λ-LATTICES

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Abstract. In this paper, we generalize the notion of supremum and infimum in a poset.

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1. λ-posets and λ-lattices

For terminology and notation throughout the paper see [MMT] and [GR].

Let $P = (P, \leqslant)$ be an ordered set. If $A \subseteq P$, denote by $L(A)$ and $U(A)$

$$L(A) = \{x \in P;\ x \leqslant a \text{ for all } a \in A\}$$
$$U(A) = \{x \in P;\ x \geqslant a \text{ for all } a \in A\}.$$

Call $L(A)$ or $U(A)$ the lower or upper cone of A, respectively. If B is a finite family of elements of P, say $A = \{a_1, a_2, \ldots, a_n\}$, we write briefly $L(a_1, a_2, \ldots, a_n)$ or $U(a_1, a_2, \ldots, a_n)$ for $L(A)$ or $U(A)$, respectively.

A λ-poset is a poset $(P, \leqslant)$, where $L(a, b) \neq \emptyset \neq U(a, b)$ for every two elements $a, b \in P$, with a choice function λ where λ is choosing a single element from $L(a, b)$ as well from $U(a, b)$ and λ satisfies the following condition:

$$(*) \qquad \text{if } a \leqslant b \text{ then } \lambda(L(a, b)) = a \text{ and } \lambda(U(a, b)) = b.$$

The chosen element $\lambda(L(a, b))$ is denoted by $a \cdot b$ and $\lambda(U(a, b))$ by $a + b$. After the choice of λ, the elements $a \cdot b$ and $a + b$ are fixed. Because $L(a, b) = L(b, a)$ and $U(a, b) = U(b, a)$, the choice of λ is independent on the order of the elements a and b. On the other hand, the choice is not assumed to be consequential, i.e. if

$L(a,b) = L(c,d)$ for some elements $a,b,c,d \in P$, $(a,b) \neq (c,d)$, $a \cdot b$ and $c \cdot b$ need not be equal; and analogously for $a+b$ and $c+d$. Thus the choice of λ depends on the elements a and b only.

Definition 1.1. A λ-lattice is an algebra $P = (P, \cdot, +)$ where $+$ and $\cdot$ are two binary operations on P, satisfying the following laws for all $a,b,c \in P$:

i.) $a \cdot a = a$	i_+) $a + a = a$
c.) $a \cdot b = b \cdot a$	c_+) $a + b = b + a$
t.) $a \cdot ((a \cdot b) \cdot c) = (a \cdot b) \cdot c$	t_+) $a + ((a+b)+c) = (a+b)+c$
a.) $a \cdot (a+b) = a$	a_+) $a + (a \cdot b) = a$

Theorem 1.1. *Let $(P, \leqslant, \lambda)$ be a λ-poset. Then the algebra $P = (P, \cdot, +)$ with binary operations $\cdot$ and $+$, where $a \cdot b = \lambda(L(a,b))$ and $a + b = \lambda(U(a,b))$, is a λ-lattice.*

Proof. i.): $a \cdot a = \lambda(L(a,a)) = a$. c.) is true because the choice λ is independent of the order of the elements a and b. t.): Because $a \cdot b$ is from $L(a,b)$, $a \cdot b \leqslant a$ in $(P, \leqslant)$; analogously $(a \cdot b) \cdot c \leqslant a \cdot b$, and from transitivity $(a \cdot b) \cdot c \leqslant a$. According to $(*)$ we have $a \cdot ((a \cdot b) \cdot c) = \lambda(L((a \cdot b) \cdot c, a)) = (a \cdot b) \cdot c$. a.): Because $a + b \in U(a,b)$, $a + b \leqslant a$. According to $(*)$ we have $\lambda(L(a+b,a)) = a$. The identities i_+), c_+), t_+) and a_+) can be proved analogously. □

Theorem 1.2. *Let $(P, \cdot, +)$ be a λ-lattice. A λ-poset $(P, \leqslant)$ is obtained by putting $a \cdot b = a \iff a \leqslant b$. Moreover, if $(P, \leqslant)$ is a λ-poset $(P, \cdot, +)$ the λ-lattice induced by $(P, \leqslant)$, and $(P, \preceq)$ the λ-poset induced by $(P, \cdot, +)$ then $(P, \leqslant) = (P, \preceq)$.*

Proof. The validity of the latter assertion is clear after proving the first one because $a \leqslant b \iff a \cdot b = a \iff a \preceq b$. So it remains to show that the order $a \leqslant b$ induced by $a \cdot b = a$ is a partial order. In fact, we begin with proving that $a \cdot b = a \iff a + b = b$.

$\Rightarrow$: $a + b = (a \cdot b) + b = b$ according to the absorption law in a_+)

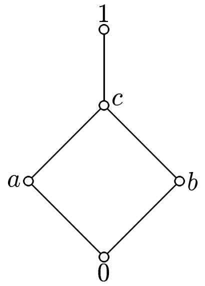
$\Leftarrow$: $a \cdot b = (a+b) \cdot a = a$ according to the absorption law in a.)

