x + \int_\Omega (p(r) - p(\varrho))\, \mathrm{div}_x \mathbf{U} \ \mathrm{d}x$$

$$+ \int_\Omega \left[(r - \varrho)\partial_t P'(r) + \nabla_x P'(r) \cdot (r\mathbf{U} - \varrho \mathbf{u})\right] \ \mathrm{d}x$$

Applications

Weak-strong uniqueness

Weak and strong solutions of the compressible Navier-Stokes/Euler system emanating from the same initial data coincide as long as the latter exists (**EF, Jin, Novotný, Sun [2014]**)

Conditional regularity

Weak solution to the Navier-Stokes system with bounded density component emanating from smooth initial data are smooth (**EF, Jin, Novotný, Sun [2014]**)

Singular limits

Rotating fluids

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon} \varrho \mathbf{b} \times \mathbf{u} + \frac{1}{\varepsilon^{2M}} \nabla_x p(\varrho)$$

$$= \varepsilon^R \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^{2F}} \nabla_x G$$

Path dependent singular limit

$$\varepsilon \to 0, \text{ certain relation between } M, R, F > 0$$

- low Mach $\Rightarrow$ compressible $\to$ incompressible
- high Rossby $\Rightarrow$ 3D $\to$ 2D
- high Reynolds $\Rightarrow$ viscous $\to$ inviscid

Convergence to singular limit system

Target problem - Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \ \mathrm{div}_x \mathbf{v} = 0, \ x \in R^2$$

Convergence results (EF, Lu, Novotný 2014)

- **Spatial geometry - infinite strip:**

$$\Omega = R^2 \times (0, \pi)$$

- **Complete slip (Navier) boundary conditions:**

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$$

Limits on domains with variable geometry

Channel like domains

$$\Omega_\varepsilon = \left\{ (\mathbf{x}, z) \ \middle| \ z \in (0,1), \ |\mathbf{x} - \varepsilon \mathbf{X}(z)|^2 < \varepsilon^2 R^2(z) \right\}, \ |\mathbf{X}(z)| < R(z)$$

Boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_\Sigma = 0, \ (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_\Sigma = 0$$

$$\Sigma = \partial\Omega \cap \{z \in (0,1)\}$$

$$\mathbf{u}|_{z=0,1} = 0$$

Target systems

Inviscid limit

$$\partial_t(\varrho_E A) + \partial_z(\varrho_E u_E A) = 0$$

$$\partial_t(\varrho_E u_E A) + \partial_z(\varrho_E u_E^2 A) + A\partial_z p(\varrho_E) = 0$$

Viscous limit

$$\partial_t(\varrho_{NS} A) + \partial_z(\varrho_{NS} u_{NS} A) = 0$$

$$\partial_t(\varrho_{NS} u_{NS} A) + \partial_z(\varrho_{NS} u_{NS}^2 A) + A\partial_z p(\varrho_{NS})$$

$$= A\nu\partial_z^2 u_{NS} + \nu\partial_z\left(R'(z)/R(z)u_{NS}\right), \ \nu = \frac{4}{3}\mu + \eta > 0$$

$$A = R^2$$

Convergence

Korn-Poincaré inequality

$$\int_{\Omega_\varepsilon} |\mathbf{v}|^2 \, \mathrm{d}x \leq c_{KP} \int_{\Omega_\varepsilon} \left| \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} \right|^2 \, \mathrm{d}x$$

Convergence (Bella, EF, Lewicka, Novotný 2015)

- Convergence to the target Euler system with geometric terms in the inviscid limit
- Convergence to the Navier-Stokes system in the viscous limit provided the bulk viscosity in the primitive system is positive

Navier-Stokes system driven by stochastic forces

Navier–Stokes system with stochastic forcing

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\ \mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + [\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)]\ \mathrm{d}t = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\ \mathrm{d}t + \mathbb{G}(\varrho, \varrho \mathbf{u}) \mathrm{d}W,$$

White–noise forcing

$$\mathbb{G}(\varrho, \varrho \mathbf{u})\ \mathrm{d}W = \sum_{k \geq 1} \mathbf{G}_k(\varrho, \varrho \mathbf{u})\ \mathrm{d}W_k.$$

