TIETZE EXTENSION THEOREM FOR PAIRWISE ORDERED FUZZY EXTREMALLY DISCONNECTED SPACES

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Abstract. In this paper a new class of fuzzy topological spaces called pairwise ordered fuzzy extremally disconnected spaces is introduced. Tietze extension theorem for pairwise ordered fuzzy extremally disconnected spaces has been discussed as in the paper of Kubiak (1987) besides proving several other propositions and lemmas.

Keywords: pairwise ordered fuzzy extremally disconnected space, ordered T_1-fuzzy continuous function, lower (upper) T_1-fuzzy continuous functions

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1. Introduction and preliminaries

The fuzzy concept has invaded almost all branches of Mathematics since the introduction of the concept by Zadeh [11]. Fuzzy sets have applications in many fields such as information [8] and control [9]. The theory of fuzzy topological spaces was introduced and developed by Chang [5] and since then various notions in classical topology have been extended to fuzzy topological spaces [2], [3], [4]. A new class of fuzzy topological spaces called pairwise ordered fuzzy extremally disconnected spaces is introduced in this paper by using the concepts of fuzzy extremally disconnected space [1], fuzzy open sets [4], ordered fuzzy topology [7] and fuzzy bitopology [6]. Some interesting properties and characterizations are studied. Tietze extension theorem for pairwise ordered fuzzy extremally disconnected spaces has been discussed as in [10] besides proving several other propositions and lemmas.

Definition 1 [5]. Let $f\colon (X,T) \to (Y,S)$ be a mapping from a fuzzy topological space X to another fuzzy topological space Y. A mapping f is called

(a) a fuzzy continuous mapping [1] if $f^{-1}(\lambda) \in T$ for each $\lambda \in S$ or equivalently $f^{-1}(\mu)$ is a fuzzy closed subset of X for each fuzzy closed set μ of Y.

(b) a fuzzy open (fuzzy closed) mapping [1] if $f(\lambda)$ is a fuzzy open (fuzzy closed) subset of X for each fuzzy open (fuzzy closed) set λ of X.

Definition 2 [7]. A fuzzy set λ in (X, T) is called increasing (decreasing) if $\lambda(x) \leqslant \lambda(y)(\lambda(x) \geqslant \lambda(y))$ whenever $x \leqslant y$ in (X, T) and $x, y \in X$.

Definition 3 [7]. An ordered set on which a fuzzy topology is given is called an ordered fuzzy topological space.

Definition 4 [6]. A fuzzy bitopological space is a triple (X, T_1, T_2) where X is a set and T_1, T_2 are any two fuzzy topologies on X.

2. Pairwise ordered fuzzy extremally disconnected spaces

In this section the concept of a pairwise ordered fuzzy extremally disconnected space is introduced. Its characterizations and properties are studied.

Definition 5. Let λ be any fuzzy set in the ordered fuzzy topological space $(X, T, \leqslant)$. Then we define

$I(\lambda)$ = increasing fuzzy closure of $\lambda = \bigwedge\{\mu\colon\ \mu$ is a fuzzy closed increasing set and $\mu \geqslant \lambda\}$;

$D(\lambda)$ = decreasing fuzzy closure of $\lambda = \bigwedge\{\mu\colon\ \mu$ is a fuzzy closed decreasing set and $\mu \geqslant \lambda\}$;

$I^0(\lambda)$ = increasing fuzzy interior of $\lambda = \bigvee\{\mu\colon\ \mu$ is a fuzzy open increasing set and $\mu \leqslant \lambda\}$;

$D^0(\lambda)$ = decreasing fuzzy interior of $\lambda = \bigvee\{\mu\colon\ \mu$ is a fuzzy open decreasing set and $\mu \leqslant \lambda\}$.

Clearly, $I(\lambda)(D(\lambda))$ is the smallest fuzzy closed increasing (decreasing) set containing λ and $I^0(\lambda)(D^0(\lambda))$ is the largest fuzzy open increasing (decreasing) set contained in λ.

