STRUCTURE OF FOURIER EXPONENTS OF ALMOST PERIODIC FUNCTIONS AND PERIODICITY OF ALMOST PERIODIC FUNCTIONS

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Abstract. The paper deals with almost periodic functions which are limits of sequences of continuous periodic functions, and determines the structure of their Fourier exponents and their ranges. It is shown that the class $CP(\mathscr{X})$ of continuous periodic functions is not densely distributed in the space $AP(\mathscr{X})$.

Keywords: almost periodicity (Bohr), Fourier coefficient, Fourier exponent, Bochner transformation

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1. Introduction

1.1. Preliminaries. Oscillating motions enjoy a privileged position in many technical as well as theoretical domains. The central role is played here by periodic motions represented by periodic functions. Unfortunately, the class of periodic functions is not linear since the sum of two periodic functions, which do not have a non-zero period in common, gives rise to a non-periodic function. This lack of linearity is circumvented by introducing almost periodic functions.

1.2. Properties of continuous periodic functions. The definition of an almost periodic function is based upon two properly generalized concepts: the period to the so-called almost period and the periodic distribution of periods to the so-called relative density of almost periods.

An equivalent definition makes use of the generalization of the relative compactness of translates of a continuous periodic function.

2. Notation and definitions

2.1. Sets and set-theoretical operations. Standard set-theoretical notation is used. The symbol $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{Z}$ the set of all integers, $\mathbb{Q}$ the set of all rational numbers, $\mathbb{R}$ the set of all real numbers, $\mathbb{C}$ the set of all complex numbers. If M is a non-empty set of numbers then $G(M)$ denotes the smallest additive group which contains M as a subset. $G(\emptyset) = \emptyset$.

2.2. Spaces, mappings, functions. A Banach space is also called a B-space and is denoted by capital letters $\mathscr{X}$, $\mathscr{Y}$. The norm in the B-space $\mathscr{X}$ is denoted by $\|\cdot\|_{\mathscr{X}}$ or simply $\|\cdot\|$ if no confusion may arise.

If $\mathscr{X}$ is a B-space (e. g. also $\mathbb{R}$ or $\mathbb{C}$ normed by the absolute value), then $C(\mathscr{X})$ denotes all mappings (functions) $f\colon \mathbb{R} \to \mathscr{X}$ continuous on $\mathbb{R}$. The symbol $CB(\mathscr{X})$ denotes the set of all functions from $C(\mathscr{X})$ the range R_f of which is a bounded set in $\mathscr{X}$. In the space $CB(\mathscr{X})$ the norm is introduced by the formula

$$\|f\| = \|f\|_{CB(\mathscr{X})} = \sup_t \|f(t)\|_{\mathscr{X}} = \sup\{\|f(t)\|_{\mathscr{X}} : t \in \mathbb{R}\}$$

for $f \in CB(\mathscr{X})$. This makes $CB(\mathscr{X})$ into a B-space. The symbol $CP(\mathscr{X})$ stands for the set of all periodic functions from $CB(\mathscr{X})$.

2.3. Translates. If a function $f \in C(\mathscr{X})$ and a number $s \in \mathbb{R}$ are given, then the function $g \in C(\mathscr{X})$ defined by the formula $g(t) = f(t+s)$, $t \in \mathbb{R}$, is called the s-translate (s-shift) or simply the translate (the shift) of the function f.

In the sequel we often handle translates and limits of sequences of translates, hence for a given sequence $\alpha = \{\alpha_m\}$ of real numbers and for a given function $f \in C(\mathscr{X})$ the symbol $T_\alpha f$ is introduced for convenience in writing by the formula

$$T_\alpha f(t) = \lim_{m\to\infty} f(t+\alpha_m) = \lim f(t+\alpha_m)$$

for $t \in \mathbb{R}$. The statement "$T_\alpha f$ exists (or $T_\alpha f = g$) uniformly" means that the limit exists (or exists and is equal to $g(t)$) uniformly for $t \in \mathbb{R}$.

If the sequence α is constant and $\alpha_m = h$, $m = 1, 2, \ldots$, then the h-translate of the function f is denoted alternatively by $T_h f$.

2.4. Relative compactness. Let M be a non-empty subset of a B-space $\mathscr{X}$. M is called relatively compact (in $\mathscr{X}$) if the closure $\overline{M}$ is compact.

3.1. Definition and basic properties. Harald Bohr (1887–1951), creator of the theory of almost periodic functions, published his results in the twenties of this century. Since then his theory has been developed by a number of outstanding mathematicians.

Definition 3.1. Let $f \in C(\mathscr{X})$, where $\mathscr{X}$ is a B-space, let ε be a positive number. A real number τ is called an *ε-almost period* or merely an *almost period* of the function f if $\|f(t+\tau) - f(t)\| \leqslant \varepsilon$ for all $t \in \mathbb{R}$.

Definition 3.2. A set M of real numbers is said to be *relatively dense* in $\mathbb{R}$ if there exists a positive number l such that for any real number a the intersection $M \cap \langle a, a+l \rangle$ is non-empty.

Definition 3.3. A function $f \in C(\mathscr{X})$ is said to be *$\mathscr{X}$-almost periodic* or merely *almost periodic* if for any positive number ε the set of all ε-almost periods of the function f is relatively dense in $\mathbb{R}$.

