THE PICONE IDENTITY FOR A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS

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(Received February 16, 2001)

Abstract. The Picone-type identity for the half-linear second order partial differential equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi\left(\frac{\partial u}{\partial x_i}\right) + c(x)\Phi(u) = 0, \quad \Phi(u) := |u|^{p-2}u,\ p > 1,$$

is established and some applications of this identity are suggested.

Keywords: Picone's identity, half-linear PDE, p-Laplacian, variational technique

MSC 2000: 35J20

1. Introduction

The classical Picone identity established by Picone [24] concerns a pair of second order Sturm-Liouville operators

$$\begin{aligned} l[x] &:= (r(t)x')' + c(t)x, \quad r(t) > 0, \\ L[y] &:= (R(t)y')' + C(t)y, \quad R(t) > 0 \end{aligned} \tag{1}$$

and reads as follows. Let x, y be differentiable functions such that rx', Ry' are also differentiable and $y(t) \neq 0$ on an interval $I \subset \mathbb{R}$. Then in this interval

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{x}{y}(yrx' - xRy')\right\} = (r-R)x'^2 + (C-c)x^2 + R\Big(x' - \frac{x}{y}y'\Big)^2 + \frac{x}{y}\left\{yl[x] - xL[y]\right\}.$$

Research supported by the Grant A1019902 of the Grant Agency of the Czech Academy of Sciences (Prague).

Since 1910, when the original paper of Picone was published, this identity has been extended to various equations (not only to ODE's but also to PDE's and to difference equations, see [1], [3], [12], [13], [14], [16], [17], [23], [27] and the references given therein), and it has turned out to be a very useful tool in the oscillation theory of these equations.

In our paper we follow this idea and establish a Picone-type identity for the second order partial differential equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi\left(\frac{\partial u}{\partial x_i}\right) + c(x)\Phi(u) = 0, \tag{2}$$

where $\Phi(u) := |u|^{p-2}u$, $p > 1$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $c\colon \mathbb{R}^n \to \mathbb{R}$ is a Hölder continuous function (Theorem 1). We also present some basic consequences of this identity (Theorems 2–4).

If $p = 2$, i.e. $\Phi(\frac{\partial u}{\partial x_i}) = \frac{\partial u}{\partial x_i}$, then (2) reduces to the classical Laplace equation

$$\Delta u + c(x)u = 0. \tag{3}$$

There also exists a voluminous literature dealing with the PDE's with the so-called p-Laplacian

$$\operatorname{div}(\|\nabla u\|_2^{p-2}\nabla u) + c(x)\Phi(u) = 0, \tag{4}$$

where ∇ is the usual nabla operator $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$ and $\|x\|_2 = \big(\sum\limits_{i=1}^{n} x_i^2\big)^{\frac{1}{2}}$ is the Euclidean norm in $\mathbb{R}^n$, see e.g. [7], [11] and the references given therein. In these papers and monographs, it was shown that solutions of (4) have many of the properties typical for the equation with the classical Laplacian (3). Let us mention at least the papers [1], [2], [10], [12], [14] dealing with Picone's identity for (4). This identity is used in [1], [2] to study the Dirichlet eigenvalue problem associated with (4) and the papers [10], [12], [14] deal with the application of Picone's identity in oscillation theory and show that oscillatory properties of (4) are very similar to those of (3).

PDE of the form (2) has been investigated in a series of papers of G. Bognár [4], [5], [6], and basic properties of the eigenvalue problem associated with (2) have been established. Our investigation can be regarded as a continuation of this research and also as an attempt to extend the results concerning (3) and (4) to (2). In particular, using the Picone identity established in this paper we show that the basic facts of the classical Sturmian theory (like Sturm comparison and separation theorems) do extend to (2).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u \in W_0^{1,p}(\Omega)$. Then multiplying the left-hand side of (2) by u and integrating the obtained formula over Ω (using the Gauss theorem) we see that (2) is closely related to the p-degree functional

