ON THE RANGE-KERNEL ORTHOGONALITY OF ELEMENTARY OPERATORS

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Abstract. Let $L(H)$ denote the algebra of operators on a complex infinite dimensional Hilbert space H. For $A,B\in L(H)$, the generalized derivation $\delta_{A,B}$ and the elementary operator $\Delta_{A,B}$ are defined by $\delta_{A,B}(X)=AX-XB$ and $\Delta_{A,B}(X)=AXB-X$ for all $X\in L(H)$. In this paper, we exhibit pairs (A,B) of operators such that the range-kernel orthogonality of $\delta_{A,B}$ holds for the usual operator norm. We generalize some recent results. We also establish some theorems on the orthogonality of the range and the kernel of $\Delta_{A,B}$ with respect to the wider class of unitarily invariant norms on $L(H)$.

Keywords: derivation; elementary operator; orthogonality; unitarily invariant norm; cyclic subnormal operator; Fuglede-Putnam property

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1. Introduction

Let H be a complex infinite dimensional Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators acting on H into itself. Given $A,B\in L(H)$, we define the generalized derivation $\delta_{A,B}\colon L(H)\to L(H)$ by $\delta_{A,B}(X)=AX-XB$, and the elementary operator $\Delta_{A,B}\colon L(H)\to L(H)$ by $\Delta_{A,B}(X)=AXB-X$. Let $\delta_{A,A}=\delta_A$ and $\Delta_{A,A}=\Delta_A$.

In [1], Anderson shows that if A is normal and commutes with T, then for all $X\in L(H)$

$$\|\delta_A(X)+T\|\geqslant\|T\|, \tag{1.1}$$

where $\|\cdot\|$ is the usual operator norm. In view of [1], Definition 1.2, the inequality (1.1) says that the range $R(\delta_A)$ of δ_A is orthogonal to its kernel $\ker(\delta_A)$, which is just the commutant $\{A\}'$ of A.

If A and B are normal operators such that $AT = TB$ for some $T \in L(H)$, notice that if we consider the operators $A \oplus B$, $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ on $H \oplus H$, then for all $X \in L(H)$ we have

$$\|\delta_{A,B}(X) + T\| \geqslant \|T\|.$$

Inequality (1.1) has a Δ_A analogue. Thus, Duggal [6] proved that if A is a normal operator such that $\Delta_A(T) = 0$ for some $T \in L(H)$, then for all $X \in L(H)$ we have

$$\|\Delta_A(X) + T\| \geqslant \|T\|.$$

The orthogonality of the range and the kernel of elementary operators with respect to the wider class of unitarily invariant norms on $L(H)$ has been considered by many authors [3], [5], [6], [8], [10] and [11].

The purpose of this paper is to study the range-kernel orthogonality of the operators $\delta_{A,B}$ and $\Delta_{A,B}$. We give pairs (A, B) of operators such that the range and the kernel of $\delta_{A,B}$ are orthogonal. We exhibit pairs (A, B) of operators such that $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$.

We investigate the orthogonality of the range and the kernel of $\Delta_{A,B}$ in norm ideals. Related results on orthogonality for certain elementary operators are also given.

Given $X \in L(H)$, we shall denote the kernel, the orthogonal complement of the kernel and the closure of the range of X by $\ker(X)$, $\ker^{\perp}(X)$, and $\overline{R(X)}$, respectively. The spectrum of X will be denoted by $\sigma(X)$, and $X|M$ will denote the restriction of X to an invariant subspace M.

2. Main results

Definition 2.1. Let E be a normed linear space and $\mathbb{C}$ the complex numbers.

1) We say that $x \in E$ is orthogonal to $y \in E$ if $\|x - \lambda y\| \geqslant \|\lambda y\|$ for all $\lambda \in \mathbb{C}$.

2) Let F and G be two subspaces in E. If $\|x + y\| \geqslant \|y\|$ for all $x \in F$ and for all $y \in G$, then F is said to be orthogonal to G.

Remark 2.1.

- ▷ Note that if x is orthogonal to y, then y need not be orthogonal to x.
- ▷ This definition generalizes the idea of orthogonality in Hilbert space.
- ▷ It is shown in [1] that if F is orthogonal to G, and F, G are closed subspaces of E, then the algebraic direct sum $F \oplus G$ is a closed subspace in E.

