SOME NOTES ON OSCILLATION OF TWO-DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. Oscillatory properties of solutions to the system of first-order linear difference equations

$$\Delta u_k = q_k v_k$$
$$\Delta v_k = -p_k u_{k+1},$$

are studied. It can be regarded as a discrete analogy of the linear Hamiltonian system of differential equations.

We establish some new conditions, which provide oscillation of the considered system. Obtained results extend and improve, in certain sense, results presented in Opluštil (2011).

Keywords: two-dimensional system; linear difference equation; oscillatory solution

MSC 2010: 39A10, 39A21

1. Introduction

We consider the two-dimensional system of linear difference equations

$$\begin{aligned} \Delta u_k &= q_k v_k \\ \Delta v_k &= -p_k u_{k+1}, \end{aligned} \tag{1.1}$$

where

$$\Delta x_k = x_{k+1} - x_k, \quad p_k, q_k \in \mathbb{R} \quad \text{for } k \in \mathbb{N}.$$

By a solution of system (1.1) we understand a vector sequence $\{(u_k, v_k)\}_{k=1}^{\infty}$ satisfying system (1.1) for every natural k.

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System (1.1) is a possible, the best one in certain sense, discrete analogy of the linear Hamiltonian system of differential equations

$$\begin{aligned} u' &= q(t)v \\ v' &= -p(t)u, \end{aligned}$$

and discrete analogy of the second-order linear differential equation

$$\Big(u'\frac{1}{q(t)}\Big)' + p(t)u = 0.$$

Oscillation theory for linear ordinary differential equations is a widely studied and well-developed topic of the general theory of differential equations. We should mention, in particular, the results which are closely related to those of this paper, see e.g., [4], [2], [5], [6], [7], [9]. On the other hand, there are many interesting and open problems in the difference case.

Definition 1.1. A nontrivial solution $\{(u_k, v_k)\}_{k=1}^{\infty}$ of system (1.1) is said to be oscillatory if there exists an infinite set $\mathbb{N}_0 \subseteq \mathbb{N}$ such that

$$u_k u_{k+1} \leqslant 0 \quad \text{for } k \in \mathbb{N}_0.$$

If the sequence $\{q_k\}^{\infty}$ is nonnegative and system (1.1) has at least one oscillatory solution, then it is known (see, e.g., [1]) that all solutions of (1.1) are oscillatory. So it is possible to introduce the following definition.

Definition 1.2. System (1.1) is said to be oscillatory if all its solutions are oscillatory, it is said to be and nonoscillatory otherwise.

Remark 1.1. Oscillatory properties of system (1.1) are known in the case where

$$0 < m \leqslant q_k \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad \sum_{j=1}^{\infty} p_j = \infty$$

or in the case where the following conditions

$$0 < m \leqslant q_k \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad -\infty = \liminf_{k\to\infty} \sum_{j=1}^{k} p_j < \limsup_{k\to\infty} \sum_{j=1}^{k} p_j$$

are fulfilled (see, e.g., [1]). System (1.1) is oscillatory in both cases above. We note that the original version (for the second-order linear differential equation) of these oscillation criteria can be found in [3], [10], [11].

We can see that one of the cases which is not covered in the above mentioned criteria is that $\sum_{j=1}^{\infty} p_j$ converges to a finite number, i.e.,

$$\sum_{j=1}^{\infty} p_j = c_0, \tag{1.2}$$

where $c_0 \in \mathbb{R}$. In this case, some oscillatory criteria are presented in [8]. Actually, we build on the work done in [8] and we establish new conditions, which guarantee that system (1.1) is oscillatory.

Consequently, in what follows, we assume (1.2) is fulfilled and the sequence $\{q_k\}^{\infty}$ is bounded, i.e.,

$$0 < m \leqslant q_k \leqslant M < \infty \quad \text{for } k \in \mathbb{N}, \tag{1.3}$$

where m, M are real positive constants.

