# DOMINATION NUMBERS ON THE BOOLEAN FUNCTION GRAPH OF A GRAPH

T. N. Janakiraman, Tiruchirappalli,
S. Muthammai, M. Bhanumathi, Pudukkottai

(Received April 16, 2004)

*Abstract.* For any graph $G$, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$ respectively. The Boolean function graph $B(G, L(G), \mathrm{NINC})$ of $G$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(G, L(G), \mathrm{NINC})$ are adjacent if and only if they correspond to two adjacent vertices of $G$, two adjacent edges of $G$ or to a vertex and an edge not incident to it in $G$. For brevity, this graph is denoted by $B_1(G)$. In this paper, we determine domination number, independent, connected, total, cycle, point-set, restrained, split and non-split domination numbers of $B_1(G)$ and obtain bounds for the above numbers.

*Keywords*: domination number, point-set domination number, split domination number, Boolean function graph

*MSC 2000*: 05C15

## 1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph $G$, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. For a connected graph $G$, the eccentricity $e_G(v)$ of a vertex $v$ in $G$ is the distance to a vertex farthest from $v$. Thus, $e_G(v) = \max\{d_G(u, v)\colon\ u \in V(G)\}$, where $d_G(u, v)$ is the distance between $u$ and $v$ in $G$. If there is no confusion, then we simply denote the eccentricity of a vertex $v$ in $G$ as $e(v)$ and use $d(u, v)$ to denote the distance between two vertices $u$, $v$ in $G$ respectively. The minimum and maximum eccentricities are the radius and diameter of $G$, denoted $r(G)$ and $\mathrm{diam}(G)$ respectively. The neighborhood $N_G(v)$ of a vertex $v$ is the set of all vertices adjacent to $v$ in $G$. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the closed neighborhood of $v$. A set $S$ of edges in a graph $G$ is said to be independent if no two of the edges in $S$ are adjacent. A set of independent edges covering all the vertices of a graph $G$ is called a perfect matching. An edge $e = (u, v)$

is a dominating edge in a graph $G$ if every vertex of $G$ is adjacent to at least one of $u$ and $v$.

The concept of domination in graphs was introduced by Ore [12]. A set $D \subseteq V(G)$ is said to be a dominating set of $G$, if every vertex in $V(G) - D$ is adjacent to some vertex in $D$. $D$ is said to be a minimal dominating set if $D - \{u\}$ is not a dominating set for any $u \in D$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set. We call a set of vertices a $\gamma$-set if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set $D$ is called a connected (independent) dominating set, if the induced subgraph $\langle D \rangle$ is connected (independent) [14], [2]. $D$ is called a total dominating set if every vertex in $V(G)$ is adjacent to some vertex in $D$ [5]. An dominating set $D$ is called a cycle dominating set if the subgraph $\langle D \rangle$ has a Hamiltonian cycle, and is called a perfect dominating set, if every vertex in $V(G) - D$ is adjacent to exactly one vertex in $D$ [6]. $D$ is called a restrained dominating set if every vertex in $V(G) - D$ is adjacent to another vertex in $V(G) - D$ [7]. By $\gamma_{\mathrm{c}}$, $\gamma_{\mathrm{i}}$, $\gamma_{\mathrm{t}}$, $\gamma_{\mathrm{o}}$, $\gamma_{\mathrm{p}}$ and $\gamma_{\mathrm{r}}$, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, cycle dominating set, perfect dominating set and restrained dominating set respectively.

Sampathkumar and Pushpalatha [13] introduced the concept of point-set domination number of a graph. A set $D \subseteq V(G)$ is called a point-set dominating set (psd-set), if for every set $T \subseteq V(G) - D$, there exists a vertex $v \in D$ such that the subgraph $\langle T \cup \{v\} \rangle$ induced by $T \cup \{v\}$ is connected. The point-set domination number $\gamma_{\mathrm{ps}}(G)$ is the minimum cardinality of a psd-set of $G$. Kulli and Janakiram [11] introduced the concept of split and non-split domination in graphs. An dominating set $D$ of a connected graph $G$ is a split (non-split) dominating set, if the induced subgraph $\langle V(G) - D \rangle$ is disconnected (connected). The split (non-split) domination number $\gamma_{\mathrm{s}}(G)$ ($\gamma_{\mathrm{ns}}(G)$) of $G$ is the minimum cardinality of a split (non-split) dominating set. A set $F \subseteq E(G)$ is an edge dominating set, if each edge in $E$ is either in $F$ or is adjacent to an edge in $F$. The edge domination number $\gamma'(G)$ is the smallest cardinality among all minimal edge dominating sets.

When a new concept is developed in graph theory, it is often first applied to particular classes of graphs. Afterwards more general graphs are studied. As for every graph (undirected, uniformly weighted) there exists an adjacency $(0, 1)$ matrix, we call the general operation as Boolean operation. Boolean operation on a given graph uses the adjacency relation between two vertices or two edges and incidence relationship between vertices and edges and define new structure from the given graph. This adds extra bit information of the original graph and encode it new structure. If it is possible to decode the given graph from the encoded graph in polynomial time, such operation may be used to analyze various structural properties of original graph in terms of the Boolean graph. If it is not possible to decode the original graph in polynomial time, then that operation can be used in graph coding or coding of certain grouped signal.

Whitney [16] introduced the concept of the line graph $L(G)$ of a given graph $G$ in 1932. The first characterization of line graphs is due to Krausz. The Middle graph $M(G)$ of a graph $G$ was introduced by Hamada and Yoshimura [8]. Chikkodimath and Sampathkumar [4] also studied it independently and they called it the semi-total graph $T_1(G)$ of a graph $G$. Characterizations were presented for middle graphs of any graph, trees and complete graphs in [1]. The concept of total graphs was introduced by Behzad [3] in 1966. Sastry and Raju [15] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. This motivates us to define and study other graph operations. Using $L(G)$, $G$, incident and non-incident, complementary operations, complete and totally disconnected structures, one can get thirty-two graph operations. As already total graphs, semi-total edge graphs, semi-total vertex graphs and quasi-total graphs and their complements (8 graphs) are defined and studied, we have studied all other similar remaining graph operations. This is illustrated below.

$$\boxed{G/\overline{G}/K_p/\overline{K}_p} \text{——} \boxed{\text{incident (INC), not incident (NINC)}} \text{——} \boxed{L(G)/\overline{L}(G)/K_q/\overline{K}_q}.$$

Here, $\overline{G}$ and $L(G)$ denote the complement and the line graph of $G$ respectively. $K_p$ is the complete graph on $p$ vertices.

The Boolean function graph $B(G, L(G), \mathrm{NINC})$ of $G$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(G, L(G), \mathrm{NINC})$ are adjacent if and only if they correspond to two adjacent vertices of $G$, two adjacent edges of $G$ or to a vertex and an edge not incident to it in $G$ [10]. For brevity, this graph is denoted by $B_1(G)$. In other words, $V(B1(G)) = V(G) \cup V(L(G))$; and $E(B_1(G)) = [E(T(\overline{G})) - (E(\overline{G}) \cup E(\overline{L}(G)))] \cup (E(G) \cup E(L(G)))$, where $\overline{G}$, $L(G)$ and $T(G)$ denote the complement, the line graph and the total graph of $G$ respectively. The vertices of $G$ and $L(G)$ are referred to as point and line vertices respectively.

In this paper, we determine the domination numbers mentioned above for this graph $B_1(G)$. The definitions and details not furnished in this paper may be found in [9].