Theorem 2.1. *Let $A, B \in L(H)$. If B is invertible and $\|A\| \cdot \|B^{-1}\| \leqslant 1$, then*

$$\|\delta_{A,B}(X) + T\| \geqslant \|T\|$$

for all $X \in L(H)$ and for all $T \in \ker(\delta_{A,B})$.

Proof. Let $T \in L(H)$, such that $AT = TB$. This implies that $ATB^{-1} = T$. Since $\|A\| \cdot \|B^{-1}\| \leqslant 1$, it follows from [11], Corollary 1.4, that

$$\|AYB^{-1} - Y + T\| \geqslant \|T\|$$

for all $Y \in L(H)$. If we set $X = YB^{-1}$, then we get

$$\|AX - XB + T\| \geqslant \|T\|.$$

Hence $\|\delta_{A,B}(X) + T\| \geqslant \|T\|$ for all $T \in \ker(\delta_{A,B})$ and for all $X \in L(H)$. □

Theorem 2.2. *Let $A, B \in L(H)$. If either*

1) *A is an isometry and the operator B is a contraction or*
2) *A is a contraction and B is co-isometric, then*

$$\|\delta_{A,B}(X) + T\| \geqslant \|T\|$$

for all $X \in L(H)$ and for all $T \in \ker(\delta_{A,B})$.

Proof. 1) Given $T \in \ker(\delta_{A,B})$, we have

$$\delta_{A,B}(T) = 0 \;\Rightarrow\; T = A^*TB \;\Rightarrow\; A^*T = A^*(A^*T)B.$$

Moreover, we see that

$$\|\delta_{A,B}(X) + T\| \geqslant \|A^*(\delta_{A,B}(X) + T)\| = \|\Delta_{A^*,B}(X) - A^*T\|.$$

Since A is an isometry and B is a contraction, it follows from [11], Corollary 1.4, that

$$\|\delta_{A,B}(X) + T\| \geqslant \|\Delta_{A^*,B}(X) - A^*T\| \geqslant \|A^*T\| \geqslant \|A^*TB\| = \|T\|.$$

Then, $\|\delta_{A,B}(X) + T\| \geqslant \|T\|$ for all $X \in L(H)$.

2) Let $T \in \ker(\delta_{A,B})$ and $X \in L(H)$. By taking adjoints, observe that

$$\|\delta_{A,B}(X) + T\| = \|\delta_{B^*,A^*}(X^*) - T^*\|.$$

Since B^* is isometric and A^* is a contraction, the result follows from the first part of the proof. □

As an application of Theorem 2.2 we have a well known result.

Corollary 2.1. *Let U, V be isometries such that $\delta_{U,V}(T) = 0$ for some $T \in L(H)$. Then*

$$\|\delta_{U,V}(X) + T\| \geqslant \|T\|$$

for all $X \in L(H)$.

Remark 2.2. Let $A, B \in L(H)$. If A is an isometry and B is a contraction, then

$$\overline{R(\delta_{A,B})} \cap \ker(\delta_{A,B}) = \{0\}.$$

Definition 2.2 ([7])**.** A proper two-sided ideal $\mathcal{J}$ in $L(H)$ is said to be a norm ideal if there is a norm on $\mathcal{J}$ possessing the following properties:

i) $(\mathcal{J}, \|\|_{\mathcal{J}})$ is a Banach space.
ii) $\|AXB\|_{\mathcal{J}} \leqslant \|A\|\|X\|_{\mathcal{J}}\|B\|$ for all $A, B \in L(H)$ and for all $X \in \mathcal{J}$.
iii) $\|X\|_{\mathcal{J}} = \|X\|$ for X a rank one operator.

Remark 2.3. If $(\mathcal{J}, \|\|_{\mathcal{J}})$ is a norm ideal, then the norm $\|\|_{\mathcal{J}}$ is unitarily invariant, in the sense that $\|UAV\|_{\mathcal{J}} = \|A\|_{\mathcal{J}}$ for all $A \in \mathcal{J}$ and for all unitary operators $U, V \in L(H)$.

Corollary 2.2. *Let $(\mathcal{J}, \|\|_{\mathcal{J}})$ be a norm ideal and $A, B \in L(H)$. If A is an isometry and the operator B is a contraction, then*

$$\|\delta_{A,B}(X) + T\|_{\mathcal{J}} \geqslant \|T\|_{\mathcal{J}}$$

for all $X \in \mathcal{J}$ and for all $T \in \ker(\delta_{A,B}) \cap \mathcal{J}$.