Because of i.) $a \leqslant a$ for all $a \in P$. Let $a \leqslant b$ and $b \leqslant a$. Then $a \cdot b = a$ and $b \cdot a = a$; $a = b$ follows from c.). Moreover $a \leqslant b$ and $b \leqslant c$, give $a + b = b$ and $b + c = c$. Now

$$c = b + c = (a+b) + c = a + ((a+b)+c) = a + (b+c) = a + c$$

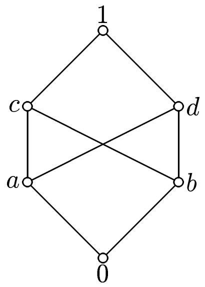
according to t_+), whence $a \leqslant c$. Thus $\leqslant$ is a partial order in P. It remains to show that $L(a,b) \neq \emptyset \neq U(a,b)$ for every two elements $a,b \in P$. In fact, we will show that $a + b \in U(a,b)$; the proof is analogous to that of $a \cdot b \in L(a,b)$. From $(a+b) \cdot a = a$ it follows that $a \leqslant a + b$ and, analogously, $b \leqslant a + b$. Thus $a + b \in U(a,b)$. □

Examples.



$\cdot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$+$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	c	1
b	b	1	b	c	1
c	c	c	c	c	1
1	1	1	1	1	1



$\cdot$	0	a	b	c	d	1
0	0	0	0	0	d	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	a	c
d	0	a	b	a	d	d
1	0	a	b	c	d	1

$+$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	d	c	d	1
b	b	d	b	c	d	1
c	c	c	c	c	1	1
d	d	d	d	1	d	1
1	1	1	1	1	1	1

2. Ideals and strong ideals

A subset $I \subseteq P$ of a λ-lattice $(P, \cdot, +)$ is called an ideal of P, if $i \in I$ and $a \in P \Rightarrow i \cdot a \in I$, and if $i, j \in I \Rightarrow i + j \in I$.

The set theoretical intersection of an arbitrary system of ideals is clearly an ideal again, thus the set $\mathrm{Id}(P)$ of all ideals of P forms a complete lattice with respect to set inclusion. Evidently $I \wedge J = I \cap J$ and $I \vee J = \bigcap\{K;\, I, J \subseteq K$ and K is an ideal of $P\}$.

If B is any subset of P then the ideal generated by B, denoted by $I(B)$, is the intersection of all ideals of P containing B. If B is a finite set $\{a_1 \dots a_n\}$ we will write $I(a_1 \dots a_n)$ for $I(\{a_1 \dots a_n\})$. An ideal J is said to be strong if it satisfies the following condition:

$$a \cdot b \in J \Rightarrow I(a) \cap I(b) \subseteq J.$$

In a lattice every ideal is strong.

Note that $(a] = \{x;\, x \leqslant a, a \in P\}$ need not be an ideal of P.

Theorem 2.1. *Let $(P, \cdot, +)$ be a λ-lattice induced by a λ-poset $(P, \leqslant, \lambda)$. If $(a]$ is an ideal for every $a \in P$, then $(P, \leqslant)$ is a join-semilattice.*

Proof. If $a \leqslant x$ and $b \leqslant x$ then since $a, b \in (x]$ and since $(x]$ is an ideal, we have $a + b \in (x]$ whence $a + b \leqslant x$. $\square$

Theorem 2.2. *Let $(P, \cdot, +)$ be a λ-lattice induced by a λ-poset $(P, \leqslant, \lambda)$. If $(a]$ is a strong ideal for every $a \in P$, then $(P, \leqslant)$ is a lattice.*

Proof. According to Theorem 2.1, P is a join-semilattice and, we need only to show that P is also a meet-semilattice. If $x \leqslant a$ and $x \leqslant b$ then $(x] \subset (a]$ and $(x] \subseteq (b]$. Since $a \cdot b \leqslant (a \cdot b]$ and $(a \cdot b]$ is a strong ideal, then $(x] \subseteq (a] \cap (b] \subseteq (a \cdot b]$. Hence we have $x \leqslant a \cdot b$. $\square$

3. On congruences on a λ-lattice

Denote by $\operatorname{Con} P$ the lattice of all congruences on a λ-lattice P. Let $\Theta \in \operatorname{Con} P$. As usually, ω is the least element ($x \equiv y(\omega) \iff x = y$) and $\imath$ is the greatest element of $\operatorname{Con} P$ ($x \equiv y(\imath)$ for every elements $x, y \in P$). Further $x \equiv y(\Theta \wedge \Phi) \iff x \equiv y(\Theta)$ and $x \equiv y(\Phi)$. Moreover $x \equiv y(\Theta \vee \Phi) \iff$ there is a sequence $x = z_0, z_1, \ldots, z_n = y$ of elements such that $z_{j-1} \equiv z_j(\Theta)$ or $z_{j-1} \equiv z_j(\Phi)$ for every $i, i = 1, \ldots, n$. A subset $K \subseteq P$ of a λ-lattice P is called convex, if $a, b \in K$, $t \in P$ and $a \leqslant t \leqslant b \Rightarrow t \in K$.