Proposition 1. *For any fuzzy subset λ of an ordered fuzzy topological space $(X, T, \leqslant)$, the following identities hold:*

(a) $1 - I(\lambda) = D^0(1 - \lambda)$,

(b) $1 - D(\lambda) = I^0(1 - \lambda)$,

(c) $1 - I^0(\lambda) = D(1 - \lambda)$,

(d) $1 - D^0(\lambda) = I(1 - \lambda)$.

Definition 6. Let $(X, T_1, T_2, \leqslant)$ be an ordered fuzzy bitopological space. Let λ be any T_1-fuzzy open increasing (decreasing) set. If $I_{T_2}(\lambda)(D_{T_2}(\lambda))$ is T_2-fuzzy open increasing (decreasing) set then $(X, T_1, T_2, \leqslant)$ is said to be T_1-upper (T_1-lower) fuzzy extremally disconnected. Similarly we can define the T_2-upper (T_2-lower) fuzzy extremally disconnected space. A fuzzy topological space $(X, T_1, T_2, \leqslant)$ is said to be pairwise upper fuzzy extremally disconnected if it is both T_1-upper fuzzy extremally disconnected and T_2-upper fuzzy extremally disconnected. Similarly we can define the pairwise lower fuzzy extremally disconnected space. A fuzzy topological space $(X, T_1, T_2, \leqslant)$ is said to be pairwise ordered fuzzy extremally disconnected if it is both pairwise upper fuzzy extremally disconnected and pairwise lower fuzzy extremally disconnected.

Example 1. Let $X = \{a, b, c\}$, $T_1 = \{0, 1, \lambda_1, \lambda_2)$ and $T_2 = \{0, 1, \mu_1, \mu_2)$, where $\lambda_1\colon X \to [0, 1]$ is such that

$$\lambda_1(a) = 3/4, \quad \lambda_1(b) = 1/2, \quad \lambda_1(c) = 1/4,$$
$$\lambda_2(a) = 0, \quad \lambda_2(b) = 1/4, \quad \lambda_2(c) = 1/4,$$

and $\mu_1\colon X \to [0, 1]$ is such that

$$\mu_1(a) = 1, \quad \mu_1(b) = 9/10, \quad \mu_1(c) = 8/10,$$
$$\mu_2(a) = 0, \quad \mu_2(b) = 7/10, \quad \mu_2(c) = 8/10.$$

Clearly, λ_1 is a T_1-fuzzy open decreasing set and $D_{T_2}(\lambda_1) = 1$ is a T_2-fuzzy open decreasing set, μ_2 is a T_2-fuzzy open decreasing set and $D_{T_1}(\mu_2) = 1$ is a T_1-fuzzy open decreasing set. Hence $(X, T_1, T_2, \leqslant)$ is a pairwise lower fuzzy extremally disconnected space. Similarly we can prove that $(X, T_1, T_2, \leqslant)$ is a pairwise upper fuzzy extremally disconnected space. Therefore $(X, T_1, T_2, \leqslant)$ is a pairwise ordered fuzzy extremally disconnected space.

Proposition 2. *For an ordered fuzzy bitopological space $(X, T_1, T_2, \leqslant)$, the following assertions are equivalent.*

(a) *$(X, T_1, T_2, \leqslant)$ is pairwise upper fuzzy extremally disconnected.*

(b) *For each T_1-fuzzy closed decreasing set λ, $D^0_{T_2}(\lambda)$ is T_2-fuzzy closed decreasing. Similarly, for each T_2-fuzzy closed decreasing set λ, $D^0_{T_1}(\lambda)$ is T_1-fuzzy closed decreasing.*

(c) *For each T_1-fuzzy open increasing set λ we have $I_{T_2}(\lambda) + D_{T_2}(1 - I_{T_2}(\lambda)) = 1$. Similarly, for each T_2-fuzzy open increasing set λ we have $I_{T_1}(\lambda) + D_{T_1}(1 - I_{T_1}(\lambda)) = 1$.*