Relative energy inequality

Relative energy inequality - (Breit, EF, Hofmanová 2015)

$$-\int_0^T \partial_t \psi \, \mathcal{E}\left(\varrho, \mathbf{u} \Big| r, \mathbf{U}\right) \, \mathrm{d}t$$

$$+\int_0^T \psi \int_\Omega (\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \psi(0)\mathcal{E}\left(\varrho, \mathbf{u} \Big| r, \mathbf{U}\right)(0) + \int_0^T \psi \mathrm{d}M_{RE} + \int_0^T \psi \mathcal{R}\left(\varrho, \mathbf{u} \Big| r, \mathbf{U}\right) \mathrm{d}t$$

$$\psi \in C_c^\infty[0, T) \text{ (deterministic)}, \ \psi \geq 0.$$

Test functions

$$\mathrm{d} \, r = D_t^d r \, \mathrm{d}t + D_t^s r \, \mathrm{d}W, \ \mathrm{d}\mathbf{U} = D_t^d \mathbf{U} \, \mathrm{d}t + D_t^s \mathbf{U} \, \mathrm{d}W$$

Stochastic remainder

Remainder

$$\mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)$$

$$= \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, \mathrm{d}x + \int_\Omega \varrho \Big(D_t^d \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x$$

$$+ \int_\Omega \Big((r - \varrho) H''(r) D_t^d r + \nabla_x H'(r)(r\mathbf{U} - \varrho \mathbf{u}) \Big) \, \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U} (p(\varrho) - p(r)) \, \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_\Omega \varrho \left| \frac{\mathbf{G}_k(\varrho, \varrho \mathbf{u})}{\varrho} - D_t^s \mathbf{U}_k \right|^2 \mathrm{d}x$$

$$+ \frac{1}{2} \sum_{k \geq 1} \int_\Omega \varrho H'''(r) |D_t^s r_k|^2 \, \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_\Omega p''(r) |D_t^s r_k|^2 \, \mathrm{d}x$$

Results for stochastic Navier-Stokes system

Weak–strong uniqueness (Breit, EF, Hofmanová 2015)

- Pathwise weak-strong uniqueness
- Weak-strong uniqueness in law

Inviscid–incompressible limit in the stochastic setting (Breit, EF, Hofmanová 2015)

Convergence to the limit stochastic Euler system for vanishing viscosity and the Mach number. Results for well-prepared data.

Possible extensions

Numerical analysis (Gallouet, Herbin, Maltese, Novotný 2014)

Relative energy inequality for the numerical scheme proposed by K.Karlsen and T. Karper. Error estimates.

Measure–valued solutions

Weak–strong uniqueness for measure-valued solutions (**EF, Gwiazda, Swierczewska-Gwiazda, Wiedemann 2015**)

Preliminaries to measure-valued solutions

Families of integrable solutions

$$[\varrho_n, \mathbf{u}_n] : \underbrace{(0,T)\times\Omega}_{\text{physical space}} \mapsto \underbrace{[0,\infty)\times R^N}_{\text{phase space}}$$

$$\{\varrho_n\},\ \{\mathbf{u}_n\} \text{ bounded in } L^1((0,T)\times\Omega)$$

Nonlinear compositions - Young measure

$$F \text{ bounded continuous, } F(\varrho_n, \mathbf{u}_n) \to \overline{F(\varrho, \mathbf{u})} \text{ weakly in } L^1((0,T)\times\Omega)$$

$$\Rightarrow\ \overline{F(\varrho, \mathbf{u})} = \langle \nu_{t,x}; F(s, \mathbf{v})\rangle \text{ for a.a. } (t,x)$$

Biting limit

$$\int_0^T \int_\Omega |F(\varrho_n, \mathbf{u}_n)\ \mathrm{d}x\ \mathrm{d}t \leq c \ \Rightarrow\ \langle \nu_{t,x}; F(s,\mathbf{v})\rangle \in L^1((0,T)\times\Omega)$$

Biting limit decomposition

Bounded integrable compositions

$$\int_0^T \int_\Omega |F(\varrho_n, \mathbf{u}_n)| \, \mathrm{d}x \, \mathrm{d}t \leq c$$

$$\underbrace{\Rightarrow}_{\text{up to a subsequence}}$$

$$F(\varrho_n, \mathbf{u}_n) \to \overline{F(\varrho, \mathbf{u})} \text{ weakly-(*) in } \mathcal{M}([0,T] \times \overline{\Omega})$$

