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Abstract

We consider a flow of heat conducting fluid inside a moving domain whose shape in time is prescribed. The flow in this case is governed by the Navier-Stokes-Fourier system consisting of equation of continuity, momentum balance, entropy balance and energy equality. The velocity is supposed to fulfill the full-slip boundary condition and we assume that the fluid is thermally isolated. In the presented article we show the existence of a variational solution.

Keywords: compressible Navier-Stokes-Fourier equations, entropy inequality, time-varying domain, slip boundary conditions

1 Introduction

The flow of heat conducting fluid inside a moving domain is an interesting problem with a lot of practical applications and deserves its attention. Let us mention modeling of the motion of a piston in a cylinder filled by a viscous heat conducting gas. There are many references on this problem in the statistical physics. We can mention works of Lieb [21], Gruber et al. [17, 16, 18], Wright [33, 34, 35], etc. The problem was investigated by Shelukhin [30], Antman and Wilber [1] in the case of homogeneous boundary conditions for barotropic case. The extension to the case of non-homogeneous boundary conditions can be found in the work of Maity et al. [23]. The motion of a piston in a cylinder filled by a viscous heat conducting gas was studied by Shelukhin [31]. His results coincide with the statistical mechanics scenario for the thermally insulting piston. Rather complex behavior of piston was proven by Feireisl et al. [14].

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Although the full Navier-Stokes-Fourier system in a moving domain has been already investigated (see [19]) it turns out that it is more useful to examine the following form of the NSF system (under more general hypotheses on the pressure) at least from the point of view of further analysis like asymptotic limits, dimension reduction and so on:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.2}$$

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s \mathbf{u}) + \operatorname{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{1.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \left(\varrho |\mathbf{u}|^2 + \varrho e\right) \, \mathrm{d}x = 0. \tag{1.4}$$

These equations, which are considered on a time-space domain $(0,T) \times \Omega_t \subset (0,\infty) \times \mathbb{R}^3$ where Ω_t is a time dependent domain, are mathematical formulations of the balance of mass, linear momentum, entropy and total energy respectively. Unknowns are the density $\varrho : (0,T) \times \Omega_t \mapsto [0,\infty)$, the velocity $\mathbf{u} : (0,T) \times \Omega_t \mapsto \mathbb{R}^3$ and the temperature $\vartheta : (0,T) \times \Omega_t \mapsto [0,\infty)$. Other quantities appearing in these equations are functions of the unknowns, namely the stress tensor $\mathbb{S}$, the internal energy e, the pressure p, the entropy s, and the entropy production rate σ. Their needed properties are mentioned later on. For simplicity we do not consider in this work any external forces.

The time dependent domain Ω_t is prescribed by movement of its boundary on the time interval $[0,T]$. Namely, the boundary of the domain Ω_t occupied by the fluid is described by a given velocity field $\mathbf{V}(t,x)$ where $t \geq 0$ and $x \in \mathbb{R}^3$. Supposing $\mathbf{V}$ is regular enough we can associate the following system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,x) = \mathbf{V}\Big(t, \mathbf{X}(t,x)\Big), \ t > 0, \ \mathbf{X}(0,x) = x,$$

with our domain, then we set

$$\Omega_\tau = \mathbf{X}\left(\tau, \Omega_0\right), \text{ where } \Omega_0 \subset \mathbb{R}^3 \text{ is a given domain}, \ \Gamma_\tau = \partial \Omega_\tau, \text{ and } Q_\tau = \cup_{t \in (0,\tau)} \{t\} \times \Omega_t.$$

We assume that the volume of the domain can not degenerate in time, namely

$$\text{there exists } V_0 > 0 \text{ such that } |\Omega_\tau| \geq V_0 \text{ for all } \tau \in [0,T]. \tag{1.5}$$

Moreover, we assume that

$$\operatorname{div}_x \mathbf{V} = 0 \text{ on the neighborhood of } \Gamma_\tau, \tag{1.6}$$

see Remark 5.1.

We assume that the boundary of the physical domain is impermeable. This is described by the condition

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \text{ for any } \tau \geq 0, \tag{1.7}$$

where $\mathbf{n}(t,x)$ denotes the unit outer normal vector to the boundary Γ_t. Moreover, we study the problem with the full slip boundary conditions in the form

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n} = 0. \tag{1.8}$$

The heat flux satisfies the conservative boundary conditions

$$\mathbf{q} \cdot \mathbf{n} = 0 \text{ for all } t \in [0,T], \ x \in \Gamma_t. \tag{1.9}$$

For the physical motivation of correct description of the fluid boundary behavior, see Bulíček, Málek and Rajagopal [3], Priezjev and Troian [28] and the references therein.