(d) *For each pair of a T_1-fuzzy open increasing set λ and a T_1-fuzzy open decreasing set μ with $I_{T_2}(\lambda)+\mu=1$ we have $I_{T_2}(\lambda)+D_{T_2}(\mu)=1$. Similarly, for each pair of a T_2-fuzzy open increasing set λ and a T_2-fuzzy open decreasing set μ with $I_{T_1}(\lambda)+\mu=1$ we have $I_{T_1}(\lambda)+D_{T_1}(\mu)=1$.*

Proof. (a) $\Rightarrow$ (b). Let λ be any T_1-fuzzy closed decreasing set. We claim that $D^0_{T_2}(\lambda)$ is a T_2-fuzzy closed decreasing set. Now, $1-\lambda$ is T_1-fuzzy open increasing and so by assumption (a), $I_{T_2}(1-\lambda)$ is T_2-fuzzy open increasing. That is, $D^0_{T_2}(\lambda)$ is T_2-fuzzy closed decreasing.

(b) $\Rightarrow$ (c). Let λ be any T_1-fuzzy open increasing set. Then

$$1-I_{T_2}(\lambda)=D^0_{T_2}(1-\lambda). \tag{1}$$

Consider $I_{T_2}(\lambda)+D_{T_2}(1-I_{T_2}(\lambda))=I_{T_2}(\lambda)+D_{T_2}(D^0_{T_2}(1-\lambda))$. As λ is T_1-fuzzy open increasing, $1-\lambda$ is T_1-fuzzy closed decreasing and by assumption (b), $D^0_{T_2}(1-\lambda)$ is T_2-fuzzy closed decreasing. Therefore, $D_{T_2}(D^0_{T_2}(1-\lambda))=D^0_{T_2}(1-\lambda)$. Now,

$$I_{T_2}(\lambda)+D_{T_2}(D^0_{T_2}(1-\lambda))=I_{T_2}(\lambda)+D^0_{T_2}(1-\lambda)=I_{T_2}(\lambda)+1-I_{T_2}(\lambda)=1.$$

That is, $I_{T_2}(\lambda)+D_{T_2}(1-I_{T_2}(\lambda))=1$.

(c) $\Rightarrow$ (d). Let λ be any T_1-fuzzy open increasing set and μ any T_2-fuzzy open decreasing set such that

$$I_{T_2}(\lambda)+\mu=1. \tag{2}$$

By assumption (c),

$$I_{T_2}(\lambda)+D_{T_2}(1-I_{T_2}(\lambda))=1=I_{T_2}(\lambda)+\mu. \tag{3}$$

That is, $\mu=D_{T_2}(1-I_{T_2}(\lambda))$. Since $\mu=1-I_{T_2}(\lambda)$ (by (2)), we have

$$D_{T_2}(\mu)=D_{T_2}(1-I_{T_2}(\lambda)). \tag{4}$$

From (3) and (4) we obtain $I_{T_2}(\lambda)+D_{T_2}(\mu)=1$.

(d) $\Rightarrow$ (a). Let λ be any T_1-fuzzy open increasing set. Put $\mu=1-I_{T_2}(\lambda)$. Clearly, μ is T_2-fuzzy open decreasing and from the construction of μ it follows that $I_{T_2}(\lambda)+\mu=1$. By assumption (d), we have $I_{T_2}(\lambda)+D_{T_2}(\mu)=1$ and so $I_{T_2}(\lambda)=1-D_{T_2}(\mu)$ is T_2-fuzzy open increasing. Therefore $(X,T_1,T_2,\leqslant)$ is upper T_1-fuzzy extremally disconnected. Similarly we can prove that $(X,T_1,T_2,\leqslant)$ is upper T_1-fuzzy extremally disconnected. Hence the Proposition.