Biting limit decomposition

$$\overline{F(\varrho, \mathbf{u})} = \underbrace{\overline{F(\varrho, \mathbf{u})} - \langle \nu_{t,x}; F(s, \mathbf{v}) \rangle}_{\text{concentration part}} + \underbrace{\langle \nu_{t,x}; F(s, \mathbf{v}) \rangle}_{\text{oscillatory part}}$$

Measure-valued solutions

Parameterized (Young) measure

$$\nu_{t,x} \in L^\infty_{\rm weak}((0,T)\times\Omega; \mathcal{P}([0,\infty)\times R^N)),\ [s,\mathbf{v}] \in [0,\infty)\times R^N$$

$$\varrho(t,x) = \langle \nu_{t,x}; s \rangle\ ,\ \mathbf{u} = \langle \nu_{t,x}; \mathbf{v} \rangle$$

Navier-Stokes/Euler, velocity/momentum

Navier-Stokes $\mathbf{u} \in L^2(0,T; W^{1,2}_0(\Omega; R^N))$,

Euler $\mathbf{u} \approx \mathbf{m} \approx \varrho\mathbf{u}$

Initial data

$$\nu_0 = \nu_{0,x}$$

Regular initial data

$$\nu_{0,x} = \delta_{\varrho_0(x),\mathbf{u}_0(x)} \text{ for a.a. } x$$

Field equations

Equation of continuity

$$\left[\int_{\Omega} \langle \nu_{t,x}, s \rangle \, \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \langle \nu_{t,x}; s \rangle \, \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t + \langle R_1; \nabla_x \varphi \rangle$$

Momentum balance

$$\left[\int_{\Omega} \langle \nu_{t,x}, s\mathbf{v} \rangle \, \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle : \nabla_x \varphi + \langle \nu_{t,x}; p(s) \rangle \, \mathrm{div}_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t + \langle R_2; \nabla_x \varphi \rangle$$

Dissipativity

Energy inequality

$$\left[\int_\Omega \left\langle \nu_{\tau,x}; \left(\frac{1}{2}s|\mathbf{v}|^2 + P(s)\right)\right\rangle \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$+\int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t + \boxed{\mathcal{D}(\tau)} \leq 0$$

Compatibility

$$\left|R_1[0,\tau] \times \overline{\Omega}\right| + \left|R_2[0,\tau] \times \overline{\Omega}\right| \leq \xi(\tau)\mathcal{D}(\tau), \ \xi \in L^1(0,T)$$

$$\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; |\mathbf{v} - \mathbf{u}|^2 \right\rangle \ \mathrm{d}x \ \mathrm{d}t \leq c_P \mathcal{D}(\tau)$$

Truly measure-valued solutions

Truly measure-valued solutions for the Euler system (EF, Chiodaroli, Kreml, Wiedemann)

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded L^p weak solutions to the Euler system.

Do we need measure valued solutions?

Limits of problems with higher order viscosities

Multipolar fluids with complex rheologies (Nečas - Šilhavý)

$$\mathbb{T}(\mathbf{u}, \nabla_x \mathbf{u},\ \nabla_x^2 \mathbf{u}, \dots)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) + \delta \sum_{j=1}^{k-1} \left((-1)^j \mu_j \Delta^j (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) + \lambda_j \Delta^j \mathrm{div}_x \mathbf{u}\ \mathbb{I} \right)$$

$$+ \text{ non-linear terms}$$

Limit for $\delta \to 0$

Limits of numerical solutions

Numerical solutions resulting from Karlsen-Karper and other schemes

Sub-critical parameters

$p(\varrho) = a\varrho^\gamma$, $\gamma < \gamma_{\mathrm{critical}}$

Weak (mv) - strong uniqueness

Theorem - EF, Gwiazda, Swierczewska-Gwiazda, Wiedemann 2015

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

Relative energy (entropy)

Relative energy functional

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)(\tau)$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v} - \mathbf{U}|^2 + P(s) - P'(r)(s-r) - P(r) \right\rangle \, \mathrm{d}x$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v}|^2 + P(s) \right\rangle \, \mathrm{d}x - \int_\Omega \langle \nu_{\tau,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \, \mathrm{d}x$$

$$+ \int_\Omega \frac{1}{2} \langle \nu_{\tau,x}; s \rangle \, |\mathbf{U}|^2 \, \mathrm{d}x$$

$$- \int_\Omega \langle \nu_{\tau,x}; s \rangle \, P'(r) \, \mathrm{d}x + \int_\Omega p(r) \, \mathrm{d}x$$