Note that, since $\sum_{j=1}^{\infty} p_j$ converges to a finite number, there exists

$$\lim_{k\to\infty} c_k = c_0,$$

where

$$c_k = \frac{1}{\sum_{j=1}^{k-1} q_j} \sum_{j=1}^{k-1} q_j \sum_{i=1}^{j-1} p_i \quad \text{for } k \in \mathbb{N}. \tag{1.4}$$

Let us introduce the following notations for simpler formulation of the main results:

$$Q_k = \left(c_0 - \sum_{j=1}^{k-1} p_j\right) \sum_{j=1}^{k-1} q_j = \sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} p_j \quad \text{for } k \in \mathbb{N}, \tag{1.5}$$

$$H_k = \frac{1}{\sum_{j=1}^{k-1} q_j} \sum_{j=1}^{k-1} p_j \left(\sum_{i=1}^{j} q_i\right)^2 \quad \text{for } k \in \mathbb{N}, \tag{1.6}$$

$$\begin{aligned} Q_* &= \liminf_{k\to\infty} Q_k, \qquad Q^* = \limsup_{k\to\infty} Q_k, \\ H_* &= \liminf_{k\to\infty} H_k, \qquad H^* = \limsup_{k\to\infty} H_k. \end{aligned} \tag{1.7}$$

2. Main results

The statements formulated below complement criteria established in [8] and can be regarded as a difference analogy of oscillatory theorems for ordinary differential equations presented in [9].

Theorem 2.1. *Let* $Q_* > -\infty$ *and*

$$\limsup_{k\to\infty} \frac{\sum\limits_{j=1}^{k-1} p_j \sum\limits_{i=1}^{j} q_i}{\sum\limits_{j=1}^{k} \Big(q_j \Big/ \sum\limits_{i=1}^{j} q_i\Big)} > \frac{1}{4}. \tag{2.1}$$

Then system (1.1) *is oscillatory.*

R e m a r k 2.1. It follows from the proof of Theorem 2.1 (see bellow), particulary from (4.13), that a sufficient condition for the system (1.1) to be oscillatory has also the form

$$\limsup_{k\to\infty} \frac{(c_0 - c_k) \sum\limits_{j=1}^{k-1} q_j}{\sum\limits_{j=k_0}^{k-1} \Big(q_j \Big/ \sum\limits_{i=1}^{j} q_i\Big)} > \frac{1}{4}.$$

Theorem 2.2. *Let*

$$\limsup_{k\to\infty} (Q_k + H_k) > 1. \tag{2.2}$$

Then system (1.1) *is oscillatory.*

Theorem 2.3. *Let the conditions*

$$0 \leqslant Q_* \leqslant \frac{1}{4} \quad \text{and} \quad 0 \leqslant H_* \leqslant \frac{1}{4}$$

be fulfilled and let either

$$Q^* > Q_* + \frac{1}{2}\big(\sqrt{1 - 4Q_*} + \sqrt{1 - 4H_*}\big), \tag{2.3}$$

or

$$H^* > H_* + \frac{1}{2}\big(\sqrt{1 - 4Q_*} + \sqrt{1 - 4H_*}\big). \tag{2.4}$$

Then system (1.1) *is oscillatory.*

R e m a r k 2.2. The condition (2.4) improves, under the additional assumption $0 \leqslant H_* \leqslant 1/4$, the second inequality of

$$0 \leqslant Q_* \leqslant \frac{1}{4} \quad \text{and} \quad H^* > \frac{1}{2}\big(1 + \sqrt{1 - 4Q_*}\big) \tag{2.5}$$

presented in [8], Theorem 2.1, which also guarantees oscillation of system (1.1).

Indeed, if we put $H_* = 0$ in (2.4) then we get exactly the second inequality in (2.5). Moreover, for $0 < H_* \leqslant 1/4$, the condition (2.4) improves the second inequality in (2.5). Analogically, the condition (2.3) improves the condition (5) in [8], Theorem 2.2, under the additional assumption $0 \leqslant Q_* \leqslant 1/4$.

R e m a r k 2.3. All the above statements can be regarded as discrete analogies of known results for two-dimensional system of differential equations (see [9], Theorem 1.2, Corollary 1.1, Theorem 1.3, Theorem 1.5).

3. Auxiliary propositions

In [8], the following properties and estimates of nonoscilatory solutions of system (1.1) were established. The lemmas presented below presented lemmas are used to prove the main results.