$$\mathcal{F}_p(u;\Omega) := \int_\Omega \left\{ \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^p - c(x)|u(x)|^p \right\} \mathrm{d}x = \int_\Omega \left\{ \|\nabla u\|_p^p - c(x)|u|^p \right\} \mathrm{d}x, \tag{5}$$

where $\|x\|_p = \bigl(\sum_{i=1}^n |x_i|^p\bigr)^{\frac{1}{p}}$ denotes the p-norm in $\mathbb{R}^n$. Another important object associated with (2) is a Riccati-type equation which we obtain as follows. Let u be a solution of (2) which is nonzero in Ω and denote

$$v := \left(\Phi\left(\frac{\partial u}{\partial x_1}\right), \dots, \Phi\left(\frac{\partial u}{\partial x_n}\right) \right), \quad w := \frac{v}{\Phi(u)}.$$

Then, using the fact that (2) can be written in the form $\operatorname{div} v = -c(x)\Phi(u)$, we have

$$\begin{aligned}
\operatorname{div} w &= \frac{1}{\Phi^2(u)} \left\{ \Phi(u) \operatorname{div} v - \Phi'(u)\langle \nabla u, v\rangle \right\} \\
&= -c(x) - (p-1)\frac{|u|^{p-2}}{|u|^{2p-2}} \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^p \\
&= -c(x) - (p-1) \sum_{i=1}^n \left| \Phi\left(\frac{\partial u / \partial x_i}{u} \right) \right|^q \\
&= -c(x) - (p-1)\|w\|_q^q,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^n$, $q := \frac{p}{p-1}$ is the conjugate exponent of p and $\|x\|_q = \bigl(\sum_{i=1}^n |x_i|^q\bigr)^{\frac{1}{q}}$ denotes the q-norm in $\mathbb{R}^n$. Consequently, the vector variable w satisfies the Riccati-type equation

$$\operatorname{div} w + c(x) + (p-1)\|w\|_q^q = 0. \tag{6}$$

Of course, if $p = q = 2$ and $n = 1$, then the last equation reduces to the classical Riccati equation $w' + c(t) + w^2 = 0$ associated with (1) (where $r(t) \equiv 1$) via the substitution $w = u'/u$. Recall also that if we apply the same procedure as above to equation (4) and the vector variable $z := \frac{\nabla u}{u}$, we get the functional

$$\widetilde{\mathcal{F}}_p(u;\Omega) := \int_\Omega \left\{ \|\nabla u\|_2^p - c(x)|u|^p \right\} \mathrm{d}x \tag{7}$$

and the Riccati-type equation

$$\operatorname{div} z + c(x) + (p-1)\|w\|_2^q = 0. \tag{8}$$

Now we are in a position to formulate a Picone-type identity for (2), more precisely, for the p-degree functional (5) and the Riccati type equation (6) associated with (2). Before formulating it, observe that if $R \equiv r$, $C \equiv c$ in (1), y is a solution of $l(y) = 0$ and $w := \frac{ry'}{y}$, then Picone's identity for (1) mentioned in the first section can be written in the form

$$r(t)x'^2 - c(t)x^2 = [x^2 w(t)]' + \frac{1}{r(t)}[r(t)x' - w(t)x]^2,$$

and integrating this identity over $I = [a,b]$ we get the integral form of this identity

$$\begin{aligned} \mathcal{F}(x;a,b) &:= \int_a^b [r(t)x'^2 - c(t)x^2]\,\mathrm{d}t \\ &= x^2 w(t)\big|_a^b + \int_a^b r^{-1}(t)[r(t)x' - w(t)x]^2\,\mathrm{d}t. \end{aligned} \tag{9}$$