## 2. Prior results

**Theorem 2.1** [13]**.** *Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a point-set dominating set of $G$ if and only if for every independent set $W$ in $V - S$, there exists a vertex $u$ in $S$ such that $W \subseteq N_G(u) \cap (V - S)$.*

**Observation** [10].

2.2. *$G$ and $L(G)$ are induced subgraphs of $B_1(G)$.*

2.3. *Number of vertices in $B_1(G)$ is $p+q$ and if $d_i = \deg_G(v_i)$, $v_i \in V(G)$, then the number of edges in $B_1(G)$ is $q(p-2) + \frac{1}{2} \sum\limits_{1 \leqslant i \leqslant p} d_i^2$.*

2.4. *The degree of a point vertex in $B_1(G)$ is $q$ and the degree of a line vertex in $B_1(G)$ is $\deg_{L(G)}(e') + p - 2$. Also if $d^*(e')$ is the degree of the line vertex $e'$ in $B_1(G)$, then $0 \leqslant d^*(e') \leqslant p+q-3$. The lower bound is attained if $G \cong K_2$, and the upper bound is attained if $G \cong K_{1,n}$, for $n \geqslant 2$.*

**Theorem 2.5** [10]. *$B_1(G)$ is disconnected if and only if $G$ is one of the following graphs: $nK_1$, $K_2$, $2K_2$ and $K_2 \cup nK_1$, for $n \geqslant 1$.*

**Theorem 2.6** [10]. *For any graph $G$, if $B_1(G)$ has a dominating edge, then the end vertices of this edge correspond to a point vertex and a line vertex.*

## 3. Main results

In this section, we denote the line vertex in $B_1(G)$ corresponding to an edge $e$ in $G$ by $e'$. For any $(p,q)$ graph $G$, the domination number of $B_1(G)$ is at least two, since there is no vertex of degree $p+q-1$ in $B_1(G)$. We prove that the (independent) domination number of $B_1(G)$ is 2.

**Theorem 3.1.** *For any graph $G$ having at least one edge,*

$$\gamma(B_1(G)) = \gamma_{\rm i}(B_1(G)) = 2.$$

Proof. Let $e = (u,v)$ be an edge in $G$ where $u, v \in V(G)$ and let $e'$ be the corresponding line vertex in $B_1(G)$. Then $u, v, e' \in V(B_1(G))$ and the set $D = \{u, e'\}$ or $\{v, e'\}$ is a dominating set of $B_1(G)$ and is also independent. Hence, $\gamma(B_1(G)) = \gamma_{\rm i}(B_1(G)) \leqslant 2$. Thus, $\gamma(B_1(G)) = \gamma_{\rm i}(B_1(G)) = 2$. □

The next theorem relates $\gamma(B_1(G))$ to the domination number $\gamma(G)$ of $G$.

**Theorem 3.2.** *For any graph $G, \gamma(B_1(G)) \leqslant \gamma(G)$, if $\gamma(G) \geqslant 3$.*

Proof. Assume $\gamma(G) \geqslant 3$.

Let $D$ be a $\gamma$-set of $G$. Since $G$ is an induced subgraph of $B_1(G)$, it dominates all the point vertices in $B_1(G)$. Also any line vertex in $B_1(G) - D$ is adjacent to at least one vertex in $D$, since $|D| \geqslant 3$ and $D$ is a dominating set of $B_1(G)$. Hence $\gamma(B_1(G)) \leqslant \gamma(G)$ if $\gamma(G) \geqslant 3$. □

In the following theorems, we prove that the connected domination number $\gamma_c$ of $B_1(G)$ is either 2 or 3.

**Theorem 3.3.** *For any graph $G$ with at least one edge, $\gamma_c(B_1(G)) = 2$ if and only if $G$ contains a triangle in which at least one vertex has degree 2.*

P r o o f. Assume $\gamma_c(B_1(G)) = 2$. Then $B_1(G)$ has a dominating edge. By Theorem 2.6, vertices of the dominating edge correspond to a point vertex and a line vertex. Let $D = \{v, e'\}$, where $v \in V(G)$ and $e'$ is the line vertex corresponding to an edge $e$ in $G$ not incident with $v$ in $G$. Then $D$ is a connected dominating set of $B_1(G)$. Assume $e = (u_1, v_1)$, where $u_1$, $v_1 \in V(G)$. Then $u_1$, $v_1 \in V(B_1(G)) - D$ and are not adjacent to $e'$ in $D$. Since $D$ is a dominating set of $B_1(G)$, both $u_1$ and $v_1$ must be adjacent to $v$ in $D$. That is, $e = (u_1, v_1)$ lies on a triangle in $G$ and hence $G$ contains a triangle. If $\deg_G(v) \geqslant 3$, then there exists at least one edge, say $e_1$, in $E(G)$ incident with $v$ other than the edges $(v, u_1)$ and $(v, v_1)$. If $e'_1$ is the line vertex corresponding to $e_1$, then $e'_1 \in V(B_1(G)) - D$ is not adjacent to any of the vertices in $D$, which is a contradiction. Hence, $\deg_1(v) = 2$. Thus, $G$ contains a triangle in which the degree of at least one vertex is 2. Conversely, let $v, e$ be, respectively, a vertex and an edge of a triangle in $G$ such that $e$ is not incident with $v$ and $\deg_G(v) = 2$. Then $D = \{v, e'\} \subseteq V(B_1(G))$ is a connected dominating set of $B_1(G)$, where $e'$ is the line vertex in $B_1(G)$ corresponding to $e$, and hence $\gamma_c(B_1(G)) \leqslant 2$. But since $\gamma_c(B_1(G)) \geqslant \gamma(G) = 2$, $\gamma_c(B_1(G)) = 2$. □

**Theorem 3.4.** *Let $G$ be any $(p, q)$ graph such that $B_1(G)$ is connected and $p \geqslant 3$, $q \geqslant 2$. Then $\gamma_c(B_1(G)) = 3$ if and only if either $G$ is triangle-free or the degree of each 3-cyclic vertex in $G$ is greater than or equal to 3.*

P r o o f. Let $\gamma_c(B_1(G)) = 3$. If $G$ contains triangles in which the degree of at least one vertex is 2, then $\gamma_c(B_1(G)) = 2$ by Theorem 3.3. Therefore, either $G$ is triangle-free or the degree of each 3-cyclic vertex in $G$ is greater than or equal to 3. Conversely, assume that $G$ is either triangle-free or the degree of each 3-cyclic vertex in $G$ is greater than or equal to 3. Let $e_{12} = (v_1, v_2)$ and $e_{23} = (v_2, v_3)$ be two adjacent edges in $G$, where $v_1, v_2, v_3 \in V(G)$. Then $v_1, v_2, v_3$, $e'_{12}$, $e'_{23} \in V(B_1(G))$ and the set $D = \{v_1, v_2, e'_{23}\} \subseteq V(B_1(G))$ is a minimal connected dominating set of $B_1(G)$. Hence, $\gamma_c(B_1(G)) \leqslant 3$. By the assumption, there exists no connected dominating set of two elements in $B_1(G)$. Thus, $\gamma_c(B_1(G)) = 3$. □

R e m a r k 3.1. By Theorems 3.3 and 3.4, the connected domination number of $B_1(G)$ is either 2 or 3. Similarly, the total domination number of $B_1(G)$ is either 2 or 3.

In the following the cycle domination number $\gamma_o$ of $B_1(G)$ is determined.