Any continuous periodic function, *à fortiori* any constant function, is readily seen to be almost periodic. It is easy to derive that an almost periodic function has a relatively compact range in $\mathscr{X}$, is uniformly continuous and that the limit of a uniformly convergent sequence of almost periodic functions is again an almost periodic function.

An equivalent definition of an almost periodic function is due to Salomon Bochner:

Definition 3.4. A function $f \in C(\mathscr{X})$ is called *almost periodic* if any sequence α' of real numbers contains a subsequence α such that $T_\alpha f$ exists uniformly.

This Bochner definition ensures that the sum or the product (if it exists) of a finite number of almost periodic functions or the limit of a uniformly on $\mathbb{R}$ convergent sequence of almost periodic functions are again almost periodic functions. This means that the space $AP(\mathscr{X})$ of all $\mathscr{X}$-almost periodic functions is a closed linear subspace of the space $CB(\mathscr{X})$ with the induced norm, i. e. $AP(\mathscr{X})$ is a B-space.

3.2. Trigonometric polynomials. Given N pairwise different real numbers $\lambda_1, \ldots, \lambda_N$ and N elements $b_1, \ldots, b_N$ from the B-space $\mathscr{X}$, where $N \in \mathbb{N}$, then the function Q defined by the formula

$$Q(t) = b_1 \exp(\mathrm{i}\lambda_1 t) + \ldots + b_N \exp(\mathrm{i}\lambda_N t), \quad t \in \mathbb{R},$$

is called an *$\mathscr{X}$-trigonometric* or simply *trigonometric polynomial.* Any trigonometric polynomial Q is an almost periodic function. (The value e^t of the exponential function is denoted here by $\exp(t)$.)

3.3. Harmonic analysis. Given an almost periodic function f then the number

$$M(f) = M_t\{f(t)\} = \lim_{T\to+\infty} \frac{1}{T}\int_a^{a+T} f(t)\,\mathrm{d}t,$$

which exists uniformly with respect to $a \in \mathbb{R}$, is called the *mean value* of the function f. This means that $M_t\{f(t+a)\} = M(f)$ for every $a \in \mathbb{R}$.

The function

$$a(\lambda) = M_t\{f(t)\exp(-\mathrm{i}\lambda t)\}, \quad \lambda \in \mathbb{R},$$

is called the *Bohr transformation* of an almost periodic function f, and a real number λ for which $a(\lambda) = a(\lambda, f)$ is non-zero, is called the *Fourier exponent* and $a(\lambda)$ is called the *Fourier coefficient* of the almost periodic function f. The inequality $\|a(\lambda, f)\|_{\mathscr{X}} \leqslant \|f\|$ holds for any Fourier exponent $a(\lambda, f)$.

The set of all Fourier exponents of an almost periodic function f will be denoted by Λ_f. The set Λ_f is at most countable. If f is a zero function then $\Lambda_f = \emptyset$.

The theory of almost periodic functions shows (see [1], [7]) that for any $\mathscr{X}$-almost periodic function f and for any $\varepsilon > 0$ there exists an $\mathscr{X}$-trigonometric polynomial $Q = Q_{\varepsilon,f}$ such that $\|Q - f\| \leqslant \varepsilon$ and $\Lambda_Q \subset \Lambda_f$. Hence, the set of all $\mathscr{X}$-trigonometric polynomials is densely distributed in $AP(\mathscr{X})$.

3.4. Kronecker's theorem. If $\lambda_1, \dots, \lambda_N$ and $\theta_1, \dots, \theta_N$, where $N \in \mathbb{N}$, are real numbers, then a necessary and sufficient condition that for any positive number δ the system of (congruent) inequalities

$$|\lambda_j t - \theta_j| \leqslant \delta \pmod{2\pi}, \quad j = 1, \dots, N,$$

have at least one solution is that each equality $q_1\lambda_1 + \ldots + q_N\lambda_N = 0$ where $q_1, \dots, q_N$ are integers, imply the (congruent) equality $q_1\theta_1 + \ldots + q_N\theta_N = 0 \pmod{2\pi}$.

R e m a r k 3.5. The elements of the theory of almost periodic functions including the proof of the Kronecker theorem may be found in [1], [5], [6], [7].

4. Regular sequences

4.1. Definitions and basic properties. Given any function defined on $\mathbb{R}$ then with every sequence of real numbers we associate a sequence of translates of the given function.

Definition 4.1. Let $f \in C(\mathscr{X})$ and let α be a sequence of real numbers. The sequence α is called a *regular sequence* of the function f if $T_\alpha f$ exists uniformly. The set of all regular sequences of the function f will be denoted by the symbol S_f.

E x a m p l e 4.2. 1) For the function $f(t) = t$, $t \in \mathbb{R}$, the set S_f coincides with the set of all convergent sequences of real numbers. The same is true for the function $f(t) = \arctan t$, $t \in \mathbb{R}$.

2) If $f \in C(\mathscr{X})$ is a constant function then S_f is the set of all sequences of real numbers.

3) If $f \in AP(\mathscr{X})$ then every sequence α of real numbers contains a subsequence from S_f, (Bochner's definition).

If we are given a sequence $\theta_1, \theta_2, \ldots$ of real numbers then the statement "there exists $\lim \theta_m (\operatorname{mod} 2\pi)$" means that there exists a real number θ and a sequence of integers $k_1, k_2, \ldots$ such that $\lim(\theta_m + 2\pi k_m - \theta) = 0$, which is denoted by $\lim \theta_m = \theta (\operatorname{mod} 2\pi)$. It may be easily proved that the existence of $\lim \theta_m (\operatorname{mod} 2\pi)$ is equivalent to the existence of $\lim \exp(\mathrm{i}\theta_m)$.