Finally we supplement our considered system with given initial data

$$\varrho_0, \ (\varrho\mathbf{u})_0, \ \vartheta_0.$$

In comparison with [19], we consider entropy (1.3) and energy (1.4) balance in place of thermal energy equation – cf. [19, (1.3)]. Even though strong solutions for both variants coincide as far as ϑ is bounded below away from zero, results concerning weak solutions cannot be simply transferred between them – for example a relative entropy inequality which plays a significant role in weak-strong uniqueness and in various singular limits – see [2, 7, 10, 11]. Hence, it is reasonable to complete the theory by showing the existence of variational solution to system (1.1)–(1.4) provided $\Omega_t \subset \mathbb{R}^3$ is time-dependent. This is the main goal of this paper.

The existence theory for the barotropic Navier-Stokes system on *fixed* spatial domains in the framework of weak solutions dates back to the seminal work by Lions [22], who worked with certain growth of pressure, and Feireisl et. al [12] where the existence to a class of physically relevant pressure-density state equations was shown. These results were later extended to the full Navier-Stokes-Fourier system in [5, 6], where the formulation with thermal energy equation is used.

Feireisl and Novotný [10] also proved the existence of weak solutions to the Navier-Stokes-Fourier system formulated with an entropy inequality instead of the thermal energy equation. This approach on the one hand allows for proving existence of solutions with more general hypotheses on the pressure on the other hand it requires the presence of a radiative term $a\vartheta^4$ in the pressure law.

The investigation of *incompressible* fluids in time dependent domains started with a seminal paper of Ladyzhenskaya [20], Fujita et al.[15], see also [24, 25, 26, 29] for more recent results in this direction.

Compressible fluid flows in time dependent domains in barotropic case were examined in [8] for the *no-slip* boundary conditions and in [9] for the slip boundary conditions. As mentioned earlier, the existence of solution to the NSF system was the content of [19].

In order to give a proof of existence of variational solutions, we proceed in the following way.

1. In order to be able to start our consideration of the penalised system in a fixed time independent domain B large enough, such that
$$\overline{\Omega_t} \subset B \text{ for any } t \in [0,T] \tag{1.10}$$
we add to the momentum equation an extra term
$$\frac{1}{\varepsilon}\int_0^T \int_{\Gamma_t} (\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\ \boldsymbol{\varphi}\cdot\mathbf{n}\,\mathrm{dS}_x\,\mathrm{d}t, \ \varepsilon > 0 \text{ small}, \tag{1.11}$$
which was originally proposed by Stokes and Carey in [32]. This penalizes the flux through the interface Γ_t and allows to deal with the slip boundary conditions. Namely, as $\varepsilon \to 0$, this additional term yields the boundary condition $(\mathbf{u}-\mathbf{V})\cdot\mathbf{n} = 0$ on Γ_t. Consequently the large domain $(0,T)\times B$ becomes divided by an impermeable interface $\cup_{t\in(0,T)}\{t\}\times\Gamma_t$ to a *fluid domain* Q_T and a *solid domain* Q_T^c. Next to handle the behaviour of the solution in the solid domain Q_T^c we use a penalization scheme which is represented by parameters $\omega, \ \nu, \ \delta, \ \lambda$ and η.

2. In addition to (1.11), we introduce *variable* coefficients: the shear viscosity coefficient $\mu_\omega(t,x,\vartheta)$, the heat conductivity coefficient $\kappa_\nu(t,x,\vartheta)$, and coefficient in a radiation counterpart of the pressure, the specific entropy, and internal energy function $a = a_\eta(t,x)$. All of them remain strictly positive in the fluid domain Q_T, but vanish in the solid domain $B \setminus \overline{Q_T^c}$ as ω, ν, and η converge to zero respectively.

3. We add a term $\lambda\vartheta^5$ into the energy balance and $\lambda\vartheta^4$ into the entropy balance. These terms yield a control over a temperature on the solid domain. The exact choice of power ϑ^5 is not essential, however it is important that the power is larger than ϑ^4.

4. Similarly to the existence theory developed in [10], we introduce the *artificial pressure* related to coefficient δ:

$$p_\delta(\varrho,\vartheta) = p(\varrho,\vartheta) + \delta\varrho^\beta,\ \beta \geq 4,\ \delta > 0.$$

This gives an extra information abut the density.

5. Keeping ε, η, ω, ν, λ, and $\delta > 0$ fixed, we solve the modified problem in a (bounded) reference domain $B \subset \mathbb{R}^3$ such that (1.10) is satisfied. To this end, we adapt the existence theory for the compressible Navier-Stokes-Fourier system with variable coefficients developed in [10].