**Proposition 3.1.** *For any graph $G$ having cycles, $\gamma_o(B_1(G)) \leqslant \gamma_o(G)$.*

Proof. Any cycle dominating set $D$ of $G$ contains at least three elements and by Theorem 3.2, it is a dominating set of $B_1(G)$. Since $G$ is an induced subgraph of $B_1(G)$, $D$ is also a cycle dominating set of $B_1(G)$. Hence, $\gamma_o(B_1(G)) \leqslant \gamma_o(G)$. □

**Proposition 3.2.** *For any graph $G$, $\gamma_o(B_1(G)) \leqslant \gamma_o(L(G))$, where $\gamma_o(L(G)) \geqslant 4$.*

Proof. Any cycle dominating set of $L(G)$ dominates both point and line vertices of $B_1(G)$ and hence is a cycle dominating set of $B_1(G)$, since $L(G)$ is an induced subgraph of $B_1(G)$. □

Remark 3.2. Let $D'$ be a cycle dominating set of $L(G)$ such that $|D'| = 3$. If the edges in $G$ corresponding to the vertices in $D'$ are not the edges of $K_{1,3}$ in $G$, then $D'$ is also a cycle dominating set of $B_1(G)$.

In the following theorem, we find the graphs for which $\gamma_o(B_1(G))$ is 3.

**Theorem 3.5.** *For any $K_{1,3}$-free graph, $\gamma_o(B_1(G)) = 3$ if and only if one of the following is true.*

(i) $\gamma_o(G) = 3$;

(ii) $\gamma_o(L(G)) = 3$; *and*

(iii) *$G$ contains $K_{1,3}$ as a subgraph and there exists at least one vertex $v$ in $K_{1,3}$ other than the center vertex of $K_{1,3}$ such that $N(v)$ is a subset of $V(K_{1,3})$.*

Proof. Assume $\gamma_o(B_1(G)) = 3$. Then there exists a cycle dominating set $D$ of $B_1(G)$ with $|D| = 3$. If $D$ contains point or line vertices only, then $D$ is a cycle dominating set of $G$ or $L(G)$, since both $G$ and $L(G)$ are induced subgraphs of $B_1(G)$. Hence $\gamma_o(G) = 3$ or $\gamma_o(L(G)) = 3$. Let $D$ contain two line vertices, say $e_1'$, $e_2'$, and one point vertex, say $v$. Then $v \in V(G)$ and the edges $e_1$ and $e_2$ in $G$ corresponding to $e_1'$, $e_2'$ are adjacent edges and $v$ is not incident with both $e_1$ and $e_2$ in $G$. Let $e_1 = (u_1, u)$, $e_2 = (u_2, u)$, where $u$, $u_1$, $u_2 \in V(G)$. Then $u$, $u_1$, $u_2 \in V(B_1(G))$ and $u \in V(B_1(G)) - D$ is not adjacent to both $e_1'$ and $e_2'$ in $D$. Since $D$ is a dominating set of $B_1(G)$, $u$ must be adjacent to $v$ in $D$ and hence in $G$. Thus, $G$ contains $K_{1,3}$ as a subgraph (not induced) with $u$ as the center vertex. Let $w$ be a vertex in $G$ adjacent to $v$ such that $w \notin V(K_{1,3})$ and $(v, w) = e \in E(G)$. Then $e' \in V(B_1(G)) - D$ is not adjacent to any of the vertices in $D$, which is a contradiction to the assumption that $D$ is a dominating set of $B_1(G)$. Thus, $N(v) \in V(K_{1,3})$. If $D$ contains one line vertex and two point vertices, then $D$ is not a dominating set of $B_1(G)$. □

Conversely, if $G$ is one of the graphs given in the theorem, then there exists a cycle dominating set of $B_1(G)$ having three vertices. Hence, $\gamma_o(B_1(G)) = 3$.

Remark 3.3. If $G$ contains $P_4$ (a path on 4 vertices) as a subgraph or $K_{1,3}$ as an induced subgraph, then $\gamma_o(B_1(G)) \leqslant 4$.

In the following, we obtain the graphs $G$ for which the perfect domination number $\gamma_p$ of $B_1(G)$ is 2 or 3.

**Theorem 3.6.** *For any graph $G$, $\gamma_p(B_1(G)) = 2$ if and only if $G$ is one of the following graphs*
(i) *$G \cong K_{1,n} \cup K_{1,m}$, for $m, n \geqslant 2$; and*
(ii) *$G$ contains $K_2$ or $C_3$ as one of its components.*

Proof. Assume $\gamma_p(B_1(G)) = 2$. Let $D$ be a perfect dominating set of $B_1(G)$ such that $|D| = 2$.

Case (i): $D$ contains two point vertices. Let $D = \{v_1, v_2\} \subseteq V(B_1(G))$, where $v_1, v_2 \in V(G)$. If $v_1, v_2$ are adjacent in $G$, then $D$ is not a dominating set of $B_1(G)$. So $v_1$ and $v_2$ are nonadjacent in $G$ and hence in $B_1(G)$ and $D$ must be a perfect dominating set of $G$, since $G$ is an induced subgraph of $B_1(G)$. Hence, $G \cong K_{1,n} \cup K_{1,m}$, $m, n \geqslant 2$, where $v_1$ and $v_2$ are the center vertices of $K_{1,n}$ and $K_{1,m}$ respectively.

Case (ii): $D$ contains one point and one line vertex. Let $D = \{v, e'\} \subseteq V(B_1(G))$, where $v$, $e'$ are point and line vertices respectively. Then $v \in V(G)$. Let $e$ be the edge in $G$ corresponding to $e'$.

Subcase (i): $e \in E(G)$ is incident with $v \in V(G)$. Let $e = (v, u)$, $u \in V(G)$. If there exists a vertex $w \in V(G)$ adjacent to $v$ in $G$, then $w \in V(B_1(G)) - D$ is adjacent to both $v$ and $e'$ in $D$, which is a contradiction and hence there exists no vertex in $G$ adjacent to $v$. Similarly, if there exists an edge $e_1$ in $G$ incident with $u$ in $G$, then $e_1' \in V(B_1(G)) - D$ is adjacent to both $v$ and $e'$ in $D$. This is a contradiction, since $D$ is a perfect dominating set of $B_1(G)$. Hence, $G$ contains $K_2$ as one of its components.

Subcase (ii): $e \in E(G)$ is not incident with $v \in V(G)$. Then $D = \{v, e'\} \subseteq V(B_1(G))$ is a connected dominating set of $B_1(G)$. By Theorem 3.3, $\gamma_c(B_1(G)) = 2$ if and only if $G$ contains a triangle in which at least one vertex has degree two. Let $e = (u, w) \in E(G)$ be such that $\langle\{u, v, w\}\rangle \cong C_3$ in $G$ and $\deg_G(v) = 2$, where $u, w \in V(G)$. If there exists a vertex adjacent to $u$ or $w$ in $G$, then the corresponding line vertex in $B_1(G)$ is adjacent to both $v$ and $e'$ in $D$. Hence, there exists no vertex adjacent to $u$ or $w$ in $G$ and $G$ contains $C_3$ as one of its components.