Theorem 4.3. *If a function $f \in AP(\mathscr{X})$ and a sequence $\alpha = \{\alpha_m\} \in S_f$ then the limit $\lim \exp(\mathrm{i}\lambda\alpha_m)$ exists for any $\lambda \in \Lambda_f$.*

P r o o f. For any $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \|a(\lambda, f)\| \cdot |\exp(\mathrm{i}\lambda\alpha_m) - \exp(\mathrm{i}\lambda\alpha_n)| &= \big\| M_t\{[f(t+\alpha_m) - f(t+\alpha_n)] \exp(-\mathrm{i}\lambda t)\} \big\| \\ &\leqslant \sup_t \|f(t+\alpha_m) - f(t+\alpha_n)\| \to 0 \text{ for } m, n \to \infty. \end{aligned}$$

If $\lambda \in \Lambda_f$ then $a(\lambda, f) \neq 0$ and the limit $\lim \exp(\mathrm{i}\lambda\alpha_m)$ exists. □

Corollary 4.4. *If Q is an $\mathscr{X}$-trigonometric polynomial and $\alpha = \{\alpha_m\} \subset \mathbb{R}$ then the existence of $\lim \exp(\mathrm{i}\lambda\alpha_m)$ for any $\lambda \in \Lambda_Q$ is a necessary and sufficient condition for the validity of the incidence $\alpha \in S_Q$.*

Definition 4.5. For a non-empty set $M \subset C(\mathscr{X})$ we define $S_M = \bigcap\limits_{f \in M} S_f$.

Theorem 4.6. *If a sequence of $\mathscr{X}$-trigonometric polynomials $M = \{Q_p\}$ converges uniformly on $\mathbb{R}$ to a function $f \in AP(\mathscr{X})$ and if $\Lambda_{Q_p} \subset \Lambda_f$, $p = 1, 2, \ldots$, then $S_M = S_f$.*

P r o o f. The inclusion $S_f \subset S_M$ results from Theorem 4.3 and Corollary 4.4. We prove the reversed inclusion. Let a sequence $\alpha \in S_M$ and let $Q_{p\alpha} = T_\alpha Q_p$, $p = 1, 2, \ldots$. For any positive number ε there exists a real number $p(\varepsilon)$ such that for any positive integers p, q greater than $p(\varepsilon)$ the following inequalities are valid:

$$\sup_t \|Q_p(t+\alpha_m) - Q_q(t+\alpha_m)\| \leqslant \varepsilon, \quad m = 1, 2, \ldots$$

(uniform convergence on $\mathbb{R}$). The passage to the limit for $m \to \infty$ yields

$$\sup_t \|Q_{p\alpha}(t) - Q_{q\alpha}(t)\| \leqslant \varepsilon.$$

This means that the sequence of trigonometric polynomials $\{Q_{p\alpha}\}$ converges uniformly on $\mathbb{R}$. Denote $Q_\alpha = \lim Q_{p\alpha}$. Evidently Q_α is an almost periodic function. Taking into account the uniform convergence of $\mathbb{R}$ we get the equality

$$T_\alpha f = T_\alpha(\lim_{p\to\infty} Q_p) = \lim_{p\to\infty} T_\alpha Q_p = \lim_{p\to\infty} Q_{p\alpha} = Q_\alpha$$

uniformly on $\mathbb{R}$. This yields $\alpha \in S_f$ and $S_M = S_f$. □

In the publications [1], [7] the following assertion is proved:

Theorem 4.7. *Let $\mathscr{X}$, $\mathscr{Y}$ be two B-spaces and let $f \in AP(\mathscr{X})$, $g \in AP(\mathscr{Y})$. A necessary and sufficient condition for $S_f \subset S_g$ to be valid is that $\Lambda_g \subset G(\Lambda_f)$ (the smallest additive group containing Λ_f as a subset).*

Corollary 4.8. *If $f \in AP(\mathscr{X})$ and M is the closure of the set of all trigonometric polynomials Q for which $\Lambda_Q \subset G(\Lambda_f)$ then $S_M = S_f$.*

4.2. The $\mathscr{H}$ property. Here we deal with the class of almost periodic functions for which uniformly on $\mathbb{R}$ convergent sequences of translates have the same properties enjoyed by continuous periodic functions.

Definition 4.9. A function $f \in AP(\mathscr{X})$ is said *to have the $\mathscr{H}$ property*, symbolically $f \in \mathscr{H}$, if for any sequence $\alpha \in S_f$ there exists a real number $h = h(\alpha) = h(\alpha, f)$ such that $T_\alpha f = T_h f$ uniformly ($T_h f(t) = f(t+h)$, $t \in \mathbb{R}$).

Example 4.10. 1) The function $f(t) = t$, $t \in \mathbb{R}$, has the $\mathscr{H}$ property and for $\alpha \in S_f$ there exists a unique $h = h(\alpha, f)$. The same is true for the function $g(t) = \arctan t$, $t \in \mathbb{R}$.

2) Any continuous periodic function has the $\mathscr{H}$ property, but $h(\alpha)$ corresponding to a regular sequence α need not be determined uniquely. In fact, the set of such numbers is relatively dense in $\mathbb{R}$.