Proposition 3. *Let $(X, T_1, T_2, \leqslant)$ be an ordered fuzzy bitopological space. Then $(X, T_1, T_2, \leqslant)$ is a pairwise ordered fuzzy extremally disconnected space if and only if for a T_1-fuzzy decreasing open set λ and a T_2-fuzzy decreasing closed set μ such that $\lambda \leqslant \mu$, we have $D_{T_1}(\lambda) \leqslant D^0_{T_1}(\mu)$.*

Proof. Suppose $(X, T_1, T_2, \leqslant)$ is an upper pairwise fuzzy extremally disconnected space. Let λ be any T_1-fuzzy open decreasing set, μ be any T_2-fuzzy closed decreasing set such that $\lambda \leqslant \mu$. Then by (b) of Proposition 2, $D^0_{T_1}(\mu)$ is T_1-fuzzy closed decreasing. Also, since λ is T_1-fuzzy open decreasing and $\lambda \leqslant \mu$, it follows that $\lambda \leqslant D^0_{T_1}(\mu)$. Again, since $D^0_{T_1}(\mu)$ is T_1-fuzzy closed decreasing, it follows that $D_{T_1}(\lambda) \leqslant D^0_{T_1}(\mu)$. To prove the converse, let μ be any T_2-fuzzy closed decreasing set. By Definition 5, $D^0_{T_1}(\mu)$ is T_1-fuzzy open decreasing and it is also clear that $D^0_{T_1}(\mu) \leqslant \mu$. Therefore it follows by assumption that $D_{T_1}(D^0_{T_1}(\mu)) \leqslant D^0_{T_1}(\mu)$. This implies that $D^0_{T_1}(\mu)$ is T_1-fuzzy closed decreasing. Hence, by (b) of Proposition 2, it follows that $(X, T_1, T_2, \leqslant)$ is upper T_1-fuzzy extremally disconnected. Similarly we can prove also the other cases.

Notation. An ordered fuzzy set which is both fuzzy decreasing (increasing) open and closed is called a fuzzy decreasing (increasing) clopen set.

Remark 1. Let $(X, T_1, T_2, \leqslant)$ be a pairwise upper fuzzy extremally disconnected space. Let $\{\lambda_i, 1 - \mu_i \colon i \in \mathbb{N}\}$ be a collection such that λ_i's are T_1-fuzzy open decreasing sets, μ_i's are T_2-fuzzy closed decreasing sets and let λ, $1 - \mu$ be T_1-fuzzy open decreasing and T_2-fuzzy open increasing sets, respectively. If $\lambda_i \leqslant \lambda \leqslant \mu_j$ and $\lambda_i \leqslant \mu \leqslant \mu_j$ for all $i, j \in \mathbb{N}$, then there exists a T_1- and T_2-fuzzy clopen decreasing set γ such that $D_{T_1}(\lambda_i) \leqslant \gamma \leqslant D^0_{T_1}(\mu_j)$ for all $i, j \in \mathbb{N}$. By Proposition 3, $D_{T_1}(\lambda_i) \leqslant D_{T_1}(\lambda) \wedge D^0_{T_1}(\mu) \leqslant D^0_{T_1}(\mu_j)$ $(i, j \in \mathbb{N})$. Put $\gamma = D_{T_1}(\lambda) \wedge D^0_{T_1}(\mu)$. Now, γ satisfies the required condition.

Proposition 4. *Let $(X, T_1, T_2, \leqslant)$ be a pairwise ordered fuzzy extremally disconnected space. Let $\{\lambda_q\}_{q \in Q}$ and $\{\mu_q\}_{q \in Q}$ be monotone increasing collections of T_1-fuzzy open decreasing sets and T_2-fuzzy closed decreasing sets of $(X, T_1, T_2, \leqslant)$, respectively, and suppose that $\lambda_{q_1} \leqslant \mu_{q_2}$ whenever $q_1 < q_2$ (Q is the set of rational numbers). Then there exists a monotone increasing collection $\{\gamma_q\}_{q \in Q}$ of T_1- and T_2-fuzzy clopen decreasing subsets of $(X, T_1, T_2, \leqslant)$ such that $D_{T_1}(\lambda_{q_1}) \leqslant \gamma_{q_2}$ and $\gamma_{q_1} \leqslant D^0_{T_1}(\mu_{q_2})$ whenever $q_1 < q_2$.*

Proof. Let us arrange the rational numbers into a sequence $\{q_n\}$ without repetitions. For every $n \geqslant 2$, we define inductively a collection $\{\gamma_{q_i} \colon 1 \leqslant i < n\} \subset I^X$

such that

$$(\mathrm{S}_n) \qquad \begin{aligned} D_{T_1}(\lambda_q) &\leqslant \gamma_{q_i} \quad \text{if } q < q_i, \\ \gamma_{q_i} &\leqslant D^0_{T_1}(\mu_q) \quad \text{if } q_i < q, \end{aligned}$$

for all $i < n$.