Case (iii): $D$ contains two line vertices. Let $D = \{e_1', e_2'\} \subseteq V(B_1(G))$, where $e_1, e_2 \in E(G)$. If $e_1$ and $e_2$ are adjacent edges in $G$, then $D$ is not a dominating set of $B_1(G)$; hence $e_1$ and $e_2$ are independent edges in $G$. Also $D$ is a dominating set of $B_1(G)$ if and only if each edge in $G$ is adjacent to $e_1$, $e_2$ or both. The fact that $D$ is a perfect dominating set of $B_1(G)$ implies that there exists no edge in $G$ adjacent to at least one of $e_1$ and $e_2$. Hence, $G \cong 2K_2$. Conversely, if $G$ is one of the graphs given in the theorem, then $\gamma_{\rm o}(B_1(G)) = 2$. $\square$

**Proposition 3.3.** *Any dominating set of $G$ having at least three vertices is not a perfect dominating set of $B_1(G)$.*

Proof. Let $D$ be a dominating set of $G$ such that $|D| \geqslant 3$ and $v \in V(G) - D$. Then there exists a vertex $u \in D$ such that $e = (u, v) \in E(G)$ and $e' \in V(B_1(G)) - D$ is adjacent to at least two vertices in $D$. Hence, $D$ is not a perfect dominating set of $B_1(G)$. $\square$

**Proposition 3.4.** *For any $(p, q)$ graph $G$ with $p \geqslant 5$, any subset of $V(L(G))$ having at least two independent vertices is not a perfect dominating set of $B_1(G)$.*

Proof. Let $D$ be a subset of $V(L(G))$ having at least two independent vertices, say $e'_1$ and $e'_2$. Then the corresponding edges $e_1$ and $e_2$ are independent in $G$. Let $v$ be a vertex in $G$ not incident with both $e_1$ and $e_2$. Then $v \in V(B_1(G)) - D$ is adjacent to both $e'_1$ and $e'_2$ and hence $D$ is not a perfect dominating set of $B_1(G)$. $\square$

**Theorem 3.7.** *For any graph $G, \gamma_{\rm p}(B_1(G)) = 3$ if and only if $G \cong P_3$, a path on three vertices.*

Proof. Assume that $\gamma_{\rm p}(B_1(G)) = 3$ and $D$ is a perfect dominating set of $B_1(G)$ with cardinality 3.

Case (i): $D$ contains three point vertices. Then $D$ cannot be a perfect dominating set of $B_1(G)$ by Proposition 3.3.

Case (ii): $D$ contains three line vertices. If $\beta_0(\langle D\rangle) \geqslant 2$ in $B_1(G)$, then $D$ is not a perfect dominating set of $B_1(G)$ by Proposition 3.4. Hence, $\beta_0(\langle D\rangle) = 1$ in $B_1(G)$. Then $D$ is a perfect dominating set of $B_1(G)$ if $G \cong C_3$. But in that case $\gamma_{\rm p}(B_1(G)) = 2$.

Case (iii): $D$ contains two line vertices and one point vertex. Then it is clear that $D$ is a perfect dominating set of $B_1(G)$ implies that $G \cong 2K_2$ or $P_3$. But, if $G \cong 2K_2$, then $\gamma_{\rm p}(B_1(G)) = 2$.

Case (iv): $D$ contains two point vertices and one line vertex. Then $D$ is a perfect dominating set of $B_1(G)$ implies that $G \cong K_2$. But $\gamma_{\rm p}(B_1(K_2)) = 2$.

From (i), (ii) and (iii) it follows that $\gamma_{\rm p}(B_1(G) = 3$ only if $G \cong P_3$. The converse is obvious. $\square$

Remark 3.4. There is no perfect dominating set of $B_1(G)$ having at least four vertices.

In the following, the point-set domination number $\gamma_{\rm ps}$ of $B_1(G)$ is determined by applying Theorem 2.1. The graphs $G$ for which $\gamma_{\rm ps}(B_1(G))$ is 2 are obtained below.

**Theorem 3.8.** $\gamma_{\rm ps}(B_1(G)) = 2$ *if and only if* $G$ *is one of the following graphs.*

(i) *There exists an edge* $e$ *in* $G$ *such that all the edges of* $G$ *are adjacent to* $e$*. That is,* $r(L(G)) = 1$*, where* $r(L(G))$ *is the radius of* $L(G)$*.*

(ii) $G \cong K_4$ *or* $K_4 - x$*, where* $x$ *is an edge in* $K_4$*.*

Proof. Assume $\gamma_{\rm ps}(B_1(G)) = 2$. Let $D$ be a point set dominating (psd) set of $B_1(G)$ with $|D| = \gamma_{\rm ps}(B_1(G))$.

Case (i): Both vertices of $D$ are point vertices. Let $D = \{v_1, v_2\} \subseteq V(G)$. $v_1$ and $v_2$ must be nonadjacent in $G$. Otherwise, $D \subseteq V(B_1(G))$ is not a dominating set of $B_1(G)$. Also each vertex in $V(G) - D$ is adjacent to both $v_1$ and $v_2$. If not, let $v_3$ be a vertex in $G$ adjacent to exactly one of the vertices $v_1$ and $v_2$, say $v_1$ and let $e_{13} = (v_1, v_3) \in E(G)$. Then $W = \{v_3, e'_{13}\} \subseteq V(B_1(G)) - D$ is an independent set in $V(B_1(G)) - D$ and there exists no vertex $v$ in $D$ such that $W \subseteq N(v) \cap V((B_1(G)) - D)$. Therefore, $v_3$ is adjacent to both $v_1$ and $v_2$. Also $v_3$ is the only vertex in $V(G) - D$ adjacent to both $v_1$ and $v_2$. Otherwise, let $v_4 \in V(G) - D$ be adjacent to both $v_1$ and $v_2$ and $e_{24} = (v_2, v_4) \in E(G)$. Then $W = \{e'_{13}, e'_{24}\} \subseteq V(B_1(G)) - D$ is an independent set in $V(B_1(G)) - D$ and there exists no vertex $v$ in $D$ such that $W \subseteq N(v) \cap V(B_1(G)) - D)$, which is a contradiction to Theorem 2.1. Hence, $G \cong P_3$.

Case (ii): One vertex of $D$ is a point vertex and the other is a line vertex. Let $D = \{v, e'\} \subseteq V(B_1(G))$, where $v, e'$ are point, line vertices respectively. Then $v \in V(G)$. Let $e$ be the edge in $G$ corresponding to $e'$.

Subcase (i): $e \in E(G)$ is incident with $v \in V(G)$. Then the fact that $D$ is a psd-set of $B_1(G)$ implies that either each edge in $G$ is adjacent to $e$ or if there exists an edge $e_1 = (u_1, v_1)$ not adjacent to $e$, then both $u_1, v_1$ must be adjacent to $v$ in $G$ and there exists no edge (other than $e$) in $G$ incident with $v$ and independent of $e_1$, where $u_1$, $v_1 \in V(G)$. In the first case, $r(L(G)) = 1$, and in the second case, $G \cong K_4$, $K_4 - x$ or $K_{1,3} + y$, where $x$ and $y$ are edges. But if $G \cong K_{1,3} + y$, then $r(L(G)) = 1$. Therefore, $G \cong K_4 - x$ or $K_4$.

Subcase (ii): $e \in E(G)$ is not incident with $v \in V(G)$. Let $e = (u, w)$, where $u, w \in V(G)$. $u$ and $w$ must be adjacent to $v$, since otherwise $D$ is not a dominating set of $B_1(G)$. Also, $\deg_G(v) = 2$. Let $(u, v) = e_1 \in E(G)$. Then $W = \{u, e'_1\} \subseteq V(B_1(G)) - D$ is an independent set and there exists no vertex in $D$ adjacent to both $u$ and $e'_1$. Hence, $D$ is not a psd-set of $B_1(G)$.