Theorem 4.11. *If an $\mathscr{X}$-almost periodic function f has the property $\mathscr{H}$ then its range R_f is a closed set.*

Proof. Let z_0 be an arbitrarily given point from the closure of R_f. There exists a sequence $\{z_m\}$ of values of the function f such that $\lim z_m = z_0$; $\{z_m\}$ being the sequence of values of the function f there exists a sequence $\alpha' = \{\alpha'_m\}$ of real numbers such that $f(\alpha'_m) = z_m$, $m = 1, 2, \ldots$. The almost periodicity of the function f yields the existence of a sequence $\alpha \subset \alpha'$ such that $\alpha \in S_f$. In virtue of $f \in \mathscr{H}$ there exists a real number $h = h(\alpha, f)$ such that $T_\alpha f = T_h f$ uniformly. For $t = 0$ we have $f(h) = T_\alpha f(0) = \lim z_m = z_0$, i. e. $z_0 = f(h) \in R_f$. Since $z_0 \in \overline{R}_f$ has been chosen arbitrarily it follows that $\overline{R}_f = R_f$. □

Theorem 4.12. *A necessary and sufficient condition for an $\mathscr{X}$-almost periodic function f to have the $\mathscr{H}$ property is that for every sequence $\alpha \in S_f$ there exists a number h_α such that* $\lim \exp(\mathrm{i}\lambda\alpha_m) = \exp(\mathrm{i}\lambda h_\alpha)$ *for any $\lambda \in \Lambda_f$. In the affirmative case of the existence we have $h_\alpha = h(\alpha, f)$.*

P r o o f. The validity of the assertion is immediate for the trigonometric polynomial f. In case f is not a trigonometric polynomial there exists a sequence of trigonometric polynomials Q_p uniformly convergent on $\mathbb{R}$ to f for which $\Lambda_{Q_p} \subset \Lambda_f$, $p = 1, 2, \ldots$. It remains to use Theorem 4.6. □

Let us denote by S_0 the set of all sequences of real numbers with constant members starting from a certain index which depends upon the particular sequence.

Definition 4.13. A non-empty set $M \subset AP(\mathscr{X})$ is said *to have the $\mathscr{H}$ property*, symbolically $M \subset \mathscr{H}$, provided each function from M has the $\mathscr{H}$ property, $S_M \neq S_0$ and for any sequence $\alpha \in S_M$ there exists a real number $h = h(\alpha) = h(\alpha, M)$ such that for every function $f \in M$ the equality $T_\alpha f = T_h f$ holds uniformly.

If the set $M \subset AP(\mathscr{X})$ is at most countable and $M \subset \mathscr{H}$ then every sequence of real numbers contains a subsequence belonging to S_M. (By Cantors's diagonal method.) Theorem 4.12 yields the following statements.

Corollary 4.14. *Let the sequence of trigonometric polynomials $M = \{Q_p\}$ converge uniformly on $\mathbb{R}$ to an almost periodic function f and let $\Lambda_{Q_p} \subset \Lambda_f$, $p = 1, 2, \ldots$. If the set M has the $\mathscr{H}$ property then also the function f has the $\mathscr{H}$ property and $\{f\} \cup M \subset \mathscr{H}$ holds.*

Corollary 4.15. *Let $\mathscr{X}$, $\mathscr{Y}$ be B-spaces. If a function $f \in AP(\mathscr{X})$ has the $\mathscr{H}$ property and the set $M \subset AP(\mathscr{Y})$ is the closure in $AP(\mathscr{Y})$ of the set of all functions $g \in AP(\mathscr{Y})$ for which $\Lambda_g \subset G(\Lambda_f)$ then M has the $\mathscr{H}$ property.*

5.1. Structure of Fourier exponents. Here we shall show the analogy between the structure of Fourier exponents of continuous periodic functions and a certain class of almost periodic functions.

Definition 5.1. 1) Real numbers $\lambda_1, \dots, \lambda_N$, where $N \in \mathbb{N}$, are said to be *linearly dependent (over $\mathbb{Z}$)* if there exist integers $n_1, \dots, n_N$ such that $n_1\lambda_1 + \dots + n_N\lambda_N = 0$ while $|n_1| + \dots + |n_N| > 0$.

2) A non-empty set M of real numbers is said to be a *dependent set* provided any two numbers from M are linearly dependent.

3) By definition, the empty set is a dependent set.

Each one-point set $M \subset \mathbb{R}$ is a dependent set. Each subset of a dependent set is a dependent set. The union of a dependent set and the set $\{0\}$ is a dependent set.