Lemma 3.1 ([8], Lemma 3.1). *Let* $\{(u_k, v_k)\}^\infty$ *be a nonoscillatory solution of system* (1.1). *Then*

$$\sum^{\infty} R_j < \infty,$$

where

$$w_j = \frac{v_j}{u_j} \quad \textit{and} \quad R_j = \frac{w_j^2 q_j}{1 + w_j q_j}. \tag{3.1}$$

Lemma 3.2 ([8], Lemma 3.2). *Let* $0 \leqslant Q_* \leqslant 1/4$ *and* $\{(u_k, v_k)\}^\infty$ *be a nonoscillatory solution of system* (1.1). *Then*

$$\liminf_{k\to\infty} \frac{v_k}{u_k} \sum_{j=1}^{k-1} q_j \geqslant \frac{1}{2}\big(1 - \sqrt{1 - 4Q_*}\big).$$

Lemma 3.3 ([8], Lemma 3.3). *Let* $0 \leqslant H_* \leqslant 1/4$ *and* $\{(u_k, v_k)\}^\infty$ *is a nonoscillatory solution of system* (1.1). *Then*

$$\limsup_{k\to\infty} \frac{v_k}{u_k} \sum_{j=1}^{k-1} q_j \leqslant \frac{1}{2}\big(1 + \sqrt{1-4H_*}\big).$$

4. Proofs of main results

Proof of Theorem 2.1. Let us suppose on the contrary that system (1.1) is nonoscillatory. Then there exists a solution $\{u_k, v_k\}^\infty$ of (1.1) and $k_0 \in \mathbb{N}$ such that

$$u_k u_{k+1} > 0 \quad \text{for } k \geqslant k_0.$$

If we put $w_k = v_k/u_k$ for $k \geqslant k_0$, then system (1.1) can be rewritten as

$$\Delta w_k + p_k + R_k = 0 \quad \text{for } k \geqslant k_0, \tag{4.1}$$

where R_k is defined by (3.1). Moreover, it is clear that

$$R_k = \frac{w_k^2 q_k}{1 + w_k q_k} \geqslant 0 \quad \text{for } k \geqslant k_0. \tag{4.2}$$

Sum of equality (4.1) from k to l results in

$$w_k - w_{l+1} = \sum_{j=k}^{l} p_j + \sum_{j=k}^{l} R_j \quad \text{for } k \geqslant k_0, \tag{4.3}$$

On the other hand, according to Lemma 3.1 and (1.3) we have

$$\lim_{l\to\infty} w_l = 0. \tag{4.4}$$

Hence, we obtain from (4.3) by letting $l \to \infty$ that

$$w_k = \sum_{j=k}^{\infty} p_j + \sum_{j=k}^{\infty} R_j \quad \text{for } k \geqslant k_0. \tag{4.5}$$

Consequently, by virtue of (1.2), we get

$$w_k = c_0 - \sum_{j=1}^{k-1} p_j + \sum_{j=k}^{\infty} R_j \quad \text{for } k \geqslant k_0.$$

The multiplication of this relation by q_k and the summation from k_0 to $k-1$ lead to

$$\sum_{j=k_0}^{k-1} w_j q_j = c_0 \sum_{j=k_0}^{k-1} q_j - \sum_{j=k_0}^{k-1} q_j \sum_{i=1}^{j-1} p_i + \sum_{j=k_0}^{k-1} q_j \sum_{i=j}^{\infty} R_i \quad \text{for } k > k_0. \tag{4.6}$$

Let us denote

$$C_{k,k_0} = (c_0 - c_k) \sum_{j=1}^{k-1} q_j - c_0 \sum_{j=1}^{k_0-1} q_j \quad \text{for } k > k_0, \tag{4.7}$$

where c_k is defined by (1.4). Now we can write equality (4.6) in the form

$$\sum_{j=k_0}^{k-1} w_j q_j = C_{k,k_0} + \sum_{j=1}^{k_0-1} q_j \sum_{i=1}^{j-1} p_i + \sum_{j=k_0}^{k-1} q_j \sum_{i=j}^{\infty} R_i \quad \text{for } k > k_0. \tag{4.8}$$