Case (iii): Both vertices of $D$ are line vertices. Let $D = \{e'_1, e'_2\} \subseteq V(B_1(G))$, where the corresponding edges $e_1$ and $e_2$ in $G$ are independent. If $e_1$ and $e_2$ are adjacent edges in $G$, then $D$ cannot be a dominating set of $B_1(G)$. Also each edge in $G$ is adjacent to both $e_1$ and $e_2$. Indeed, if there exists an edge either adjacent to exactly one of $e_1$ and $e_2$ or independent of $e_1$ and $e_2$, then $D$ cannot be a psd-set of $B_1(G)$. If $e_1 = (u_1, v_1)$, $e_2 = (u_2, v_2)$, where $u_1, v_1, u_2, v_2 \in V(G)$, then $(u_1, u_2), (u_1, v_2)$, $(v_1, u_2)$ and $(v_1, v_2) \in E(G)$, since $D$ is a psd-set of $B_1(G)$. Thus, $G \cong K_4$.

By Case (i), Case (ii) and Case (iii), it follows that $\gamma_{\rm ps}(B_1(G)) = 2$ only if either there exists an edge $e$ in $G$ such that all the edges of $G$ are adjacent to $e$ or $G \cong K_4$. The converse is obvious. □

**Corollary 3.8.1.** *If $D$ is an independent dominating set of $G$ having exactly two vertices, then $D$ is a psd-set of $B_1(G)$ if and only if $G \cong P_3$.*

**Corollary 3.8.2.** *Any independent dominating set of $L(G)$ having exactly two vertices is a psd-set of $B_1(G)$ if and only if $G \cong K_4$.*

In the following, we obtain a necessary and sufficient condition for a psd-set of $G$ to be also a psd-set of $B_1(G)$.

**Theorem 3.9.** *Let $G$ be any graph such that $B_1(G)$ is connected. Then $\gamma_{\rm ps}(B_1(G)) \leqslant \gamma_{\rm ps}(G)$ with $\gamma_{\rm ps}(G) \geqslant 3$ if and only if there exists a minimal psd-set $D$ of $G$ having at least three vertices satisfying:*

(a) *$G$ has no perfect matching and for any set $S$ of independent edges in $G$, there is a vertex in $D$ not incident with edges in $S$; and*

(b) *each vertex in $V(G) - D$ is adjacent to at least two vertices in $D$.*

Proof. Assume $\gamma_{\rm ps}(B_1(G)) \leqslant \gamma_{\rm ps}(G)$, $\gamma_{\rm ps}(G) \geqslant 3$. Let $D$ be a psd-set of both $G$ and $B_1(G)$ with $|D| = \gamma_{\rm ps}(G)$. Let $G$ have a perfect matching, say $e_1, e_2, \ldots, e_t$, where $e_1, e_2, \ldots, e_t \in E(G)$. Then $W = \{e'_1, e'_2, \ldots, e'_t\} \subseteq V(B_1(G)) - D$ is an independent set in $B_1(G) - D$ and there exists no vertex $u$ in $D$ such that $W \subseteq N(u) \cap V(B_1(G)) - D$, which is a contradiction to the fact that $D$ is a psd-set of $B_1(G)$. Hence, $G$ has no perfect matching. This proves (a). Let there exist a vertex $v$ in $V(G) - D$ adjacent to exactly one vertex, say $u$, in $D$ and $e = (u, v) \in E(G)$. Then $W = \{v, e'\} \subseteq V(B_1(G)) - D$ is an independent set in $B_1(G) - D$ and no vertex in $D$ is adjacent to both $v$ and $e'$ in $B_1(G)$, which is a contradiction. Hence, each vertex in $V(G) - D$ is adjacent to at least two vertices in $D$. This proves (b).

Conversely, let $D$ be a minimal psd-set of $G$. Then $D \subseteq V(B_1(G))$. Assume conditions (a) and (b). Let $W$ be an independent set in $V(B_1(G)) - D$. If $W$ contains both point and line vertices, then it has exactly one point and one line vertex. Otherwise, $W$ cannot be independent in $B_1(G)$. Therefore, all the vertices of $W$ will be point vertices, line vertices or $W$ contains one point and one line vertex.

C a s e (i): All the vertices of $W$ are point vertices. Then $W \subseteq V(G) - D$ and since $D$ is a psd-set of $G$, there exists a vertex $u$ in $D$ such that $W \subseteq N(u) \cap (V(G) - D)$ and hence $W \subseteq N(u) \cap (V(B_1(G)) - D)$.

C a s e (ii): All the vertices of $W$ are line vertices. Then the edges in $G$ corresponding to the line vertices in $W$ are independent. By (a), since $G$ has no perfect matching and there is a vertex $u$ in $D$ not incident with the edges corresponding to the line vertices in $W$. Thus, $W \subseteq N(u) \cap (V(B_1(G) - D)$.

C a s e (iii): $W$ contains one point and one line vertex. Let $W = \{v, e'\} \subseteq V(B_1(G)) - D$, where $e \in E(G)$ is incident with $v$. Then by (b), there exists a vertex in $D$ adjacent to both $v$ and $e'$ in $B_1(G)$.

By Case (i), Case (ii) and Case (iii), it follows that $D$ is a psd-set of $B_1(G)$. Hence, $\gamma_{\rm ps}(B_1(G)) \leqslant \gamma_{\rm ps}(G)$. □

R e m a r k s.

3.5. The set of all line vertices in $B_1(G)$ is a psd-set of $B_1(G)$ if and only if for every independent set $W$ in $G$, $\langle V(G) - W\rangle$ is not totally disconnected.

3.6. The set of all point vertices in $B_1(G)$ is a psd-set of $B_1(G)$ if and only if there exists no perfect matching in $G$.

3.7. If the radius of $L(G)$ is 1, then $\gamma_{\rm ps}(B_1(G)) = 2$.

3.8. If $G \cong C_{2n+1}$, for $n \geqslant 2$, then $\gamma_{\rm ps}(B_1(G)) \leqslant 2n + 1$.

In the following, we find the graphs $G$ for which the restrained domination number $\gamma_{\rm r}$ of $B_1(G)$ is 2.

**Theorem 3.10.** *For any $(p, q)$ graph with $q \geqslant 3$, $\gamma_{\rm r}(B_1(G)) = 2$.*

P r o o f. Since $\gamma(B_1(G)) = 2$, $\gamma_{\rm r}(B_1(G)) \geqslant 2$. Let $e = (u, v)$ be an edge in $G$, where $u, v \in V(G)$. Then $D = \{v, e'\} \subseteq V(B_1(G))$ is a dominating set of $B_1(G)$. The degree of a point vertex in $B_1(G)$ is $q (\geqslant 3)$ and hence any point vertex in $V(B_1(G)) - D$ is adjacent to a vertex in $V(B_1(G)) - D$. Also the degree of a line vertex $e'_1$ in $B_1(G)$ is $\deg_{L(G)}(e'_1) + p - 2$. Since $q \geqslant 3$ if $p \geqslant 4$, then $d^*(e'_1) \geqslant 3$, where $d^*(e'_1)$ is the degree of $e'_1$ in $B_1(G)$. If $G \cong C_3$, then the degree of each line vertex in $B_1(G)$ is 3 and hence each line vertex in $V(B_1(G)) - D$ is adjacent to a vertex in $V(B_1(G)) - D$. Hence, $D$ is a restrained dominating set of $B_1(G)$ and $\gamma_{\rm r}(B_1(G)) \leqslant 2$. Thus, $\gamma_{\rm r}(B_1(G)) = 2$. □

E x a m p l e 3.1. $\gamma_{\rm r}(B_1(P_3)) = 3$, $\gamma_{\rm r}(B_1(P_3 \cup nK_1)) = 2$ and $\gamma_{\rm r}(B_1(2K_2 \cup nK_1)) = 2$, for $n \geqslant 1$.