By Proposition 3, the countable collections $\{D_{T_1}(\lambda_q)\}$ and $\{D^0_{T_1}(\mu_q)\}$ satisfy $D_{T_1}(\lambda_{q_1}) \leqslant D^0_{T_1}(\mu_{q_2})$ if $q_1 < q_2$. By Remark 1, there exists a T_1- and T_2-fuzzy clopen decreasing set δ_1 such that $D_{T_1}(\lambda_{q_1}) \leqslant \delta_1 \leqslant D^0_{T_1}(\mu_{q_2})$. Setting $\gamma_{q_1} = \delta_1$ we get (S_2). Assume that T_1-fuzzy sets γ_{q_i} are already defined for $i < n$ and satisfy (S_n). Define $\Sigma = \bigvee\{\gamma_{q_i}\colon\ i < n,\ q_i < q_n\} \vee \lambda_{q_n}$ and $\Phi = \bigwedge\{\gamma_{q_j}\colon\ j < n,\ q_j > q_n\} \wedge \mu_{q_n}$. Then we have that $D_{T_1}(\gamma_{q_i}) \leqslant D_{T_1}(\Sigma) \leqslant D^0_{T_1}(\gamma_{q_j})$ and $D_{T_1}(\gamma_{q_i}) \leqslant D^0_{T_1}(\Phi) \leqslant D^0_{T_1}(\gamma_{q_j})$ whenever $q_i < q_n < q_j$ $(i, j < n)$ as well as $\lambda_q \leqslant D_{T_1}(\Sigma) \leqslant \mu_{q'}$ and $\lambda_q \leqslant D^0_{T_1}(\Phi) \leqslant \mu_{q'}$ whenever $q < q_n < q'$. This shows that the countable collections $\{\gamma_{q_i}\colon\ i < n, q_i < q_n\} \cup \{\lambda_q\colon\ q < q_n\}$ and $\{\gamma_{q_j}\colon\ j < n, q_j > q_n\} \cup \{\mu_q\colon\ q > q_n\}$ together with Σ and Φ fulfil all conditions of Remark 1. Hence, there exists a T_1- and T_2-fuzzy clopen decreasing set δ_n such that $D_{T_1}(\delta_n) \leqslant \mu_q$ if $q_n < q$, $\lambda_q \leqslant D^0_{T_1}(\delta_n)$ if $q < q_n$, $D_{T_1}(\gamma_{q_i}) \leqslant D^0_{T_1}(\delta_n)$ if $q_i < q_n$, $D_{T_1}(\delta_n) \leqslant D^0_{T_1}(\gamma_{q_j})$ if $q_n < q_j$, where $1 \leqslant i$, $j \leqslant n-1$. Now setting $\gamma_{q_n} = \delta_n$ we obtain T_1-fuzzy sets $\gamma_{q_1}, \gamma_{q_2}, \ldots, \gamma_{q_n}$ that satisfy (S_{n+1}). Therefore the collection $\{\gamma_{q_i}\colon\ i = 1, 2, \ldots\}$ has the required property. This completes the proof. □

Definition 7. Let $(X, T_1, T_2, \leqslant)$ and $(Y, S_1, S_2, \leqslant)$ be ordered fuzzy bitopological spaces. A mapping $f\colon\ (X, T_1, T_2, \leqslant) \to (Y, S_1, S_2, \leqslant)$ is called increasing (decreasing) T_1-fuzzy continuous if $f^{-1}(\lambda)$ is a T_1-fuzzy open increasing (decreasing) subset of $(X, T_1, T_2, \leqslant)$ for every S_1- or S_2-fuzzy open subset λ of $(Y, S_1, S_2, \leqslant)$. If f is both increasing and decreasing T_1-fuzzy continuous, then it is called ordered T_1-fuzzy continuous.