Theorem 5.2. *If a function $f \in AP(\mathscr{X})$ has the $\mathscr{H}$ property then Λ_f is a dependent set.*

P r o o f by contradiction. Let there exist two linerly independent Fourier exponents μ, μ' of the function $f \in \mathscr{H}$. According to the Kronecker theorem the solvability of the system of inequalities

$$|\mu t - \theta| \leqslant \delta \ (\mathrm{mod}\, 2\pi), \quad |\mu' t - \theta'| \leqslant \delta \ (\mathrm{mod}\, 2\pi)$$

is guaranteed for arbitrary real numbers θ, θ' and any positive number δ. Let us denote by α'_m a solution of this system for $\theta = 0$, $\theta' = \frac{\pi}{2}$ and $\delta = \frac{1}{m}$, $m = 1, 2, \dots$. Owing to the almost periodicity of the function f the sequence $\alpha' = \{\alpha'_m\}$ contains a subsequence $\alpha = \{\alpha_m\} \in S_f$. Hence, there exists a real number $h = h(\alpha, f)$ for which $\lim \lambda\, \alpha_m = \lambda\, h \ (\mathrm{mod}\, 2\pi)$ for every $\lambda \in \Lambda$. Consequently, the following equalities are valid:

$$\lim \mu\, \alpha_m = 0 \ (\mathrm{mod}\, 2\pi) = \mu h \ (\mathrm{mod}\, 2\pi),$$
$$\lim \mu'\, \alpha_m = \tfrac{\pi}{2} \ (\mathrm{mod}\, 2\pi) = \mu' h \ (\mathrm{mod}\, 2\pi).$$

This means that there exist integers k, k' such that $\mu h = 2\pi k$, $\mu' h = \frac{\pi}{2} + 2\pi k' = (4k'+1)\frac{2\pi}{4}$. Because of the linear independence of numbers μ, μ' we get $\mu\,\mu' \neq 0$. Since also $4k'+1 \neq 0$ we infer that $h \neq 0$ and the ratio $\mu/\mu' = 4k/(4k'+1)$. It follows that $(4k'+1)\mu - 4k\mu' = 0$ with a non-zero coefficient at μ, which is a contradiction. Thus, the numbers μ, μ' are necessarily linearly dependent. $\square$

Let $N \in \mathbb{N}$. Integers $n_1, \ldots, n_N$ are said to be *relatively prime* if their greatest common divisor is equal to one. Otherwise, integers $n_1, \ldots, n_N$ are called *non relatively prime.* For $N = 1$ and $|n_1| > 1$ the number n_1 is non relatively prime.

Definition 5.3. 1) If non-zero real numbers $\lambda_1, \ldots, \lambda_N$, where $N \in \mathbb{N}$, form a dependent set then positive numbers $\sigma_1, \ldots, \sigma_N$ are called their *primitive divisors* (with due regard to the order) if $\lambda_j = n_{jk}\sigma_k$, $j = 1, \ldots, k$, where $n_{1k}, \ldots, n_{kk}$ are relatively prime integers, $k = 1, \ldots, N$.

2) If a sequence of non-zero real numbers $\lambda_1, \lambda_2, \ldots$ forms a dependent set then a sequence of positive numbers $\sigma_1, \sigma_2, \ldots$ is called *a sequence of their primitive divisors* if for any $N \in \mathbb{N}$ the numbers $\sigma_1, \ldots, \sigma_N$ are primitive divisors of the numbers $\lambda_1, \ldots, \lambda_N$.

Theorem 5.4. *If non-zero real numbers $\lambda_1, \ldots, \lambda_N$, $N \in \mathbb{N}$ form a dependent set then*

1) *their primitive divisors $\sigma_1, \ldots, \sigma_N$ exist and are uniquely determined;*

2) *for $N > 1$ the ratio $\nu_k = \sigma_k/\sigma_{k+1}$ is a positive integer, $k = 1, \ldots, N-1$, so that $\sigma_1 \geqslant \ldots \geqslant \sigma_N$.*

P r o o f. The fact that $\{\lambda_1, \ldots, \lambda_N\}$ is a dependent set implies the existence of rational numbers $r_1, \ldots, r_k$ such that $\lambda_k = r_k|\lambda_1|$, $k = 1, \ldots, N$. Suppose that these rational numbers are of the form of ratios of two relatively prime integers and q_k is the smallest common positive denominator of fractions $r_1, \ldots, r_k$ and p_k is the greatest common divisor of integer numbers $r_1 q_k, \ldots, r_k q_k$, $k = 1, \ldots, N$. Now, we already obtain the system of equalities $\lambda_j = n_{jk}\sigma_k$, where $\sigma_k = |\lambda_1|\, p_k/q_k$ and $n_{jk} = r_j q_k/p_k$, $j = 1, \ldots, k$, are relatively prime integers, $k = 1, \ldots, N$. Thus, the existence of primitive divisors has been verified. We leave it to the reader to check their uniqueness.

Let us prove 2 by contradiction. Let $N > 1$. The set of primitive divisors $\sigma_1, \ldots, \sigma_N$ forms a dependent set since the numbers $\lambda_1, \ldots, \lambda_N$ form a dependent set. Hence $\nu_k \in \mathbb{Q}$, $k = 1, \ldots, N-1$. Let $\nu_k = \sigma_k/\sigma_{k+1} = p/q$ for some $k \in \{1, \ldots, N-1\}$, where p, q are two relatively prime positive integers and $q > 1$. But this means that $\lambda_j = n_{jk}\sigma_k = n_{jk}p\sigma_{k+1}/q = n_{jk+1}\sigma_{k+1}$, $j = 1, \ldots, k$. It follows that the numbers $n_{1k}, \ldots, n_{kk}$ are divisible by an integer $q > 1$ and this is a contradiction with the fact that they are relatively prime. We infer $q = 1$ and $\nu_k \in \mathbb{N}$, $k = 1, \ldots, N-1$. $\square$

Corollary 5.5. *If a sequence of non-zero real numbers $\lambda_1, \lambda_2, \ldots$ forms a dependent set then the sequence $\sigma_1, \sigma_2, \ldots$ of their primitive divisors exists and is determined uniquely, while the ratio $\nu_k = \sigma_k/\sigma_{k+1}$ is a positive integer, $k = 1, 2, \ldots$, so that $\sigma_1 \geqslant \sigma_2 \geqslant \ldots$.*

R e m a r k 5.6. If $\sigma_1, \sigma_2, \dots$ is a sequence of primitive divisors of a sequence of non-zero real numbers $\lambda_1, \lambda_2, \dots$ then the numbers ν_{k-1}, n_{kk} are relatively prime, $k = 2, 3, \dots$.