**Theorem 3.11.** *Let $G$ be any connected graph. Then any dominating set of $G$ having at least three vertices is a restrained dominating set of $B_1(G)$. That is, $\gamma_{\rm r}(B_1(G)) \leqslant \gamma(G)$, where $\gamma(G) \geqslant 3$.*

P r o o f. Let $D$ be a dominating set of $G$ with $|D| \geqslant 3$. Then $D \subseteq V(B_1(G))$. Let $v$ be a point vertex in $V(B_1(G)) - D$. Since $D$ is a dominating set of $G$, $v$ is adjacent to at least one vertex in $D$ and since $V(B_1(G)) - D$ contains all the line vertices, $v$ is adjacent to at least one line vertex in $V(B_1(G)) - D$. Let $e'$ be a line vertex in $V(B_1(G)) - D$. Since $|D| \geqslant 3$, $e'$ is adjacent to at least one vertex in $D$ and since $L(G)$ is connected, there exists at least one line vertex adjacent to $e'$ in $V(B_1(G)) - D$. Hence, $D$ is a restrained dominating set of $B_1(G)$. Thus, $\gamma_{\rm r}(B_1(G)) \leqslant \gamma(G)$, where $\gamma(G) \geqslant 3$. □

R e m a r k 3.9. Let $G$ be a graph having at least three vertices. Then the set of all point vertices is a restrained dominating set of $B_1(G)$ if and only if $G$ does not contain $K_2$ as one of its components.

R e m a r k 3.10. Let $G$ be a graph other than a star. Then the set of all line vertices is a restrained dominating set of $B_1(G)$ if and only if $G$ contains no isolated vertices.

In the following, several upper bounds for the split domination number $\gamma_{\rm s}$ of $B_1(G)$ are obtained. Here, the graphs $G$ for which $B_1(G)$ is connected are considered.

**Theorem 3.12.** *If there exists a path (not necessarily induced) of length 2 in $G$, then $\gamma_{\rm s}(B_1(G)) \leqslant q$.*

P r o o f. Let $P$ be a path (not necessarily induced) of length 2 in $G$, let $u$ be the initial vertex of $P$ and let $D'$ be the set of all line vertices in $B_1(G)$ corresponding to the edges not incident with $u$ in $G$. Then $D = D' \cup N_G(u) \subseteq V(B_1(G))$ is a dominating set of $B_1(G)$. Also $u$ is an isolated vertex in the induced subgraph $\langle V(B_1(G) - D\rangle$ and this subgraph is disconnected. Hence, $D$ is a split dominating set of $B_1(G)$. Thus, $\gamma_{\rm s}(B_1(G)) \leqslant |D| = q$.

This bound is attained if $G \cong C_n$, $n \geqslant 3$. □

**Theorem 3.13.** *If $G$ contains $2K_2$ as an induced subgraph, then $\gamma_{\rm s}(B_1(G)) \leqslant p + q - 6$.*

P r o o f. Let $G$ contain $2K_2$ as an induced subgraph and $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be the edges of $2K_2$, where $u_1, v_1, u_2, v_2 \in V(2K_2)$. Then $D' = \{u_1, v_1, u_2, v_2, e_1', e_2'\} \subseteq V(B_1(G)$, where $e_1'$ and $e_2'$ are the line vertices corresponding to the edges $e_1$ and $e_2$ respectively. If $D = V(B_1(G)) - D'$, then $D$ is a dominating set of $B_1(G)$. Also $\langle V(B_1(G)) - D\rangle \cong 2C_3$ in $B_1(G)$ and is disconnected. Hence, $D$ is a split dominating set of $B_1(G)$. Thus, $\gamma_s(B_1(G)) \leqslant p + q - 6$. $\square$

This bound is attained if $G \cong 3K_2$.

Remark 3.11. If $G \cong 2K_2 \cup mK_1$, then $\gamma_s(B_1(G)) = 2$.

**Theorem 3.14.** *If $G$ is a connected graph with at least three vertices, then $\gamma_s(B_1(G)) \leqslant p + \kappa'(G)$, where $\kappa'(G)$ is the vertex connectivity of $L(G)$.*

Proof. Let $G$ be a graph having at least three vertices and $E = \{e_1, e_2, \ldots, e_t\}$, where $t = \kappa'(G)$, be the vertex connectivity set of $L(G)$. Then $E \subseteq V(B_1(G))$. If $D = V(G) \cup E$, then $D \subseteq V(B_1(G))$ is a dominating set of $B_1(G)$ and $\langle V(B_1(G)) - D\rangle = \langle V(L(G)) - E\rangle$ is disconnected in $B_1(G)$. Hence, $D$ is a dominating set of $B_1(G)$. Thus, $\gamma_s(B_1(G)) \leqslant p + \kappa'(G)$. $\square$

Remark 3.12. If $\langle V(L(G)) - E\rangle$ contains isolated vertices, then $\gamma_s(B_1(G)) \leqslant p + \kappa'(G) - 2$, where $|V(G)| \geqslant 5$.

**Theorem 3.15.** *For any graph $G$ having no isolated vertices, $\gamma_s(B_1(G)) \leqslant q + \alpha_0(G) - 1$, where $\alpha_0(G)$ is the point covering number of $G$ and $\alpha_0(G) \geqslant 3$.*

Proof. Let $D$ be a point cover of $G$ such that $|D| = \alpha_0(G) \geqslant 3$. Choose a vertex $v$ in $V(G) - D$ such that $\deg_G(v)$ is maximum. Let $D'$ be the set of line vertices in $B_1(G)$ corresponding to the edges not incident with $v$. Then $D'' = D \cup D'$ is a dominating set of $B_1(G)$ and $v$ is isolated in $\langle V(B_1(G)) - D\rangle$. Hence, $D''$ is a split dominating set of $B_1(G)$. Hence, $\gamma_s(B_1(G)) \leqslant |D \cup D'| = \alpha_0(G) + q - \deg_G(v)$. But $\deg_G(v) \geqslant 1$. Thus $\gamma_s(B_1(G)) \leqslant \alpha_0(G) + q - 1$. $\square$

Similarly, the following propositions can be proved.