It is not difficult to verify that

$$\sum_{j=k_0}^{k-1} q_j \sum_{i=j}^{\infty} R_i = \sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} R_j + \sum_{j=k_0}^{k-1} R_j \sum_{i=1}^{j} q_i - \sum_{j=1}^{k_0-1} q_j \sum_{i=k_0}^{\infty} R_i \quad \text{for } k > k_0$$

and

$$\sum_{j=1}^{k_0-1} q_j \sum_{i=1}^{j-1} p_i = \sum_{j=1}^{k_0-1} q_j \sum_{j=1}^{k_0-1} p_j - \sum_{j=1}^{k_0-1} p_j \sum_{i=1}^{j} q_i \quad \text{for } k > k_0.$$

By using these equalities in (4.8) we obtain

$$\sum_{j=k_0}^{k-1} \left[w_j q_j - R_j \sum_{i=1}^{j} q_1 \right] = C_{k,k_0} + \sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} R_j - \sum_{j=1}^{k_0-1} p_j \sum_{i=1}^{j} q_i + A_{k_0} \quad \text{for } k > k_0, \tag{4.9}$$

where

$$A_{k_0} = \sum_{j=1}^{k_0-1} q_j \left(\sum_{j=1}^{k_0-1} p_j - \sum_{j=k_0}^{\infty} R_j \right).$$

On the other hand, in view of (1.3) and (4.5), A_{k_0} can be rewritten as

$$A_{k_0} = \sum_{j=1}^{k_0-1} q_j \left(c_0 + \sum_{j=k_0-1}^{\infty} \Delta w_j - \Delta w_{k_0-1} \right) = c_0 \sum_{j=1}^{k_0-1} q_j - w_{k_0} \sum_{j=1}^{k_0-1} q_j.$$

Hence, by virtue of (4.7), we get from (4.9) that

$$(4.10)\qquad (c_0 - c_k)\sum_{j=1}^{k-1} = \sum_{j=k_0}^{k-1}\left[w_j q_j - R_j\sum_{i=1}^{j} q_1\right] - \sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} R_j + \widetilde{R} \quad \text{for } k > k_0,$$

where

$$(4.11)\qquad \widetilde{R} = w_{k_0}\sum_{j=1}^{k_0-1} q_j + \sum_{j=1}^{k_0-1} p_j \sum_{i=1}^{j} q_i$$

is a finite number.

Let $\varepsilon > 0$ be arbitrary. Then, in view of relations (1.3) and (4.4), there exists $k_1(\varepsilon) > k_0$ such that

$$(4.12)\qquad |w_k q_k| \leqslant \varepsilon \quad \text{for } k \geqslant k_1(\varepsilon).$$

Obviously,

$$\left(\frac{\sqrt{q_k}w_k}{1+\varepsilon} - \frac{\sqrt{q_k}}{2\sum_{j=1}^{k} q_j}\right)^2 \geqslant 0 \quad \text{for } k \geqslant k_1(\varepsilon).$$

Hence, by using (1.3) and (4.12), we obtain

$$\frac{(1+\varepsilon)}{4}\frac{q_k}{\sum_{j=1}^{k} q_j} \geqslant w_k q_k - R_k\sum_{j=1}^{k} q_j \quad \text{for } k \geqslant k_1(\varepsilon),$$

where R_k is defined by (3.1). In view of the latter inequality, (1.3) and (4.2) we get from (4.10) that

$$(4.13)\qquad (c_0 - c_k)\sum_{j=1}^{k-1} q_j \leqslant \frac{1+\varepsilon}{4}\sum_{j=k_0}^{k-1}\frac{q_j}{\sum_{i=1}^{j} q_i} + \widetilde{R} \quad \text{for } k \geqslant k_1(\varepsilon).$$

Moreover, it follows from (1.3) that

$$(4.14)\qquad \lim_{k\to\infty}\sum_{j=k_0}^{k-1}\frac{q_j}{\sum_{i=1}^{j} q_i} = \infty.$$