Definition 8. Let $(X, T_1, T_2, \leqslant)$ be an ordered fuzzy bitopological space. A function $f\colon\ X \to \mathbb{R}(I)$ is called lower (upper) T_1-fuzzy continuous if $f^{-1}(R_t)(f^{-1}(L_t))$ is increasing or decreasing T_1-fuzzy open for each $t \in \mathbb{R}$. Similarly we can define a lower and upper T_1-fuzzy continuous function.

Lemma 1. *Let $(X, T_1, T_2, \leqslant)$ be an ordered fuzzy bitopological space, let $\lambda \in I^X$, and let*

$$f\colon\ X \to \mathbb{R}(I) \text{ be such that } f(x)(t) = \begin{cases} 1 & \text{if } t < 0, \\ \lambda(x) & \text{if } 0 \leqslant t \leqslant 1, \\ 0 & \text{if } t > 1, \end{cases}$$

for all $x \in X$. Then f is lower (upper) T_1-fuzzy continuous iff λ is a T_1-fuzzy open (closed) increasing or decreasing set.

Definition 9. The characteristic function of $\lambda \in I^X$ is the map $\chi_\lambda\colon X \to [0,1]$ defined by $\chi_\lambda(x) = (\lambda(x))$, $x \in X$.

Proposition 5. *Let $(X, T_1, T_2, \leqslant)$ be an ordered fuzzy bitopological space, and let $\lambda \in I^X$. Then χ_λ is lower (upper) T_1-fuzzy continuous iff λ is a T_1-fuzzy open (closed) increasing or decreasing set.*

P r o o f. The proof follows from Lemma 1. □

Proposition 6. *Let $(X, T_1, T_2 \leqslant)$ be an ordered fuzzy bitopological space. Then the following statements are equivalent.*

(a) *$(X, T_1, T_2, \leqslant)$ is pairwise ordered fuzzy extremally disconnected.*

(b) *If $g, h\colon X \to \mathbb{R}(I)$, g is lower T_1-fuzzy continuous, h is upper T_2-fuzzy continuous and $g \leqslant h$, then there exists an increasing T_1- and T_2-fuzzy continuous function $f\colon (X, T_1, T_2, \leqslant) \to \mathbb{R}(I)$ such that $g \leqslant f \leqslant h$.*

(c) *If $1 - \lambda$ is T_2-fuzzy open increasing and μ is T_1-fuzzy open decreasing such that $\mu \leqslant \lambda$, then there exists an increasing T_1- and T_2-fuzzy continuous function $f\colon (X, T_1, T_2, \leqslant) \to [0,1]$ such that $\mu \leqslant (1 - L_1)f \leqslant R_0 f \leqslant \lambda$.*

P r o o f. (a) $\Rightarrow$ (b). Define $H_r = L_r h$ and $G_r = (1 - R_r)g$, $r \in Q$. Thus we have two monotone increasing families of, respectively, T_1-fuzzy open decreasing and T_2-fuzzy closed decreasing subsets of $(X, T_1, T_2, \leqslant)$. Moreover, $H_r \leqslant G_s$ if $r < s$. By Proposition 4, there exists a monotone increasing family $\{F_r\}_{r \in Q}$ of T_1-and T_2-fuzzy clopen decreasing sets of $(X, T_1, T_2, \leqslant)$ such that $D_{T_1}(H_r) \leqslant F_s$ and $F_r \leqslant D^0_{T_1}(G_s)$ whenever $r < s$. Letting $V_t = \bigwedge\limits_{r<t} (1 - F_r)$ for all $t \in \mathbb{R}$, we define a monotone decreasing family $\{V_t\colon t \in \mathbb{R}\} \subset I^X$. Moreover, we have $I_{T_1}(V_t) \leqslant I^0_{T_1}(Vs)$ whenever $s < t$.