Relative energy (entropy) inequality

Relative energy inequality

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) + \int_0^\tau \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t + \mathcal{D}(\tau)$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}_0|^2 + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle \ \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \ \mathrm{d}t$$

Remainder

$$\mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)$$

$$= - \int_0^\tau \int_\Omega \langle \nu_{t,x}, s\mathbf{v} \rangle \cdot \partial_t \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_{\overline{\Omega}} [\langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(s) \rangle \,\mathrm{div}_x \mathbf{U}] \,\mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega [\langle \nu_{t,x}; s \rangle \, \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U}] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left[\left\langle \nu_{t,x}; \left(1 - \frac{s}{r}\right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \left\langle R_1; \frac{1}{2} \nabla_x \left(|\mathbf{U}|^2 - P'(r) \right) \right\rangle \ \mathrm{d}t - \int_0^\tau \langle R_2; \nabla_x \mathbf{U} \rangle \,\mathrm{d}t$$

Regularity

Theorem - EF, Gwiazda, Swierczewska-Gwiazda, Wiedemann 2015

Suppose that the initial data are smooth and satisfy the relevant compatibility conditions. Let $\nu_{t,x}$ be a measure-valued solution to the compressible Navier-Stokes system with a dissipation defect $\mathcal{D}$ such that

$$\mathrm{supp}\ \nu_{t,x} \subset \left\{ (s, \mathbf{v}) \ \middle| \ 0 \leq s \leq \overline{\varrho},\ \mathbf{v} \in R^N \right\}$$

for a.a. $(t, x) \in (0, T) \times \Omega$.
Then $\mathcal{D} = 0$ and

$$\nu_{t,x} = \delta_{\varrho(t,x), \mathbf{u}(t,x)}$$

where ϱ, $\mathbf{u}$ is a smooth solution.

Sketch of the proof

- The Navier-Stokes system admits a local-in-time smooth solution
- The measure-valued solution coincides with the smooth solution on its life-span
- The smooth solution density component remains bounded by $\overline{\varrho}$ as long as the solution exists
- Y. Sun, C. Wang, and Z. Zhang [2011]: The strong solution can be extended as long as the density component remains bounded

Corollary

Convergence of numerical solutions

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution

Convex integration - DeLellis and Shékelyhidi

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v}\otimes\mathbf{v}) + \nabla_x \Pi = 0$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0,\ x \in \mathcal{T}^N \ \text{(periodic b.c.)}$$

Reformulation

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x\left(\mathbf{v}\otimes\mathbf{v} - \frac{1}{N}|\mathbf{v}|^2\mathbb{I}\right) + \nabla_x \Pi = 0$$

Linear system vs. non-linear constitutive equation

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

$$\mathbf{U} = \mathbf{v}\otimes\mathbf{v} - \frac{1}{N}|\mathbf{v}|^2\mathbb{I},\ \mathbb{U} \in R^{N\times N}_{0,\mathrm{sym}}$$

Constitutive constraint relaxation

Goal

$$\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\frac{1}{N}|\mathbf{v}|^2\mathbb{I}-\mathbb{U}\right]=0$$

Relaxation

$$\frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]\geq\frac{1}{2}|\mathbf{v}|^2$$

Goal reformulated

$$\boxed{\frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v}-\mathbb{U}\right]=\frac{1}{2}|\mathbf{v}|^2}\Leftrightarrow\mathbb{U}=\mathbf{v}\otimes\mathbf{v}-\frac{1}{N}|\mathbf{v}|^2\mathbb{I}$$

Concept of subsolution

Equations

$\mathbf{v}$, $\mathbb{U}$ smooth in $(0, T)$

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

Extremal values

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0,\ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Energy

piece-wise smooth function e

Convex set

$$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U}\right] < e \text{ in } (0, T) \times \mathcal{T}^N$$

Oscillatory lemma - DeLellis and Shékelyhidi

Basic subsolution

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \ \mathrm{div}_x \mathbf{v} = 0$$