Theorem 1. *Let w be a solution of* (6) *which is defined in $\overline{\Omega}$ (closure of Ω) and $u \in W^{1,p}(\Omega)$. Then*

$$\begin{aligned} \mathcal{F}_p(u;\Omega) &= \int_{\partial\Omega} |u(x)|^p w(x)\,\mathrm{d}S \\ &+ p\int_\Omega \left\{\frac{\|\nabla u(x)\|_p^p}{p} - \langle \nabla u(x), \Phi(u(x))w(x)\rangle + \frac{\|w(x)\|_q^q |\Phi(u(x))|^q}{q}\right\}\mathrm{d}x. \end{aligned} \tag{10}$$

Moreover, the last integral in this formula is always nonnegative and for a function $u \not\equiv 0$ it equals zero only if $u \neq 0$ in $\overline{\Omega}$ and

$$w = \frac{1}{\Phi(u)}\left(\Phi\left(\frac{\partial u}{\partial x_1}\right), \ldots, \Phi\left(\frac{\partial u}{\partial x_n}\right)\right). \tag{11}$$

P r o o f. By a direct computation, using (6), we have

$$\begin{aligned} \operatorname{div}(w|u|^p) &= |u|^p \operatorname{div} w + p\Phi(u)\langle \nabla u, w\rangle \\ &= |u|^p\left(-c(x) - (p-1)\|w\|_q^q\right) + \|\nabla u\|_p^p - \|\nabla u\|_p^p + p\langle \nabla u, \Phi(u)w\rangle \\ &= \|\nabla u\|_p^p - c(x)|u|^p - p\left\{\frac{\|\nabla u\|_p^p}{p} - \langle \nabla u, \Phi(u)w\rangle + \frac{\|w\|_q^q|\Phi(u)|^q}{q}\right\}. \end{aligned}$$

Integrating the last formula over Ω and using the Gauss theorem we get (10).

To prove nonnegativity of the second integral in (10), consider the function $f(x) = \frac{\|x\|_p^p}{p}$. Its conjugate function

$$f^*(y) := \sup_{x\in\mathbb{R}^n} \{\langle x,y\rangle - f(x)\} \tag{12}$$

is $f^*(y) = \frac{\|y\|_q^q}{q}$ as can be verified by a direct computation. The supremum in (12) is attained for

$$x = (x_1,\dots,x_n) = (\Phi_q(y_1),\dots,\Phi_q(y_n)), \quad \Phi_q(s) := |s|^{q-2}s \tag{13}$$

being the inverse function of $\Phi(s) = |s|^{p-2}s$. The classical Fenchel inequality (see [25]) now implies that

$$\frac{\|x\|_p^p}{p} - \langle x,y\rangle + \frac{\|y\|_q^q}{q} \geqslant 0$$

for every $x,y \in \mathbb{R}^n$ with equality if and only if x and y are related by (13). Substituting there $x = \nabla u$, $y = \Phi(u)w$ we see that the (continuous and nonnegative) integrand in the second integral of (10) equals zero if and only if u and w are related by (11). $\square$

Substituting $p = 2$ in the previous theorem, we get Picone's identity as formulated e.g. in [27] and one can immediately see its similarity with (9).

Using Picone's identity (10) we can now prove the following relationship between the positivity of $\mathcal{F}_p$ (over $W_0^{1,p}(\Omega)$) and the existence of a nonzero solution (in $\overline{\Omega}$) of (2).

Corollary 1. *If there exists a solution $u = u(x)$ of (2) such that $u(x) \neq 0$ for $x \in \overline{\Omega}$ then $\mathcal{F}_p(y;\Omega) > 0$ for every $0 \not\equiv y \in W_0^{1,p}(\Omega)$.*

Proof. The proof immediately follows from Theorem 1 by setting

$$w = \frac{1}{\Phi(u)}\left(\Phi\left(\frac{\partial u}{\partial x_1}\right),\dots,\Phi\left(\frac{\partial u}{\partial x_n}\right)\right)$$

and using the fact that w and ∇y cannot be proportional since $u(x) > 0$ in $\overline{\Omega}$ and $y|_{\partial\Omega} = 0$. $\square$

3. Applications

Following the usual terminology for linear equation (3) (see e.g. [21], [22]), a bounded domain $\Omega \subset \mathbb{R}^n$ is said to be the *nodal domain* of a nontrivial solution u of (2) if $u|_{\partial\Omega} = 0$, and the curve $\Gamma = \{x \in \Omega\colon u(x) = 0\}$ is called the *nodal contour* of u in Ω. Corollary 1 immediately implies the following Sturmian-type statement.