Theorem 2.3. *Let $(\mathcal{J}, \|\|_{\mathcal{J}})$ be a norm ideal and $A \in L(H)$. Suppose that $f(A)$ is a cyclic subnormal operator, where f is a nonconstant analytic function on an open set containing $\sigma(A)$. Then*

$$\|\delta_A(X) + T\|_{\mathcal{J}} \geqslant \|T\|_{\mathcal{J}}$$

for all $X \in \mathcal{J}$ and for all $T \in \{A\}' \cap \mathcal{J}$.

Proof. Let $T \in \mathcal{J}$ be such that $AT = TA$, then we have $f(A)T = Tf(A)$ and $Af(A) = f(A)A$. Since $f(A)$ is a cyclic subnormal operator, it follows from Yoshino's result [12] that T and A are subnormal. Therefore, every compact hyponormal operator is normal [2], hence T is normal.

Consequently, $AT = TA$ implies that $AT^* = T^*A$. Hence we obtain that $\overline{R(T)}$ and $\ker^{\perp}(T)$ reduces A, and $A_0 = A/\overline{R(T)}$ and $B_0 = A/\ker^{\perp}(T)$ are normal operators.

Let $A = A_0 \oplus A_1$ with respect to $H_0 = H = \overline{R(T)} \oplus \overline{R(T)}^{\perp}$, and let $A = B_0 \oplus B_1$ with respect to $H_1 = H = \ker^{\perp}(T) \oplus \ker(T)$. Define the quasi-affinity $T_0\colon \ker^{\perp}(T) \to \overline{R(T)}$ by setting $T_0x = Tx$ for every $x \in \ker^{\perp}(T)$. Then it results that $\delta_{A_0,B_0}(T_0) = \delta_{A_0^*,B_0^*}(T_0) = 0$.

Also, we can write T and X on H_1 into H_0 as

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} X_0 & X_1 \\ X_2 & X_3 \end{pmatrix}.$$

Consequently, we have

$$\|\delta_A(X) + T\|_{\mathcal{J}} = \left\| \begin{pmatrix} \delta_{A_0,B_0}(X_0) + T_0 & * \\ * & * \end{pmatrix} \right\|_{\mathcal{J}} \geqslant \|\delta_{A_0,B_0}(X) + T\|_{\mathcal{J}}.$$

Since A_0 and B_0 are normal operators, we obtain from [4], Theorem 4, that

$$\|\delta_A(X) + T\|_{\mathcal{J}} \geqslant \|\delta_{A_0,B_0}(X_0) + T_0\|_{\mathcal{J}} \geqslant \|T_0\|_{\mathcal{J}} = \|T\|_{\mathcal{J}}.$$

□

R e m a r k 2.4. Let $A \in L(H)$ and let f be an analytic function on an open set containing $\sigma(A)$. If $f(A)$ is cyclic subnormal and T is a compact operator such that $AT = TA$, then for all $X \in L(H)$,

$$\|\delta_A(X) + T\| \geqslant \|T\|.$$

Definition 2.3. Let $A, B \in L(H)$ and let $\mathcal{J}$ be a two-sided ideal of $L(H)$. We say that the pair (A, B) possesses the Fuglede-Putnam property $\mathrm{PF}(\Delta, \mathcal{J})$, if $\ker(\Delta_{A,B}|\mathcal{J}) \subseteq \ker(\Delta_{A^*,B^*}|\mathcal{J})$.

Theorem 2.4. *Let $A, B \in L(H)$. If the pair (A, B) possesses the $\mathrm{PF}(\Delta, \mathcal{J})$ property, then*

$$\|\Delta_{A,B}(X) + T\|_{\mathcal{J}} \geqslant \|T\|_{\mathcal{J}}$$

for all $X \in \mathcal{J}$, and for all $T \in \ker(\Delta_{A,B}) \cap \mathcal{J}$.

P r o o f. Given $T \in \mathcal{J}$ such that $ATB = T$. Since the pair (A, B) possesses the $\mathrm{PF}(\Delta, \mathcal{J})$ property, $\overline{R(T)}$ reduces A, and $\ker^{\perp}(T)$ reduces B, and $A_0 = A|\overline{R(T)}$, $B_0 = B|\ker^{\perp}(T)$ are normal operators.