Oscillatory increments

$$\mathrm{div}_x \mathbf{w}_\varepsilon = 0, \ \partial_t \mathbf{w}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0$$

$$\mathbf{w}_\varepsilon, \ \mathbb{V}_\varepsilon \in C_c^\infty(Q)$$

$$\mathbf{w}_\varepsilon \to 0 \text{ weakly in } L^2(Q)$$

$$\lambda_{\max}\left[(\mathbf{v}+\mathbf{w}_\varepsilon)\otimes(\mathbf{v}+\mathbf{w}_\varepsilon) - (\mathbb{U}+\mathbb{V}_\varepsilon)\right] < e$$

Energy

$$\liminf_{\varepsilon\to 0} \int_Q \left(|\mathbf{v}+\mathbf{w}_\varepsilon|^2\right) \geq \int_Q |\mathbf{v}|^2 + c\int_Q \left(e - \frac{1}{2}|\mathbf{v}|^2\right)^\alpha$$

Infinitely many solutions

Baire category argument

- There exists e such that the set of subsolutions is non-empty
- The set of subsolutions is a complete metric space with $C_{\rm weak}([0,T]; L^2(\mathcal{T}^N; R^N))$ topology
- Points of continuity of the l.s.c. functional

$$\int_Q \left(\frac{1}{2} |\mathbf{v}|^2 - e \right) \, {\rm d}x$$

correspond to weak solutions of the Euler system

Infinitely many solutions

$${\rm div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + {\rm div}_x (\mathbf{v} \otimes \mathbf{v}) - \boxed{\nabla_x \left(\frac{1}{N} |\mathbf{v}|^2 \right)} = 0$$

Control of the limit pressure

Pressure

$$\frac{1}{2}|\mathbf{v}|^2 = e,\ \ p = -\frac{1}{N}|\mathbf{v}|^2 = -\frac{2}{N}e \text{ in } (0,T) \times \mathcal{T}^N$$

Savage-Hutter model for avalanches

joint work with Gwiazda and Świerczewska-Gwiazda

Unknowns

flow height .... $h = h(t, x)$
depth-averaged velocity .... $\mathbf{u} = \mathbf{u}(t, x)$

$$\partial_t h + \operatorname{div}_x(h\mathbf{u}) = 0$$

$$\partial_t(h\mathbf{u}) + \operatorname{div}_x(h\mathbf{u} \otimes \mathbf{u}) + \nabla_x(ah^2) = h\left(-\gamma\frac{\mathbf{u}}{|\mathbf{u}|} + \mathbf{f}\right)$$

Periodic boundary conditions

$$\Omega = \left([0,1]|_{\{0,1\}}\right)^2$$

Transformation - Step I

Helmholtz decomposition

$$h\mathbf{u} = \mathbf{v} + \mathbf{V} + \nabla_x \Psi$$

where

$$\mathrm{div}_x \mathbf{v} = 0,\ \int_\Omega \Psi \ \mathrm{d}x = 0,\ \int_\Omega \mathbf{v} \ \mathrm{d}x = 0,\ \mathbf{V} \in R^2$$

Fixing h and the potential Ψ

$$\partial_t h + \Delta \Psi = 0$$

$$h(0,\cdot) = h_0,\ -\partial_t h(0,\cdot) = \Delta \Psi_0$$

Problem I

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Psi) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Psi)}{h} + \left(a h^2 + \partial_t \Psi \right) \mathbb{I} \right)$$

$$+ \partial_t \mathbf{V}$$

$$= h \left(-\gamma \frac{\mathbf{v} + \mathbf{V} + \nabla_x \Psi}{|\mathbf{v} + \mathbf{V} + \nabla_x \Psi|} + \mathbf{f} \right),$$

Constraints and initial conditions

$$\mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \mathbf{v}(t, \cdot) \ \mathrm{d}x = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{V}(0) = \mathbf{V}_0$$

Transformation - Step II

Prescribing the kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h} = E \equiv \Lambda(t) - ah^2 - \partial_t\Psi$$

Problem II

$$\partial_t\mathbf{v}+\partial_t\mathbf{V}$$

$$+\mathrm{div}_x\left(\frac{(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)\otimes(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)}{h} - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h}\mathbb{I}\right)$$

$$= -\gamma\left(\frac{h}{2E}\right)^{1/2}(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)+h\mathbf{f}$$