Theorem 2. *Suppose that u is a nontrivial solution of* (2) *with the nodal domain $\Omega \subset \mathbb{R}^n$ and $\tilde{c}\colon \mathbb{R}^n \to \mathbb{R}$ is a Hölder continuous function such that $\tilde{c}(x) \geqslant c(x)$ in $\overline{\Omega}$. Then every solution of the (majorant) equation*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Phi\left(\frac{\partial u}{\partial x_i}\right) + \tilde{c}(x)\Phi(u) = 0 \tag{14}$$

has a nodal contour in $\overline{\Omega}$. In particular, if $\tilde{c}(x) = c(x)$, then every solution of (2) *has a nodal contour in $\overline{\Omega}$.*

P r o o f. If $\tilde{u}$ is a solution of (14) without a nodal contour in $\overline{\Omega}$, then by Corollary 1

$$\int_{\Omega} \left\{\|\nabla y\|_p^p - \tilde{c}(x)|y|^p\right\} \, \mathrm{d}x > 0$$

for every nontrivial $y \in W_0^{1,p}(\Omega)$ and in view of the inequality $\tilde{c}(x) \geqslant c(x)$ in Ω, we have $\mathcal{F}_p(y;\Omega) > 0$ for every nontrivial $y \in W_0^{1,p}(\Omega)$. On the other hand, the solution u of (2) satisfies $u \in W_0^{1,p}(\Omega)$ and by the Gauss theorem $\mathcal{F}_p(u;\Omega) = 0$, a contradiction. $\square$

Corollary 1 also suggests another application in the oscillation theory of (2), namely to compare this equation with a certain half-linear ODE. To prove that (2) possesses no positive solution in a connected (bounded or unbounded) domain $\Omega \subset \mathbb{R}^n$, it suffices to find a nontrivial function $y \in W_0^{1,p}(\Omega)$ for which $\mathcal{F}_p(y;\Omega) \leqslant 0$. It is natural to look for such a function in the radial form $y(x) = \tilde{y}(\|x\|)$, where $\|\cdot\|$ is a norm in $\mathbb{R}^n$ and $\tilde{y}\colon \mathbb{R} \to \mathbb{R}$. This idea has been used in [10], [14] for PDE with p-Laplacian (4) with the norm $\|\cdot\|_p$, for our equation (2) it is convenient to take the q-norm $\|\cdot\|_q$.

Theorem 3. *Let*

$$C(r) := \int_{\|x\|_q = r} c(x) \, \mathrm{d}S, \quad q = \frac{p}{p-1}.$$

If the half-linear second order ODE

$$(15)\qquad \left(r^{n-1}\Phi(z')\right)' + \frac{1}{\omega_n(q)}C(r)\Phi(z) = 0, \quad ' := \frac{\mathrm{d}}{\mathrm{d}r},\ \omega_n(q) := \int_{\|x\|_q=1} \mathrm{d}S$$

is oscillatory (i.e., every nontrivial solution of (15) *has arbitrarily large zeros), then* (2) *possesses no positive solution in the exterior domain* $\Omega_R = \{x \in \mathbb{R}^n :\ \|x\|_q > R\}$ *for every* $R > 0$.