**Proposition 3.5.** *Let $G$ be a graph having no isolated vertices. If there exists a line cover $D$ of $G$ containing at least two independent edges with $|D| = \alpha_1(G)$, then $\gamma_s(B_1(G)) \leqslant p + q - \alpha_1(G)$, where $\alpha_1(G)$ is the line covering number of $G$.*

**Proposition 3.6.** *If $G$ is a graph having at least five vertices which has a perfect matching, then $\gamma_s(B_1(G)) \leqslant \frac{1}{2}p + q - 2$.*

**Theorem 3.16.** *For any connected graph $G$ with $r(G) \geqslant 2$, $\gamma_s(B_1(G)) \leqslant p+q-\Delta(G)-2$, where $r(G)$ is the radius of $G$.*

P r o o f. Let $v_1$ be a vertex of maximum degree $\Delta(G)$ in $G$ and let $e_{11}, e_{12}, \ldots, e_{1\Delta}$ be the edges in $G$ incident with $v \in V(G)$. Let $G_1 = \langle V(G) - N[v_1]\rangle$ and let $v_2$ be a vertex of maximum degree $\Delta_1(G)$ in $G_1$ and $e_{21}, e_{22}, \ldots, e_{2,m}$ be the edges incident with $v_2$ in $G_1$, where $\Delta_1(G) = m$. Since $r(G) \geqslant 2$, $1 \leqslant \Delta_1(G) \leqslant \Delta(G) - 1$. If $D'$ is the set of line vertices in $B_1(G)$ corresponding to the edges $e_{11}, \ldots, e_{1,D}$, $e_{21}, \ldots, e_{2,m}$, then $D'' = D' \cup \{v_1, v_2\} \subseteq V(B_1(G))$ and $|D''| = \Delta(G) + \Delta_1(G) + 2 \geqslant \Delta(G) + 2$. If $D = V(B_1(G)) - D''$, then $D$ is a dominating set of $B_1(G)$ and $\langle V(B_1(G)) - D\rangle$ is disconnected. Thus, $D$ is a split dominating set of $B_1(G)$ and hence $\gamma_s(B_1(G)) \leqslant |D| \leqslant p+q-\Delta(G)-2$. □

This bound is sharp, since if $G \cong C_4$, $\gamma_s(B_1(G) = 4 = p+q-\Delta(G)-2$.

R e m a r k 3.13. If $r(G) = 1$, then $\gamma_s(B_1(G)) \leqslant (p+q) - (\Delta(G)+1)$ and this bound is attained if $G \cong K_{1,n}$, for $n \geqslant 2$.

**Theorem 3.17.** *If $G$ is a disconnected graph having $K_2$ as one of its components, then $\gamma_s(B_1(G)) \leqslant p-1$.*

P r o o f. Let $G$ contain $K_2$ as one of its components. Since $B_1(G)$ is connected, $G$ cannot be one of the graphs $K_2 \cup nK_1$, $2K_2$, for $n \geqslant 1$. Let $e'$ be the line vertex in $B_1(G)$ corresponding to an edge in $G$ but not in $K_2$. Then $D = \{V(G) - V(K_2), e'\} \subseteq V(B_1(G))$ is a dominating set of $B_1(G)$ and the line vertex corresponding to the edge in $K_2$ is isolated in $\langle V(B_1(G)) - D\rangle$. Hence, $D$ is a split dominating set of $B_1(G)$ and $\gamma_s(B_1(G)) \leqslant p-1$. □

This bound is attained if $G \cong K_4 \cup K_2$.

**Theorem 3.18.** *Let $G$ be a disconnected graph (not totally disconnected) other than $K_{1,n} \cup K_2 \cup mK_1$, for $n \geqslant 2$ and $m \geqslant 0$. If $G$ contains $K_2$ as one of its components, then $\gamma_s(B_1(G)) \leqslant p+q-\Delta(G)-4$.*

P r o o f. Let $v$ be a vertex of maximum degree in $G$. Then $\deg_G(v) = D(G) \geqslant 1$. Let $e'$ be the line vertex in $B_1(G)$ corresponding to the edge in $K_2$ and let $D'$ be the set of line vertices in $B_1(G)$ corresponding to the edges in $G$ incident with $v$. If $D'' = \{v, e'\} \subseteq V(K_2) \cup D''V(B_1(G))$ then $|D''| = D(G) + 4$ and $\langle D''\rangle$ is disconnected in $B_1(G)$ and $D = V(B_1(G)) - D''$. Since $G \ncong K_1 n \cup K_2 \cup mK_1$ and $B_1(G)$ is connected, there exists at least one point vertex $u$ and one line vertex such that the corresponding edge in $G$ is not incident with $u$. Then, $D$ is a dominating set of $B_1(G)$ and since $\langle V(B_1(G)) - D\rangle$ is disconnected, $D$ is a split dominating set of $B_1(G)$. Hence $\gamma_s(B_1(G)) \leqslant p+q-\Delta(G)-4$. □

This bound is attained if $G$ is one of the graphs $C_3 \cup K_2$ and $(K_{1,n}+e) \cup K_2$, for $n \geqslant 3$.

**Theorem 3.19.** *Let $G$ be any disconnected graph having no isolated vertices. If none of the components of $G$ is $K_2$, then $\gamma_{\rm s}(B_1(G)) \leqslant p-2+\delta_e(G)$, where $\delta_e(G)$ is the minimum degree of $L(G)$.*

Proof. Since none of the components of $G$ is $K_1$, $L(G)$ has no isolated vertices and hence $\delta_e(G) \geqslant 1$. Let $e'$ be a vertex in $L(G)$ such that $\deg_L(G)(e') = \delta_e(G) \geqslant 1$ and let $e = (u,v)$ be the corresponding edge in $G$, where $u$, $v \in V(G)$. Then $D = \{V(G) - \{u,v\}, N_{L(G)}(e')\} \subseteq V(B_1(G))$ is a dominating set of $B_1(G)$ and $e'$ is isolated in $\langle V(B_1(G)) - D\rangle$ and hence $D$ is a split dominating set of $B_1(G)$. Thus, $\gamma_{\rm s}(B_1(G)) \leqslant p-2+\delta_e(G)$. □

This bound is attained when $G \cong K_{1,n}$, for $n \geqslant 3$.

**Theorem 3.20.** *If a disconnected graph $G$ contains $K_{1,n}$ $(n \geqslant 2)$ as one of its components, then $\gamma_{\rm s}(B_1(G)) \leqslant p-1$.*

Proof. Let $v$ be the center vertex of $K_{1,n}$, for $n \geqslant 2$, and $D = V(G) - \{v\}$. Then $D$ is a dominating set of $B_1(G)$ and $\langle V(B_1(G)) - D\rangle$ is disconnected with exactly two components, one being $K_1$ and the other a complete graph. Thus, $D$ is a split dominating set of $B_1(G)$ and $\gamma_{\rm s}(B_1(G)) \leqslant p-1$. □

This bound is attained if $G \cong K_{1,n} \cup mK_1$, where $n \geqslant 2$ and $m \geqslant 1$.

In the following, non-split domination number $\gamma_{\rm ns}$ of $B_1(G)$ is determined. Here, again, only graphs $G$ for which $B_1(G)$ is connected are considered.

**Theorem 3.21.** *If $G$ is a $(p,q)$ graph with $q \geqslant 3$, then $\gamma_{\rm ns}(B_1(G)) = 2$.*

Proof. Let $e = (u,v) \in E(G)$, where $u, v \in V(G)$, and let $e'$ be the line vertex in $B_1(G)$ corresponding to the edge $e$. Then $D = \{v, e'\} \subseteq V(B_1(G))$ is a dominating set of $B_1(G)$. Since $G$ contains at least three edges, $V(B_1(G)) - D$ contains at least two point and two line vertices. Also, the subgraph of $B_1(G)$ induced by $V(B_1(G)) - D$ contains $P_4$ (a path on 4 vertices) as a subgraph and each line vertex in $V(B_1(G)) - D$ is adjacent to at least $p-1$ or $p-2$ point vertices and $\langle V(B_1(G)) - D\rangle$ is connected. Hence, $D$ is a non-split dominating set of $B_1(G)$ and $\gamma_{\rm ns}(B_1(G)) \leqslant 2$. But $\gamma_{\rm ns}(B_1(G)) \geqslant \gamma(B_1(G)) = 2$. Thus $\gamma_{\rm ns}(B_1(G)) = 2$. □

Example 3.2.