Lemma 5.7. *If a function $g \in AP(\mathscr{X})$ is not a trigonometric polynomial and the sequence $\Lambda_{0g} = \{\lambda_1, \lambda_2, \dots\}$, the set of all non-zero Fourier exponents of the function g, is a dependent set and $\sigma_1, \sigma_2, \dots$ is the sequence of their primitive divisors, then the sequence $\alpha_m = \sum\limits_{j=1}^{m} T_j$, where $T_j = 2\pi/\sigma_j$, $j = 1, \dots, m$, $m = 1, 2, \dots$, belongs to S_g.*

P r o o f. For $k > 1$ and $1 \leqslant j < k$ the equality

$$\frac{\sigma_k}{\sigma_j} = \frac{\sigma_k \cdot \ldots \cdot \sigma_{j+1}}{\sigma_{k-1} \cdot \ldots \cdot \sigma_j} = \frac{1}{\nu_{k-1} \cdot \ldots \cdot \nu_j}$$

is valid, hence $\sigma_k T_j = 2\pi/(\nu_{k-1} \cdot \ldots \cdot \nu_j)$. For $1 \leqslant k < j$ we have

$$\frac{\sigma_k}{\sigma_j} = \frac{\sigma_k \cdot \ldots \cdot \sigma_{j-1}}{\sigma_{k+1} \cdot \ldots \cdot \sigma_j} = \nu_k \cdot \ldots \cdot \nu_{j-1}$$

so that $\sigma_k T_j = 2\pi\nu_k \cdot \ldots \cdot \nu_{j-1} = 0 \pmod{2\pi}$. For $j = k$ we evidently get $\sigma_k T_k = 0 \pmod{2\pi}$ as well. For $m \geqslant k > 1$ we have

$$\begin{aligned} \lambda_k \alpha_m &= \sum_{j=1}^{m} \lambda_k T_j = n_{kk} \sum_{j=1}^{m} \sigma_k T_j = n_{kk} \sum_{j=1}^{k-1} \sigma_k T_j \pmod{2\pi} \\ &= 2\pi n_{kk} \sum_{j=1}^{k-1} \frac{1}{\nu_{k-1} \cdot \ldots \cdot \nu_j} \pmod{2\pi}. \end{aligned}$$

We conclude that there exists $\lim \lambda\alpha_m \pmod{2\pi}$ for every $\lambda \in \Lambda_{0g}$, namely $\lim \lambda_1 \alpha_m = 0 \pmod{2\pi}$, $\lim \lambda_k \alpha_m = 2\pi n_{kk} \sum\limits_{j=1}^{k-1} 1/(\nu_{k-1} \cdot \ldots \cdot \nu_j) \pmod{2\pi}$ for $k = 2, 3, \dots$, so that $\alpha = \{\alpha_m\} \in S_g$. $\square$

5.2. Aperiodicity. For a function $f \in AP(\mathscr{X})$ we denote again the set of all its non-zero Fourier exponents by the symbol Λ_{0f}.

Theorem 5.8. *Let a sequence of non-zero real numbers $\lambda_1, \lambda_2, \dots$ form the dependent set Λ_{0g} of an $\mathscr{X}$-almost periodic function g and let $\sigma_1, \sigma_2, \dots$ be the sequence of its primitive divisors. If $\nu_k = \sigma_k/\sigma_{k+1} \geqslant 3$, $k = 1, 2, \dots$, then $\lim \sigma_k = 0$ and the function g does not possess the $\mathscr{H}$ property and consequently g is not periodic.*

P r o o f by contradiction. Assume that $g \in \mathscr{H}$. Let $\alpha = \{\alpha_m\} \in S_g$ be the sequence defined in Lemma 5.7. For the sequence α there exists a real number $h = h(\alpha, g)$ such that $T_\alpha g = T_h g$ uniformly. We get the equality $\lim \lambda\alpha_m = \lambda h \pmod{2\pi}$ for every $\lambda \in \Lambda_g$. Hence, there exist integer numbers $n_k,\ k = 1, 2, \ldots$, such that we obtain successively the equalities

$$0 = \lambda_1 h - 2\pi n_1 = n_{11}\sigma_1 h - 2\pi n_1,$$

$$2\pi n_{kk} \sum_{j=1}^{k-1} \frac{1}{\nu_{k-1} \cdot \ldots \cdot \nu_j} = \lambda_k h - 2\pi n_k = n_{kk}\sigma_k h - 2\pi n_k, \quad k = 2, 3, \ldots .$$