Now, in view of (1.2) and (1.4), (4.13) can be rewritten in the form

$$\sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} p_j + \sum_{j=1}^{k-1} p_j \sum_{i=1}^{j} q_i \leqslant \frac{1+\varepsilon}{4}\sum_{j=k_0}^{k-1}\frac{q_j}{\sum_{i=1}^{j} q_i} + \widetilde{R} \quad \text{for } k \geqslant k_1(\varepsilon).$$

Obviously, the last relation yields

$$\frac{\sum\limits_{j=1}^{k-1} p_j \sum\limits_{i=1}^{j} q_i}{\sum\limits_{j=k_0}^{k-1} \Bigl(q_j \Big/ \sum\limits_{i=1}^{j} q_i\Bigr)} \leqslant -\frac{\sum\limits_{j=1}^{k-1} q_j \sum\limits_{j=k}^{\infty} p_j}{\sum\limits_{j=k_0}^{k-1} \Bigl(q_j \Big/ \sum\limits_{i=1}^{j} q_i\Bigr)} + \frac{1+\varepsilon}{4} + \frac{\widetilde{R}}{\sum\limits_{j=k_0}^{k-1} \Bigl(q_j \Big/ \sum\limits_{i=1}^{j} q_i\Bigr)} \quad \text{for } k \geqslant k_1(\varepsilon).$$

Hence, by virtue of the assumption $Q_* > -\infty$, (4.11) and (4.14), we get

$$\limsup_{k\to\infty} \frac{\sum\limits_{j=1}^{k-1} p_j \sum\limits_{i=1}^{j} q_i}{\sum\limits_{j=k_0}^{k-1} \Bigl(q_j \Big/ \sum\limits_{i=1}^{j} q_i\Bigr)} \leqslant \frac{1+\varepsilon}{4}$$

which, since $\varepsilon > 0$ was chosen arbitrary, contradicts (2.1). □

Proof of Theorem 2.2. Let us assume on the contrary that system (1.1) is nonoscillatory. Analogously as in the proof of Theorem 2.1 we obtain equality (4.5). Multiplication of (4.5) by $\sum\limits_{j=1}^{k-1} q_j$ leads to

$$w_k \sum_{j=1}^{k-1} q_j = \sum_{j=k}^{\infty} p_j \sum_{j=1}^{k-1} q_j + \sum_{j=k}^{\infty} R_j \sum_{j=1}^{k-1} q_j \quad \text{for } k > k_0, \tag{4.15}$$

where w_k, R_k are given by (3.1).

On the other hand, we can obtain from (4.1) (see the proof of Lemma 3.3 in [8]) the following equality

$$w_k \Bigl(\sum_{j=1}^{k-1} q_j\Bigr) = -H_k + \frac{1}{\sum_{j=1}^{k-1} q_j} \sum_{j=n}^{k-1} D_J + P_{k,n} \quad \text{for } k > n \geqslant k_0, \tag{4.16}$$

where H_k is defined by (1.6),

$$D_j = w_j q_j \Bigl(2 \sum_{i=1}^{j-1} q_i + q_j\Bigr) - R_j \Bigl(\sum_{i=1}^{j} q_i\Bigr)^2 \tag{4.17}$$

and

$$P_{k,n} = \frac{1}{\sum_{j=1}^{k-1} q_j} \Bigl(\sum_{j=1}^{n-1} q_j\Bigr)^2 w_n + \frac{1}{\sum_{j=1}^{k-1} q_j} \sum_{j=1}^{n-1} p_j \Bigl(\sum_{i=1}^{j} q_i\Bigr)^2. \tag{4.18}$$

Moreover, it is clear that

$$\limsup_{k\to\infty} P_{k,n} = 0. \tag{4.19}$$

Furthermore, the inequality $\Big(w_j\sqrt{q_j}\sum\limits_{i=1}^{j} q_i - (1+w_jq_j)\sqrt{q_j}\Big)^2 \geqslant 0$ implies that

$$D_j \leqslant q_j \quad \text{for } j \geqslant n \geqslant k_0.$$

Using this inequality in (4.16) results in

$$w_k\left(\sum_{j=1}^{k-1} q_j\right) \leqslant -H_k + 1 + P_{k,n} \quad \text{for } k > k_0, \tag{4.20}$$

where $P_{k,n}$ is defined by (4.18).