6. First, we take the initial density ϱ_0 vanishing outside Ω_0 and letting $\varepsilon \to 0$ for fixed η, ω, ν, λ, $\delta > 0$ we obtain a "two-fluid" system where the density vanishes in the solid part $((0,T) \times B) \setminus Q_T$ of the reference domain.

7. In order to get rid of the term on Q_T^c, we tend with remaining approximations to zero. We do it in the following sequence η, ω, ν, λ, and δ.

The paper is organized as follows. In Section 2 we present all assumptions and we complete system (1.1)–(1.4) by prescribing the boundary and initial conditions. We define a variational solution and we introduce the precise version of our main result. Section 3 is devoted to the penalization problem, we highlight all approximations and we discuss the existence of its solution. Finally, the proof of the main theorem is concluded in Section 4 by performing appropriate limits.

2 Preliminaries

2.1 Hypotheses

Motivated by [10] we introduce the following set of assumptions:

The stress tensor $\mathbb{S}$ is determined by the standard Newton rheological law

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I},\ \mu > 0,\ \eta \geq 0. \tag{2.1}$$

We assume the viscosity coefficients μ and η are continuously differentiable functions of the absolute temperature, namely $\mu,\ \eta \in C^1[0,\infty)$ and satisfy

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \qquad \sup_{\vartheta\in[0,\infty)} |\mu'(\vartheta)| \leq \overline{m}, \tag{2.2}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta). \tag{2.3}$$

The Fourier law for the heat flux $\mathbf{q}$ has the following form:

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x\vartheta, \tag{2.4}$$

where the heat coefficient κ can be decompose into two parts

$$\kappa(\vartheta) = \kappa_M(\vartheta) + \kappa_R(\vartheta) \tag{2.5}$$

where $\kappa_M,\ \kappa_R \in C^1[0,\infty)$ and

$$0 < \underline{\kappa_R}(1+\vartheta^3) \leq \kappa_R(\vartheta) \leq \overline{\kappa_R}(1+\vartheta^3), \tag{2.6}$$

$$0 < \underline{\kappa_M}(1+\vartheta) \le \kappa_M(\vartheta) \le \overline{\kappa_M}(1+\vartheta). \tag{2.7}$$

In the above formulas $\underline{\mu}$, $\overline{\mu}$, $\overline{m}$, $\overline{\eta}$, $\underline{\kappa_R}$, $\overline{\kappa_R}$, $\underline{\kappa_M}$, $\overline{\kappa_M}$ are positive constants. Let us remark that the existence of solutions for the fixed domain can be obtain for more general μ and κ_M (see [10]), namely with upper and lower growth described by function $(1+\vartheta^\alpha)$ with $\alpha \in (\frac{2}{5}, 1]$ instead of $(1+\vartheta)$. We believe our result can be extended to this more general one, however to simplify our consideration we assume $\alpha = 1$.

The entropy production rate σ satisfies

$$\sigma \ge \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \vartheta\right). \tag{2.8}$$

The quantities p, e, and s are continuously differentiable functions for positive values of ϱ, ϑ and satisfy Gibbs' equation

$$\vartheta D s(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right) \text{ for all } \varrho,\ \vartheta > 0. \tag{2.9}$$

Further, we assume the following state equation for the pressure and the internal energy

$$p(\varrho,\vartheta) = p_M(\varrho,\vartheta) + p_R(\vartheta), \quad p_R(\vartheta) = \frac{a}{3}\vartheta^4,\ a > 0, \tag{2.10}$$

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta), \quad \varrho e_R(\varrho,\vartheta) = a\vartheta^4, \tag{2.11}$$

and

$$s(\varrho,\vartheta) = s_M(\varrho,\vartheta) + s_R(\varrho,\vartheta), \quad \varrho s_R(\varrho,\vartheta) = \frac{4}{3}a\vartheta^3. \tag{2.12}$$

According to the hypothesis of thermodynamic stability the molecular components satisfy

$$\frac{\partial p_M}{\partial \varrho} > 0 \text{ for all } \varrho,\ \vartheta > 0 \tag{2.13}$$

and

$$0 < \frac{\partial e_M}{\partial \vartheta} \le c \text{ for all } \varrho,\ \vartheta > 0. \tag{2.14}$$

Moreover

$$\lim_{\vartheta \to 0^+} e_M(\varrho,\vartheta) = \underline{e}_M(\varrho) > 0 \text{ for any fixed } \varrho > 0, \tag{2.15}$$

and

$$\left|\varrho \frac{\partial e_M(\varrho,\vartheta)}{\partial \varrho}\right| \le c e_M(\varrho,\vartheta) \text{ for all } \varrho,\ \vartheta > 0. \tag{2.16}$$