Transformation - Step III

Determining function V

$$\partial_t \mathbf{V} - \left[\frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} \mathrm{d}x \right] \mathbf{V}$$

$$= \frac{1}{|\Omega|} \int_\Omega \left[\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \nabla_x \Psi) + h\mathbf{f} \right] \mathrm{d}x, \ \mathbf{V}(0) = \mathbf{V}_0$$

Problem III

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \odot (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)}{h} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \, \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \, \mathrm{d}x$$

$$\mathbf{v} \odot \mathbf{w} = \mathbf{v} \otimes \mathbf{w} - \frac{1}{2} \mathbf{v} \cdot \mathbf{w} \mathbb{I}$$

Transformation - Step IV

Solving elliptic problem

$$\mathrm{div}_x \mathbb{M} \equiv \mathrm{div}_x \left(\nabla_x \mathbf{m} + \nabla_x^t \mathbf{m} - \mathrm{div}_x \mathbf{m} \mathbb{I} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} \left(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi \right)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} \left(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi \right) \, \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \, \mathrm{d}x,$$

$$\int_\Omega \mathbb{M}(t, \cdot) \, \mathrm{d}x = 0 \text{ for any } t \in [0, T].$$

Abstract formulation

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Abstract operators

Boundedness

b maps bounded sets in $L^\infty((0,T)\times\Omega;R^N)$ on bounded sets in $C_b(Q,R^M)$

Continuity

$$b[\mathbf{v}_n] \to b[\mathbf{v}] \text{ in } C_b(Q;R^M) \text{ (uniformly for } (t,x)\in Q \text{)}$$

whenever

$$\mathbf{v}_n \to \mathbf{v} \text{ in } C_{\rm weak}([0,T];L^2(\Omega;R^N))$$

Causality

$\mathbf{v}(t,\cdot) = \mathbf{w}(t,\cdot)$ for $0\le t\le\tau\le T$ implies $b[\mathbf{v}] = b[\mathbf{w}]$ in $[(0,\tau]\times\Omega]$

Subsolutions

Field equations, differential constraints

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0, \ \mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Non-linear constraint

$$\mathbf{v} \in C(Q; R^N), \ \mathbb{F} \in C(Q; R^{N\times N}_{\mathrm{sym},0}),$$

$$\frac{N}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \otimes (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}] \right] < E[\mathbf{v}]$$

Subsolution relaxation

Algebraic inequality

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} \leq \frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\otimes(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}]\right]$$

$$< E[\mathbf{v}]$$

Solutions

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

$$\Rightarrow$$

$$\mathbb{F} = \frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\odot(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}]$$

Oscillatory lemma

Hypotheses:

$$U \subset R \times R^N, \ N = 2,3 \text{ bounded open set}$$

$$\tilde{\mathbf{h}} \in C(U; R^N), \ \tilde{\mathbb{H}} \in C(U; R^{N\times N}_{\mathrm{sym},0}), \ \tilde{e}, \ \tilde{r} \in C(U), \ \tilde{r} > 0, \ \tilde{e} \le \overline{e} \text{ in } U$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{\tilde{\mathbf{h}} \otimes \tilde{\mathbf{h}}}{\tilde{r}} - \tilde{\mathbb{H}}\right] < \tilde{e} \text{ in } U.$$

Conclusion:

$$\mathbf{w}_n \in C_c^\infty(U; R^N),\ \mathbb{G}_n \in C_c^\infty(U; R^{N\times N}_{\mathrm{sym},0}),\ n = 0, 1, \dots$$

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{G}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\tilde{\mathbf{h}} + \mathbf{w}_n)\otimes(\tilde{\mathbf{h}} + \mathbf{w}_n)}{\tilde{r}} - (\tilde{\mathbb{H}} + \mathbb{G}_n)\right] < \tilde{e} \text{ in } U,$$

$$\mathbf{w}_n \to 0 \text{ weakly in } L^2(U; R^N)$$

$$\liminf_{n\to\infty} \int_U \frac{|\mathbf{w}_n|^2}{\tilde{r}}\ \mathrm{d}x\mathrm{d}t \geq \Lambda(\overline{e}) \int_U \left(\tilde{e} - \frac{1}{2}\frac{|\tilde{\mathbf{h}}|^2}{\tilde{r}}\right)^2\ \mathrm{d}x\mathrm{d}t$$