In view of (1.3) and (4.2), relations (4.15) and (4.20) imply

$$Q_k + H_k \leqslant 1 + P_{k,n} \quad \text{for } k > k_0,$$

where Q_k is defined by (1.5). Hence, by virtue of (4.19), we get

$$\limsup_{k\to\infty}(Q_k + H_k) \leqslant 1,$$

which contradicts (2.2). □

P r o o f of Theorem 2.3. Let us assume on the contrary that system (1.1) is nonoscillatory. We obtain (4.15) similarly as in the proof of Theorem 2.2.

We denote

$$\alpha = \frac{1}{2}\big(1-\sqrt{1-4Q_*}\big), \quad \beta = \frac{1}{2}\big(1+\sqrt{1-4H_*}\big). \tag{4.21}$$

If $\alpha = 0$ or $\beta = 1$ then, according to Theorems 2.1 and 2.2 in [8], conditions (2.3) and (2.4) guarantee that system (1.1) is oscillatory.

Now suppose that $\alpha > 0$ and $\beta < 1$. By virtue of (1.3), (4.4), Lemmas 3.2 and 3.3, there exists $k_1(\varepsilon) \geqslant k_0$ such that the following inequalities

$$w_k\sum_{j=1}^{k-1} q_j > \alpha - \varepsilon, \quad w_k\sum_{j=1}^{k-1} q_j < \beta + \varepsilon, \quad \left|\frac{w_kq_k}{1+w_kq_k}\right| \leqslant \varepsilon \tag{4.22}$$

are satisfied for $k \geqslant k_1(\varepsilon)$, where w_k is defined by (3.1) and $\varepsilon \in]0, \min\{\alpha, \beta-1\}[$ is arbitrary.

By using inequalities (4.22) we obtain

$$\sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} R_j \geqslant \frac{(\alpha-\varepsilon)^2}{1+\varepsilon} \sum_{j=1}^{k-1} q_j \sum_{j=k}^{\infty} \frac{q_j}{\left(\sum_{i=1}^{j-1} q_i\right)^2} \geqslant \frac{(\alpha-\varepsilon)^2}{1+\varepsilon} \quad \text{for } k \geqslant k_1(\varepsilon). \tag{4.23}$$

In view of (4.22) and (4.23), we get from (4.15)

$$Q_k < \beta + \varepsilon - \frac{(\alpha-\varepsilon)^2}{1+\varepsilon} \quad \text{for } k \geqslant k_1(\varepsilon),$$

where Q_k is defined by (1.5). Since $\varepsilon > 0$ was chosen arbitrary, the last inequality leads to

$$Q^* \leqslant \beta - \alpha^2,$$

where Q^* is given by (1.7). Consequently, in view of (4.21), we have

$$Q^* \leqslant Q_* + \frac{1}{2}\left(\sqrt{1-4Q_*} + \sqrt{1-4H_*}\right)$$

which contradicts (2.3).

On the other hand, we can rewrite D_j as

$$D_j = q_j \left(w_j \sum_{i=1}^{j-1} q_i \left(2 - w_j \sum_{i=1}^{j-1} q_i\right) + \frac{w_j q_j}{1+w_j q_j} \left(w_j \sum_{i=1}^{j-1} q_i - 1\right)^2 \right) \quad \text{for } j \geqslant n \geqslant k_0,$$

where D_j is given by (4.17). Hence, by virtue of (4.22), we get from (4.16)

$$H^* \leqslant -\alpha + \varepsilon + (\beta+\varepsilon)(2-\beta-\varepsilon) + \varepsilon(\beta+\varepsilon-1)^2,$$

where H^* is given by (1.7).

Consequently, since $\varepsilon > 0$ was arbitrary, we have

$$H^* \leqslant -\alpha + \beta(2-\beta).$$

Hence, in view of (4.21), we get

$$H^* \leqslant H_* + \frac{1}{2}\left(\sqrt{1-4Q_*} + \sqrt{1-4H_*}\right),$$

which contradicts (2.4). □

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