P r o o f. If (15) is oscillatory, there exists an increasing sequence $r_n \to \infty$ and a nontrivial solution $z = z(r)$ of (15) for which $z(r_n) = 0$. Let $R > 0$ be arbitrary. Then $r_n > R$ for $n \in \mathbb{N}$ sufficiently large and for some of these n's define

$$y(x) = \begin{cases} z(\|x\|_q), & r_n \leqslant \|x\|_q \leqslant r_{n+1}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, if $\Omega_{r_n,r_{n+1}} := \{x \in \mathbb{R}^n |:\ r_n \leqslant \|x\|_q \leqslant r_{n+1}\}$, we have

$$\begin{aligned}
\mathcal{F}_p(y;\Omega_{r_n,r_{n+1}}) &= \int_{\Omega_{r_n,r_{n+1}}} \left[\|\nabla y\|_p^p - c(x)|y|^p\right] \mathrm{d}x \\
&= \int_{r_n}^{r_{n+1}} \left\{ \int_{\|x\|_q=r} [\|\nabla y\|_p^p - c(x)|y|^p] \,\mathrm{d}S \right\} \mathrm{d}r \\
&= \omega_n(q) \left\{ \int_{r_n}^{r_{n+1}} \left[r^{n-1}|z'(r)|^p - \frac{C(r)}{\omega_n(q)}|z(r)|^p \right] \mathrm{d}r \right\} \\
&= \omega_n(q) \left\{ r^{n-1}\Phi(z')z \,\Big|_{r_n}^{r_{n+1}} - \int_{r_n}^{r_{n+1}} z\left((r^{n-1}\Phi(z'))' - \frac{C(r)}{\omega_n(q)}\Phi(z) \right) \mathrm{d}r \right\} \\
&= 0.
\end{aligned}$$

Consequently, by Corollary 1, equation (2) cannot have a positive solution in Ω_R for every $R > 0$. □

The previous theorem shows that any oscillation criterion for the half-linear ordinary differential equation

$$(16)\qquad (r(t)\Phi(x'))' + c(t)\Phi(x) = 0$$

can be applied to (2). A typical example is the Leighton-Wintner criterion (see e.g. [20] for its proof) which states that (16) is oscillatory provided

$$(17)\qquad \int^{\infty} r^{1-q}(t)\,\mathrm{d}t = \infty \quad \text{and} \quad \lim_{t\to\infty} \int^{t} c(s)\,\mathrm{d}s = \infty.$$

The application of this criterion to (2) gives the following statement.

Theorem 4. *Let $p \geqslant n$. If*

$$\lim_{r\to\infty} \int_{\|x\|_q \leqslant r} c(x)\,\mathrm{d}x = \infty$$

then (2) *possesses no positive solution in the exterior domain* $\Omega_R = \{x \in \mathbb{R}^n : \|x\|_q > R\}$ *for every* $R > 0$.

P r o o f. By Theorem 3, it suffices to prove oscillation of (15), but it follows immediately from the just mentioned Leighton-Wintner criterion (17) since (because of $p \geqslant n$)

$$\int^{\infty} r^{(n-1)(1-q)}\,\mathrm{d}r = \int^{\infty} r^{-\frac{n-1}{p-1}}\,\mathrm{d}r = \infty$$

and

$$\lim_{r\to\infty} \int^{r} C(r)\,\mathrm{d}r = \lim_{r\to\infty} \int_{\|x\|_q \leqslant r} c(x)\,\mathrm{d}x = \infty.$$

□

The results of this section should be regarded only as "a sample" of the applications of Picone's identity in the oscillation theory of (2). For example, following [10], [14], any oscillation and conjugacy criterion for the half-linear equation (15) (see., e.g., [8], [9]) yields a corresponding result on nonexistence of positive solutions of (2). In addition, in our investigation we have till now completely neglected the relationship between the existence of a nonzero solution of (2) and the solvability of (6). This method of oscillation theory is usually regarded as the Riccati technique and was used in the oscillation theory of PDE's e.g. in [10], [18], [19], [21], [26] and other papers. We hope to investigate applications of the Picone identity in the oscillation theory of (2) as well as to look for a more general form of this identity in a subsequent paper.

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