Lemma 3.1. *Let P be a λ-lattice. Then $[a]\Theta$ is a convex sub-λ-lattice for every $a \in P$.*

Proof. First we prove that $[a]\Theta$ is a sub-λ-lattice. From $x \equiv a(\Theta)$ and $y \equiv a(\Theta)$ it follows that $x + y \equiv a(\Theta)$ and $x \cdot y \equiv a(\Theta)$ and we have that $[a]\Theta$ is a sub-λ-lattice. Further we prove that $[a]\Theta$ is convex. If $x \leqslant t \leqslant y$, $x, y \in [a]\Theta$ and $t \in P$ then $t = t \cdot y \equiv t \cdot a(\Theta)$, $t = t + x \equiv (t \cdot a) + x \equiv (t \cdot a) + a = a(\Theta)$, so we have $t \in [a]\Theta$. $\square$

Theorem 3.2. *Let P be a λ-lattice. A reflexive binary relation on P is a congruence on P iff the following three properties are satisfied for any $x, y, z, t \in P$.*

(i) $x \equiv y(\Theta) \iff x + y \equiv x \cdot y(\Theta)$;
(ii) $x \leqslant y \leqslant z,\ x \equiv y(\Theta),\ y \equiv z(\Theta) \Rightarrow x \equiv z(\Theta)$;
(iii) $x \leqslant y$ and $x \equiv y(\Theta) \Rightarrow x \cdot t \equiv y \cdot t(\Theta),\ x + t \equiv y + t(\Theta)$.

Proof. If Θ is a congruence on P, then it obviously satisfies the conditions (i), (ii) and (iii). Hence we will prove the converse condition only. At first we prove that if $b, c \in [a, d] = \{x;\, a \leqslant x \leqslant d\}$ and if $a \equiv d(\Theta)$ then $b \equiv c(\Theta)$. According to (iii), we obtain $b \equiv d(\Theta), a \equiv b(\Theta)$. By using of (iii) again we obtain $b \cdot c \equiv c(\Theta)$, $c \equiv c + b(\Theta)$. Because $b \cdot c \leqslant c \leqslant b + c$, (ii) implies $b \cdot c \equiv b + c(\Theta)$, and by (i) also

$b \equiv c(\Theta)$. According to (i) Θ is symmetric. To prove transitivity of Θ, we assume that $x \equiv y(\Theta), y \equiv z(\Theta)$. Then by (i) $x \cdot y \equiv x + y(\Theta)$, $y \cdot z \equiv y + z(\Theta)$, and by (iii)

$$\begin{aligned} y + z &= (y + z) + (y \cdot x) \equiv (y + z) + (y + x)(\Theta), \\ y \cdot z &= (y \cdot z) \cdot (y + x) \equiv (y \cdot z) \cdot (y \cdot x)(\Theta). \end{aligned}$$

Because $y + z \equiv (y + z) + (y + x)(\Theta)$, $y \cdot z \equiv (y \cdot z) \cdot (y \cdot x)(\Theta)$ and

$$(y \cdot z) \cdot (y \cdot x) \leqslant y \cdot z \leqslant y + z \leqslant (y + z) + (y + x),$$

we apply (ii) twice to obtain

$$(y \cdot z) \cdot (y \cdot x) \equiv (y + z) + (y + x)(\Theta).$$

Because

$$x, z \in [(y \cdot z) \cdot (y \cdot x), (y + z) + (y + x)],$$

the proof of the preceding paragraph imply that $x \equiv z(\Theta)$. Next we prove the assertion: if $x \equiv y(\Theta)$, then $x+t \equiv y+t(\Theta)$. Since $x, y \in [x \cdot y, x+y]$, (i) and the proof of the first paragraph imply that $x \equiv x+y(\Theta), y \equiv x+y(\Theta)$. Now, according to (iii), $x+t \equiv (x+y)+t(\Theta), y+t \equiv (x+y)+t(\Theta)$, and by applying transitivity proved above, we obtain $x+t \equiv y+t(\Theta)$. Now we are able to prove the substitution property of Θ for $+$: Let $x_0 \equiv y_0(\Theta), x_1 \equiv y_1(\Theta)$. Then $x_0 + x_1 \equiv x_0 + y_1(\Theta), x_0 + y_1 \equiv y_0 + y_1(\Theta)$, and according to the transitivity, also $x_0+x_1 \equiv y_0+y_1(\Theta)$. The substitution property for $\cdot$ can be proved similarly. $\square$

Theorem 3.3. *The lattice* Con P *of all congruences on a λ-lattice P is distributive.*

Proof. Consider a λ-lattice term

$$M(x, y, z) = ((x \cdot y) + (y \cdot z)) + (z \cdot x).$$

It is a routine to show that for all $a, b \in P$

$$\begin{aligned} M(a, a, b) &= a \\ M(a, b, a) &= a \\ M(b, a, a) &= a, \end{aligned}$$

i.e. $M(x, y, z)$ is a majority term and, therefore, Con P is distributive. $\square$

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