Since $\sigma_1 = |\lambda_1|$ and $|n_{11}| = 1$, $\sigma_1 h$ is an integer multiple of the number 2π. Let ϱ_1 be the integer for which $\sigma_1 h = 2\pi\varrho_1$, hence $n_1 = \varrho_1 n_{11}$. If we have determined successively the equalities $n_j = \varrho_j n_{jj}$, where $\varrho_j = (\varrho_{j-1} - 1)/\nu_{j-1}$ for $j = 2, \ldots, k-1$ is an integer for $k > 2$, then the equality $n_k = n_{kk}\,(\varrho_{k-1} - 1)/\nu_{k-1} = \varrho_k n_{kk}$ and the fact that the integer numbers ν_{k-1}, n_{kk} are relatively prime and $n_k \in \mathbb{Z}$ yields that the ratio $\varrho_k = (\varrho_{k-1} - 1)/\nu_{k-1}$ is an integer. Thus, for any positive integer k there exists an integer ϱ_k such that $n_k = \varrho_k n_{kk}$ while $\varrho_2 = (\varrho_1 - 1)/\nu_1$ and for $k > 2$ we have

$$\varrho_k = \frac{\varrho_1 - 1}{\nu_{k-1} \cdot \ldots \cdot \nu_1} - \frac{1}{\nu_{k-1} \cdot \ldots \cdot \nu_2} - \ldots - \frac{1}{\nu_{k-1}}.$$

By assumption, $\nu_k \geqslant 3$ for $k = 1, 2, \ldots$, so that

$$\sum_{k=3}^{\infty} \frac{1}{\nu_{k-1} \cdot \ldots \cdot \nu_2} \leqslant \tfrac{1}{2}$$

and for all sufficiently large $k \in \mathbb{N}$ necessarily $|\varrho_1 - 1|/(\nu_{k-1} \cdot \ldots \cdot \nu_1) < \frac{1}{2}$. Hence for all sufficiently large $k \in \mathbb{N}$ we have $|\varrho_k| < 1$. Since ϱ_k are integer numbers we conclude that $\varrho_k = 0$ starting from a certain index k_0. For every $k \geqslant k_0$ by using the equality $\varrho_k = 0$ we obtain the equality

$$\varrho_1 - 1 - \nu_1 - \nu_1\nu_2 - \ldots - (\nu_1 \cdot \ldots \cdot \nu_{k-2}) = 0.$$

However, the left hand side of this equality diverges to $-\infty$ for $k \to \infty$, which leads to a contradiction. The function g cannot have the $\mathscr{H}$ property. $\square$

Theorem 5.9. *Let a sequence of non-zero real numbers $\mu_1, \mu_2, \ldots$ form the dependent set Λ_{0f} of an $\mathscr{X}$-almost periodic function f and let $\sigma_1', \sigma_2', \ldots$ be the sequence of their primitive divisors. If $\lim \sigma_k' = 0$ then the function f has not the $\mathscr{H}$ property; consequently f is not periodic.*

P r o o f. The sequence Λ_{0f} contains a subsequence $\mu_{k_j} = \lambda_j$, $j = 1, 2, \ldots$ such that for the almost periodic function $g(t) = \sum 2^{-k} \exp(\mathrm{i}\lambda_k t)$, $k \in \mathbb{N}$, $t \in \mathbb{R}$, the assumptions of Theorem 5.8 are satisfied. So the function g does not possess the $\mathscr{H}$ property and in view of the fact that $\Lambda_g \subset G(\Lambda_f)$ we conclude by Corollary 4.15 that the function f does not have the $\mathscr{H}$ property either. $\square$

5.3. Criterion of periodicity. Now, we shall determine the relation between the $\mathscr{H}$ property and the periodicity of $\mathscr{X}$-almost periodic functions.

Theorem 5.10. *If Q is an $\mathscr{X}$-trigonometric polynomial and Λ_Q is a dependent set then Q is a periodic function and also has the $\mathscr{H}$ property.*

P r o o f. If $\Lambda_{0Q} = \emptyset$ then Q is a constant function. If $\Lambda_{0Q} = \{\lambda_1, \ldots, \lambda_N\}$, where N is a positive integer, is a dependent set with the primitive divisors $\sigma_1, \ldots, \sigma_N$ then $T = 2\pi/\sigma_N$ is the primitive period of the function Q. $\square$

Theorem 5.11. *Let the $\mathscr{X}$-almost periodic function f be not a trigonometric polynomial and let Λ_f be a dependent set. If the dependent set Λ_{0f} is ordered into a sequence and the limit of the corresponding sequence of its primitive divisors is positive then f is a periodic function and, consequently, has the $\mathscr{H}$ property.*

P r o o f. Let $\sigma_1, \sigma_2, \ldots$ be the corresponding sequence of primitive divisors and let $\lim \sigma_k = \sigma > 0$. In other words, starting from a certain index the primitive divisors remain constant and $T = 2\pi/\sigma$ is the primitive period of the function f. $\square$

Corollary 5.12. *Any $\mathscr{X}$-almost periodic function with the $\mathscr{H}$ property is periodic. (On the other hand, any continuous periodic function has the $\mathscr{H}$ property.)*

5.4. Closure of the class $CP(\mathscr{X})$. The class $CP(\mathscr{X})$ is neither linear nor closed in $CB(\mathscr{X})$. For instance, the function $f(t) = \sum\limits_{k=1}^{\infty} 3^{-k} \exp(\mathrm{i}t3^{-k})$, $t \in \mathbb{R}$, is the sum of an absolutely uniformly convergent trigonometric series the partial sums of which are periodic functions. The function f is almost periodic, however, according to Theorem 5.8 it is not periodic.