Let $T_0\colon \ker^{\perp}(T) \to \overline{R(T)}$ be the quasi-affinity defined by setting $T_0x = Tx$ for each $x \in \ker^{\perp}(T)$. Then we have $\Delta_{A_0,B_0}(T_0) = 0 = \Delta_{A_0^*,B_0^*}(T_0)$. Let $A = A_0 \oplus A_1$

with respect to $H_0 = H = \overline{R(T)} \oplus \overline{R(T)}^{\perp}$, and $B = B_0 \oplus B_1$ with respect to $H_1 = H = \ker^{\perp}(T) \oplus \ker(T)$. Let X on H_1 into H_0 have the matrix representation

$$X = \begin{pmatrix} X_0 & X_1 \\ X_2 & X_3 \end{pmatrix}.$$

Hence

$$\|\Delta_{A,B}(X) + T\|_{\mathcal{J}} = \left\| \begin{pmatrix} \Delta_{A_0,B_0}(X_0) + T_0 & * \\ * & * \end{pmatrix} \right\|_{\mathcal{J}}.$$

It follows from [7] that the diagonal part of a block matrix always has smaller norm than that of the whole matrix. Consequently, we have

$$\|\Delta_{A,B}(X) + T\|_{\mathcal{J}} = \left\| \begin{pmatrix} \Delta_{A_0,B_0}(X_0) + T_0 & * \\ * & * \end{pmatrix} \right\|_{\mathcal{J}} \geqslant \|\Delta_{A_0,B_0}(X_0) + T_0\|_{\mathcal{J}}.$$

Since A_0 and B_0 are normal, it results from [6], Theorem 2, that

$$\|\Delta_{A,B}(X) + T\|_{\mathcal{J}} \geqslant \|\Delta_{A_0,B_0}(X_0) + T_0\|_{\mathcal{J}} \geqslant \|T_0\|_{\mathcal{J}} = \|T\|_{\mathcal{J}}.$$

□

The following corollaries are consequences of the above theorem.

Corollary 2.3. *Let $A, B \in L(H)$. Let some of the following conditions be fulfilled:*

1) *$A, B \in L(H)$ such that $\|Ax\| \geqslant \|x\| \geqslant \|Bx\|$ for all $x \in H$.*
2) *A is invertible and B such that $\|A^{-1}\|\|B\| \leqslant 1$.*
3) *A is dominant and B^* is M-hyponormal.*

Then we have

$$\|\Delta_{A,B}(X) + T\|_{\mathcal{J}} \geqslant \|T\|_{\mathcal{J}}$$

for all $X \in \mathcal{J}$ and for all $T \in \ker(\Delta_{A,B}) \cap \mathcal{J}$.

Proof. It is sufficient to show that the pair (A, B) has the Fuglede-Putnam property $\mathrm{PF}(\Delta, \mathcal{J})$ in each of the preceding cases (in particular (3)).

1) It follows from [9], Lemma 1, that for all $T \in \ker(\Delta_{A,B}) \cap \mathcal{J}$, we have $\overline{R(T)}$ reduces A and $\ker^{\perp}(T)$ reduces B, and $A|\overline{R(T)}, B|\ker^{\perp}(T)$ are unitary operators. Hence, it results that the the pair (A, B) has the property $\mathrm{PF}(\Delta, \mathcal{J})$.

2) In this case, let $A_1 = \|B\|^{-1}A$ and $B_1 = \|B\|^{-1}B$, then $\|A_1x\| \geqslant \|x\| \geqslant \|B_1x\|$ for all $x \in H$. Hence, the result holds due to (1.1). □

Corollary 2.4. *Let $A, B \in L(H)$ be such that the pairs (A, A) and (B, B) have the* $\mathrm{PF}(\Delta, \mathcal{J})$ *property. If $1 \notin \sigma(A)\sigma(B)$, then*

$$\|\Delta_{A,B}(X) + T\|_{\mathcal{J}} \geqslant \|T\|_{\mathcal{J}}$$

for all $X \in \mathcal{J}$, and for all $T \in \ker(\Delta_{A,B}) \cap \mathcal{J}$.