Basic ideas of proof

Localization

Localizing the result of DeLellis and Széhelyhidi to “small” cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the “small” cubes

Eliminating singular sets

Applying Whitney’s decomposition lemma to the non-singular sets (e.g. out of the vacuum $\{h = 0\}$)

Energy and other coefficients depending on solutions

Applying compactness of the abstract operators in C

Results

Result (A)

The set of subsolutions is non-empty $\Rightarrow$ there exists infinitely many weak solutions of the problem with the same initial data

Initial energy jump

$$\frac{1}{2}\frac{|\mathbf{v}_0 + \mathbf{H}[\mathbf{v}_0]|^2}{h[\mathbf{v}_0]} \boxed{<} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Result (B)

The set of subsolutions is non-empty $\Rightarrow$ there exists a dense set of times such that the values $\mathbf{v}(t)$ give rise to non-empty subsolution set with

$$\frac{1}{2}\frac{|\mathbf{v}_0 + \mathbf{H}[\mathbf{v}_0]|^2}{h[\mathbf{v}_0]} \boxed{=} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Application to Savage-Hutter model

Theorem

(i) Let the initial data

$$h_0 \in C^2(\Omega),\ \mathbf{u}_0 \in C^2(\Omega; R^2), h_0 > 0 \text{ in } \Omega$$

be given, and let $\mathbf{f}$ and a be smooth.
Then the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$.

(ii) Let $T > 0$ and

$$h_0 \in C^2(\Omega),\ h_0 > 0$$

be given.
Then there exists

$$\mathbf{u}_0 \in L^\infty(\Omega; R^2)$$

such that the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$ satisfying the energy inequality.

Example II, Euler-Fourier system

(joint work with E.Chiodaroli and O.Kreml [2014])

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\left[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{u})\right] - \Delta \vartheta = -\varrho \vartheta \mathrm{div}_x \mathbf{u}$$

Example III, Euler-Korteweg-Poisson system

(joint work with D.Donatelli and P.Marcati [2014])

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)$$

$$= \boxed{\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho)|\nabla_x \varrho|^2 \right)} - \varrho \mathbf{u} + \varrho \nabla_x V$$

Poisson equation

$$\Delta_x V = \varrho - \overline{\varrho}$$

Example IV, Euler-Cahn-Hilliard system

Model by Lowengrub and Truskinovsky

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard equation

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x (\varrho \nabla_x c) \right)$$

Example V, models of collective behavior

(joint work with J.A. Carrillo, P.Gwiazda, A.Swierczewska–Gwiazda)

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})$$

$$= -p(\varrho) + \left(1 - H\left(|\mathbf{u}|^2\right)\right) \varrho \mathbf{u}$$

$$-\varrho \nabla_x K * \varrho + \varrho \psi * \left[\varrho \Big(\mathbf{u} - \mathbf{u}(x) \Big) \right]$$

Stochastically driven Euler system

Field equations

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p(\varrho)\mathrm{d}t = \varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W,$$

Stochastic forcing

$$\varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \sum_{k=1}^{\infty} \varrho \mathbf{G}_k(\varrho, \varrho \mathbf{u})\mathrm{d}\beta_k$$

Iconic examples

$$\varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \varrho \sum_{k=1}^{\infty} \mathbf{G}_k(x)\mathrm{d}\beta_k, \ \ \varrho \mathbf{G}(\varrho, \varrho \mathbf{u})\mathrm{d}W = \lambda \varrho \mathbf{u} \mathrm{d}\beta$$

Weak (PDE) solutions

Infinitely many weak (PDE) solutions, Breit, EF, Hofmanová [2017]

Let $T > 0$ and the initial data

$$\varrho_0 \in C^3(\Omega),\ \ \varrho_0 > 0,\ \ \mathbf{u}_0 \in C^3(\Omega)$$

be given.
There exists a sequence of *strictly positive* stopping times

$$\tau_M > 0,\ \ \tau_M \to \infty$$

a.s. such that the initial–value problem for the compressible Euler system possesses infinitely many solutions defined in $(0, T \wedge \tau_M)$. Solutions are adapted to the filtration associated to the Wiener process W.