One can easily prove that f is well defined and it is the required extension by making use of Theorem 3.7 of Kubiak (1987) [10], of the concepts of ordered fuzzy topology and fuzzy bitopology.

(b) $\Rightarrow$ (c). Suppose $1 - \lambda$ is a T_2-fuzzy open increasing set and μ is a T_1-fuzzy open decreasing set, $\mu \leqslant \lambda$. Then $\chi_\mu \leqslant \chi_\lambda$ and χ_μ, χ_λ are lower T_1- and upper T_2-fuzzy continuous functions, respectively. Hence by (b), there exists an increasing T_1- and T_2-fuzzy continuous function $f\colon (X, T, \leqslant) \to \mathbb{R}(I)$ such that $\chi_\mu \leqslant f \leqslant \chi_\lambda$. Clearly, $f(x) \in [0,1]$ for all $x \in X$ and $\mu = (1 - L_1)\chi_\mu \leqslant (1 - L_1)f \leqslant R_0 f \leqslant R_0 \chi_\lambda = \lambda$.

(c) $\Rightarrow$ (a). This follows from Proposition 3 and the fact that $(1 - L_1)f$ and $R_0 f$ are T_1-fuzzy closed decreasing and T_2 fuzzy open decreasing sets, respectively.

N o t e 1. Propositions 2 to 5 and Remark 1 can be discussed also for other cases.

3. Tietze extension theorem for pairwise ordered fuzzy extremally disconnected spaces

In this section, Tietze extension theorem for pairwise ordered fuzzy extremally disconnected spaces is studied.

Proposition 7 (Tietze Extension Theorem). *Let $(X,T_1,T_2,\leqslant)$ be a pairwise ordered fuzzy extremally disconnected space. Let $A\in X$ such that χ_A is T_1-fuzzy open increasing and T_2-fuzzy open increasing set and let $f\colon (A,T_1/A,T_2/A,\leqslant)\to I$ be an increasing T_1- and T_2-fuzzy continuous and isotone function. Then f admits an extension $F\colon (X,T_1,T_2,\leqslant)\to I$ with all its properties preserved if f satisfies the following # property:*

$$(\#)\qquad [\lambda]<[\mu]\Rightarrow f^{-1}\{\chi_{[[0],[\lambda]]}\}<f^{-1}\{\chi_{[[\mu],[1]]}\},$$

where $\delta<\theta\Leftrightarrow D_{T_2}f(\delta)\wedge I_{T_2}L(\sigma)(\theta)=0$ and

$$[[\lambda_1],[\lambda_2]]=\{[\mu]\in I(L)\colon\ [\lambda_1]\leqslant[\mu]\leqslant[\lambda_2]\}.$$

P r o o f. Define $g,h\colon X\to[0,1]$ by

$$g(x)=\begin{cases} f(x) & \text{if } x\in A,\\ [\lambda_0] & \text{if } x\notin A;\end{cases}$$
$$h(x)=\begin{cases} f(x) & \text{if } x\in A,\\ [\lambda_1] & \text{if } x\notin A\end{cases}$$

where $[\lambda_0]$ is the equivalence class [10] determined by $\lambda_0\colon R\to I$ such that

$$\lambda_0(t)=\begin{cases} 1 & \text{if } t<0,\\ 0 & \text{if } t>0,\end{cases}$$

$[\lambda_1]$ is the equivalence class [10] determined by $\lambda_1\colon R\to I$ such that

$$\lambda_1(t)=\begin{cases} 1 & \text{if } t<1,\\ 0 & \text{if } t>1,\end{cases}$$

g is lower T_1-fuzzy continuous, h is upper T_2-fuzzy continuous and $g\leqslant h$. Hence by Proposition 6 there exists an increasing T_1- and T_2-fuzzy continuous function $F\colon (X,T_1,T_2,\leqslant)\to[0,1]$ such that $g(x)\leqslant F(x))\leqslant h(x)$ for all $x\in X$. Hence for all $x\in A$ we have $g(x)\leqslant f(x))\leqslant h(x)$, so that F is the required extension of f over X. Moreover, F is isotone as f satisfies the # property. Hence the theorem.

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