Proof. It is well known that if $1 \notin \sigma(A)\sigma(B)$, then the operators $\Delta_{A,B}$ and $\Delta_{B,A}$ are invertible. Thus, a simple calculation shows that the pair $(A \oplus B, A \oplus B)$ possesses the $\mathrm{PF}(\Delta, \mathcal{J})$ property. $\square$

Remark 2.5. If $Se_n = \omega_n e_{n+1}$ is a unilateral (bilateral) weighted shift, then, it follows from [3] that the pair (S, S) has the property $\mathrm{PF}(\delta, \mathcal{J})$ if and only if

$$\sum_k \omega_k \omega_{k+1} \ldots \omega_{k+n-1} = \infty.$$

Remark 2.6. 1) Let $A, B \in L(H)$, then $\overline{R(\Delta_{A,B})} \cap \ker(\Delta_{A,B}) = \{0\}$ in each of the following cases:

i) A and B are normal.

ii) A and B are contraction.

iii) $A = B$ is cyclic subnormal.

iv) A and B^* are hyponormal.

2) If A^* and B are hyponormal, then $\overline{R(\Delta_{A,B})} \cap \ker(\Delta_{A^*,B^*}) = \{0\}$.

Corollary 2.5. *Let $A, B \in L(H)$. Then every operator in $\overline{R(\Delta_{A\oplus B})} \cap \{\ker(\Delta_{A\oplus B}) \cup \ker(\Delta_{A^*\oplus B^*})\}$ is nilpotent of order not greater than 2, in each of the following cases:*

1) *A normal and B isometric.*

2) *A normal and B cyclic subnormal.*

3) *A cyclic subnormal and B co-isometric.*

Proof. On $H \oplus H$, let T be the operator defined as $T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. A routine calculation shows that $T \in \overline{R(\Delta_{A\oplus B})} \cap \ker(\Delta_{A\oplus B})$ implies

$$P \in \overline{R(\Delta_A)} \cap \ker(\Delta_A); \quad S \in \overline{R(\Delta_B)} \cap \ker(\Delta_B);$$
$$R \in \overline{R(\Delta_{B,A})} \cap \ker(\Delta_{B,A}); \quad Q \in \overline{R(\Delta_{A,B})} \cap \ker(\Delta_{A,B}).$$

Hence, if A is normal and B is isometric, it follows from [6], Corollary 1, [11], Corollary 1.4, that $P = 0$, $S = 0$ and $R = 0$. Consequently, we obtain $T = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}$, which ensures that T is nilpotent of order not greater than 2.

By using a similar argument we get the desired result. $\square$

R e m a r k 2.7. 1) Note that Corollary 2.5 still holds if we consider the inner derivation δ_A instead of Δ_A.

2) Let $\pi\colon L(H) \to L(H)|K(H)$ denote the Calkin map. Set

$$\mathcal{S} = \{T \in L(H)\colon \|\pi(T)\| = \|T\|\}.$$

If $A \in L(H)$ satisfies one of the following conditions:

i) $A^*A - AA^*$ is compact;

ii) $A^*A - I$ or $AA^* - I$ is compact;

then $R(d_A)$ is orthogonal to $\ker(d_A) \cap \mathcal{S}$, where $d_A = \delta_A$ or $d_A = \Delta_A$.

3. A comment and some open questions

1) It is shown in [3] that if A is a cyclic subnormal operator, then $R(\delta_A)$ is orthogonal to $\{A\}'$, and this orthogonality fails in the absence of the hypothesis that the subnormal A is cyclic.

It is easy to see that if A and B are cyclic subnormal operators such that $A \oplus B$ is cyclic subnormal, then $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$.

Hence, it would be interesting to establish the range-kernel orthogonality of $\delta_{A,B}$ in the general case.

2) Let $\pi\colon L(H) \to L(H)/K(H) = \mathcal{C}(H)$ denote the Calkin map, and let

$$\mathcal{S} = \{A \in L(H)\colon \|\pi(A)\| = \|A\|\}.$$

Note that the result of Duggal [5] guarantees that if A and B are cyclic subnormal operators, then $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B}) \cap \mathcal{S}$, and $R(\Delta_{A,B})$ is orthogonal to $\ker(\Delta_{A,B}) \cap \mathcal{S}$.

From this, the following question naturally arises:

If A and B are cyclic subnormal operators, is $R(\Delta_{A,B})$ orthogonal to $\ker(\Delta_{A,B})$ for the usual operator norm?

3) Let $A \in L(H)$, and suppose that f is an analytic function on an open set containing $\sigma(A)$ such that f' does not vanish on some neighborhood of $\sigma(A)$.

If $f(A)$ is isometric or normal, what conditions on f ensure the range-kernel orthogonality of δ_A with respect to the wider class of unitarily invariant norms on $L(H)$?

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