Theorem 5.13. *If a sequence of functions $\{f_k\} \subset CP(\mathscr{X})$ converges uniformly on $\mathbb{R}$ to a function f then $f \in AP(\mathscr{X})$ and Λ_f is a dependent set.*

P r o o f. It is readily seen that $f \in AP(\mathscr{X})$. Without restricting generality we may assume that f_k are periodic trigonometric polynomials, $k = 1, 2, \ldots$. Since $\|a(\lambda, f_k)\| \geqslant \|a(\lambda, f)\| - \|a(\lambda, f_k - f)\|$ for any $\lambda \in \mathbb{R}$, for arbitrary two Fourier exponents μ, μ' of the function f there exists k sufficiently large such that μ, μ' are at the same time Fourier exponents of the periodic function f_k; consequently, they are linearly dependent. $\square$

Thus, the closure of the class $CP(\mathscr{X})$ consists of all $\mathscr{X}$-almost periodic functions f the set Λ_f of which is a dependent set. It follows that for every $\mathscr{X}$-almost periodic function f whose Λ_f is not a dependent set the distance from the element f to $CP(\mathscr{X})$ in the space $CB(\mathscr{X})$ is positive. This is of considerable theoretical as well as practical importance for the uniform approximation of almost periodic functions on $\mathbb{R}$ by means of continuous periodic functions.

5.5. Bochner's transformation. Any continuous periodic function has a compact range. This property is not a sufficient, but only a necessary condition for the periodicity of an almost periodic function. A promising conjecture naturally arises that any trigonometric polynomial with a compact range is periodic. The author has managed to verify this conjecture for scalar trigonometric polynomials of the second degree. Nonetheless, from a certain more general viewpoint the compactness of the range becomes a sufficient and necessary condition for periodicity.

Definition 5.14. Let a function $f \in C(\mathscr{X}) = \mathscr{Y}$ be given. The function $\mathscr{B}f = \tilde{f} \in C(\mathscr{Y})$ defined for any $s \in \mathbb{R}$ by the formula $\tilde{f}(s)(t) = f(t+s)$, $t \in \mathbb{R}$ (s-translate of the function f) is called the *Bochner transformation* of the function f.

If $f \in CB(\mathscr{X})$ then the inequality

$$\begin{aligned}
\sup_s \|\tilde{f}(s+\tau) - \tilde{f}(s+\nu)\|_{\mathscr{Y}} &= \sup_s \sup_t \ \|\tilde{f}(s+\tau)(t) - \tilde{f}(s+\nu)(t)\|_{\mathscr{X}} \\
&= \sup_s \sup_t \ \|f(t+s+\tau) - f(t+s+\nu)\|_{\mathscr{X}} \\
&= \sup_t \|f(t+\tau) - f(t+\nu)\|_{\mathscr{X}}, \quad s \in \mathbb{R},\ t \in \mathbb{R},
\end{aligned}$$

which is valid for any real numbers τ, ν, implies that the periodicity, the almost periodicity of the function f and the incidence $\alpha \in S_f$, respectively, is ensured by the periodicity, the almost periodicity of the function $\tilde{f}$ and by $\alpha \in S_{\tilde{f}}$, respectively. The pertinent periods or almost periods, respectively, coincide for f and $\tilde{f}$.

Theorem 5.15. *A necessary and sufficient condition for a function $f \in CB(\mathscr{X}) = \mathscr{Y}$ to be periodic is that its Bochner transformation $\mathscr{B}f = \tilde{f}$ have a compact range in the space $\mathscr{Y}$.*

P r o o f. The necessity is evident. The sufficiency follows from the fact that the compactness of the range of Bochner's transformation $\tilde{f}$ yields the $\mathscr{H}$ property and due Bochner's definition also the almost periodicity of the function f. $\square$

C o n c l u s i o n. Original results of this paper contribute to the theory of almost periodic functions. Above all, the paper deals with almost periodic functions which are limits of sequences of continuous periodic functions, and determines the structure of their Fourier exponents and their ranges. Key ingredients of the paper are contained in Sections 5.1, 5.2, 5.3 and 5.5.

It follows from the results of the paper that the class $CP(\mathscr{X})$ of continuous periodic functions is not densely distributed in the space $AP(\mathscr{X})$; considerable importance and unexchangeability of the position of almost periodic functions has been borne out again by this result.

References

[1] *Amerio, L., Prouse, G.*: Almost periodic functions and functional equations. New York, Van Nostrand Reihold Company, 1971.

[2] *Bohr, H.*: Zur Theorie der fastperiodischen Funktionen. Acta Math. *45* (1925), 29–127, *46* (1925), 101–214, *47* (1926), 237–281.

[3] *Bochner, S.*: Beiträge zur Theorie der fastperiodischen Funktionen. Math. Ann. *96* (1927), 119–147.

[4] *Bochner S.*: Abstrakte fastperiodische Funktionen. Acta Math. *61* (1933), 149–184.

[5] *Fink, A.M.*: Almost periodic differential equations. Lecture Notes in Mathematics, New York, 1978.

[6] *Levitan, B.M.*: Almost periodic functions. GIZTL, Moscow, 1953. (In Russian.)

[7] *Levitan, B.M., Žikov, V.V.*: Almost periodic functions and differential equations. IMU, Moscow, 1978. (In Russian.)

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