(i) If $G \cong P_3 \cup mK_1$, for $m \geqslant 0$, then $\gamma_{\rm ns}(B_1(G)) = 3$.

(ii) If $G \cong 2K_2 \cup mK_1$, for $m \geqslant 1$, then there exists no non-split dominating set of $B_1(G)$.

**Theorem 3.22.** *If $\gamma(G) \geqslant 3$, then $\gamma_{\rm ns}(B_1(G)) \leqslant \gamma(G)$.*

Proof. Let $D$ be a $\gamma$-set of $G$. Since $\langle D\rangle \geqslant 3$, $D$ is a dominating set of $B_1(G)$.

Case (i): $G$ is connected. Since $L(G)$ is an induced subgraph of $B_1(G)$ and each point vertex in $V(B_1(G)) - D$ is adjacent to at least one of the line vertices in $V(B_1(G))-D$, $\langle V(B_1(G)) - D\rangle$ is connected. Therefore, $D$ is a non-split dominating set of $B_1(G)$.

Case (ii): $G$ is disconnected. Let $G_1, G_2, \ldots, G_t$ $(t \geqslant 2)$ be the components of $G$. Then $L(G_i)$ is connected $(1 \leqslant i \leqslant t)$ and $V(B_1(G)) - D = \bigcup\limits_{1\leqslant i\leqslant t} V(L(G_i)) \cup (V(G) - D)$. The line vertices in $B_1(G)$ corresponding to the edges in one of the components are adjacent to the point vertices corresponding to the vertices in the remaining components and vice-versa. Hence, $\langle V(B_1(G) - D\rangle$ is connected and $D$ is a non-split dominating set of $B_1(G)$. Thus, $\gamma_{\rm ns}(B_1(G)) \leqslant \gamma(G)$, where $\gamma(G) \geqslant 3$. $\square$

Remark 3.14. Any non-split dominating set of $B_1(G)$ containing point vertices only need not be a non-split dominating set of $G$. For example, in $C_4$, the set containing any two nonadjacent vertices of $C_4$ is a non-split dominating set of $B_1(C_4)$, but is not a non-split dominating set of $C_4$.

**Theorem 3.23.** *Let $G \ncong nK_2$ $(n \geqslant 3)$ and let $G$ contain no isolated vertices. Then, $\gamma_{\rm ns}(B_1(G)) \leqslant \gamma'(G)$, if there exists a minimal edge dominating set of $G$ containing at least two independent edges, where $\gamma'(G)$ is the edge domination number of $G$.*

Proof. Let $D' \subseteq E(G)$ be an edge dominating set of $G$ containing at least two independent edges with $\langle D'\rangle = \gamma'(G)$. Let $D$ be the set of line vertices in $B_1(G)$ corresponding to the edges in $D'$. Then $D$ is a dominating set of $L(G)$ and $\beta_0(\langle D\rangle_L(G)) \geqslant 2$. Hence, $D$ is a dominating set of $B_1(G)$.

Case (i): $G$ is connected. Then the subgraph of $\langle V(B_1(G)) - D\rangle$ induced by all the point vertices in $B_1(G)-D$ is connected and each line vertex in $V(B_1(G))-D$ is adjacent to $p-2$ point vertices in $V(B_1(G))-D$; hence, $\langle V(B_1(G)) - D\rangle$ is connected and $D$ is a non-split dominating set of $B_1(G)$.

Case (ii): $G$ is disconnected. Since $G \ncong nK_2$ $(n \geqslant 3)$, at least one of the components of $G$ contains $P_3$ as a subgraph. Let $G_1$ be one of the components of $G$ having $P_3$ as a subgraph and let $e \in E(G_1)$. Then the line vertex $e'$ corresponding to $e$ is a vertex in $V(B_1(G)) - D'$ and is adjacent to at least one vertex in $V(B_1(G))$ and to all the point vertices in the remaining components. Since $G$ is an induced subgraph of $\langle V(B_1(G)) - D\rangle$, $\langle V(B_1(G)) - D\rangle$ is connected and $D$ is a non-split dominating set of $B_1(G)$, giving $\gamma_{\rm ns}(B_1(G)) \leqslant \gamma'(G)$. $\square$

Remark 3.15. If $G \cong nK_2$, for $n \geqslant 3$, then $\gamma_{\rm ns}(B_1(G)) = 2$.

A c k n o w l e d g e m e n t. Thanks are due to the referee for the modified version of this paper.

*References*


[1] *J. Akiyama, T. Hamada, I. Yoshimura*: On characterizations of the middle graphs. TRU. Math. *11* (1975), 35–39.
[2] *R. B. Allan, R. Laskar*: On domination and independent domination numbers of a graph. Discrete Math. *23* (1978), 73–76.
[3] *M. Behzad*: A criterion for the planarity of the total graph of a graph. Proc. Camb. Philos. Soc. *63* (1967), 679–681.
[4] *S. B. Chikkodimath, E. Sampathkumar*: Semi total graphs II. Graph Theory Research Report, Karnatak University *2* (1973), 5–9.
[5] *E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi*: Total domination in graphs. Networks *10* (1980), 211–219.
[6] *E. J. Cockayne, B. L. Hartnell, S. T. Hedetniemi, R. Laskar*: Perfect domination in graphs. J. Comb. Inf. Syst. Sci. *18* (1993), 136–148.
[7] *G. S. Domke, J. H. Hattingh, S. T. Hedetniemi, R. Laskar, L. R. Markus*: Restrained domination in graph. Discrete Math. *203* (1999), 61–69.
[8] *T. Hamada, I. Yoshimura*: Traversability and connectivity of the middle graph of a graph. Discrete Math. *14* (1976), 247–256.
[9] *F. Harary*: Graph Theory. Addison-Wesley, Reading, Mass., 1969.
[10] *T. N. Janakiraman, S. Muthammai, M. Bhanumathi*: On the Boolean function graph of a graph and on its complement. Math. Bohem. *130* (2005), 113–134.
[11] *V. R. Kulli, B. Janakiram*: The total global domination number of a graph. Indian J. Pure Appl. Math. *27* (1996), 537–542.
[12] *O. Ore*: Theory of Graphs. Amer. Math. Soc. Colloq. Publ. 38, Providence, AMS, 1962.
[13] *E. Sampathkumar, L. Pushpa Latha*: Point-set domination number of a graph. Indian J. Pure Appl. Math. *24* (1993), 225–229.
[14] *E. Sampathkumar, H. B. Walikar*: The connected domination number of a graph. J. Math. Phys. Sci. *13* (1979), 607–613.
[15] *D. V. S. Sastry, B. Syam Prasad Raju*: Graph equations for line graphs, total graphs, middle graphs and quasi-total graphs. Discrete Math. *48* (1984), 113–119.
[16] *H. Whitney*: Congruent graphs and the connectivity of graphs. Amer. J. Math. *54* (1932), 150–168.


*Authors' addresses*: *T. N. Janakiraman*, National Institute of Technology, Tiruchirappalli 620 015, India, e-mail: `janaki@nitt.edu`; *S. Muthammai*, *M. Bhanumathi*, Government Arts College for Women, Pudukkottai 622 001, India.