We suppose also that there is a function P satisfying

$$P \in C^1[0,\infty),\ P(0) = 0,\ P'(0) > 0, \tag{2.17}$$

and two positive constants $0 < \underline{Z} < \overline{Z}$ such that

$$p_M(\varrho,\vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) \text{ whenever } 0 < \varrho \le \underline{Z}\vartheta^{\frac{3}{2}},\ \text{or},\ \varrho > \overline{Z}\vartheta^{\frac{3}{2}} \tag{2.18}$$

and

$$p_M(\varrho,\vartheta) = \frac{2}{3}\varrho e_M(\varrho,\vartheta) \text{ for } \varrho > \overline{Z}\vartheta^{\frac{3}{2}}. \tag{2.19}$$

Finally, the problem (1.1)–(1.4) is supplemented by the initial conditions

$$\varrho(0,\cdot) = \varrho_0 \in L^{\frac{5}{3}}(\Omega_0), \quad \varrho_0 \geq 0, \quad \varrho_0 \not\equiv 0, \quad \varrho_0|_{\mathbb{R}^3\setminus\Omega_0} = 0, \tag{2.20}$$

$$(\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0, \ (\varrho\mathbf{u})_0 = 0 \text{ a.e. on the set } \{\Omega_0 \mid \varrho_0(x) = 0\}, \quad \int_{\Omega_0} \frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0}\,\mathrm{d}x < \infty, \tag{2.21}$$

$$\vartheta_0 > 0 \text{ a.e. in } \Omega_0, \qquad (\varrho s)_0 = \varrho_0 s(\varrho_0,\vartheta_0) \in L^1(\Omega_0), \tag{2.22}$$

$$E_0 = \int_{\Omega_0} \left(\frac{1}{2\varrho_0} |(\varrho\mathbf{u})_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) \right) dx < \infty. \tag{2.23}$$

2.2 Weak formulation, main result

In the weak formulation, equation (1.1) is supposed to be fulfilled in the sense of renormalized solutions introduced by DiPerna and Lions [4]:

$$\int_0^T \int_{\Omega_t} \varrho B(\varrho)(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\Omega_t} b(\varrho)\mathrm{div}_x\mathbf{u}\varphi\,\mathrm{d}x\mathrm{d}t - \int_{\Omega_0} \varrho_0 B(\varrho_0)\varphi(0)\,\mathrm{d}x \tag{2.24}$$

for any $\varphi \in C^1_c([0,T)\times\mathbb{R}^3)$, and any $b \in L^\infty \cap C[0,\infty)$ such that $b(0) = 0$ and $B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z$. Of course, we suppose that $\varrho \geq 0$ a.e. in $(0,T)\times\mathbb{R}^3$.

The momentum equation (1.2) is replaced by the following integral identity

$$\int_0^T \int_{\Omega_t} \left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p(\varrho,\vartheta)\mathrm{div}_x\boldsymbol{\varphi} - \mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi}\right)\mathrm{d}x\mathrm{d}t = -\int_{\Omega_0} (\varrho\mathbf{u})_0\cdot\boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x, \tag{2.25}$$

which should be fulfilled for any test function $\boldsymbol{\varphi} \in C^1_c(\overline{Q_T};\mathbb{R}^3)$ such that $\varphi(T,\cdot) = 0$ and

$$\boldsymbol{\varphi}\cdot\mathbf{n}|_{\Gamma_\tau} = 0 \text{ for any } \tau \in [0,T]. \tag{2.26}$$

The impermeability condition (1.7) is satisfied in the sense of traces, specifically,

$$\mathbf{u}, \nabla_x\mathbf{u} \in L^2(Q_T;\mathbb{R}^3) \text{ and } (\mathbf{u}-\mathbf{V})\cdot\mathbf{n}(\tau,\cdot)|_{\Gamma_\tau} = 0 \text{ for a.a. } \tau \in [0,T]. \tag{2.27}$$

The entropy inequality

$$\int_0^T \int_{\Omega_t} \varrho s(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\Omega_t} \frac{\kappa(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega_t} \frac{\varphi}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} + \frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right) \leq -\int_{\Omega_0} (\varrho s)_0\varphi(0)\,\mathrm{d}x \tag{2.28}$$

holds for all $\varphi \in C^1_c(\overline{Q_T})$ such that $\varphi(T,\cdot) = 0$ and $\varphi \geq 0$.

Finally, we assume the following energy inequality

$$\int_{\Omega_\tau} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right)(\tau,\cdot)\,\mathrm{d}x \leq \int_{\Omega_0} \left(\frac{1}{2}\frac{(\varrho\mathbf{u})_0^2}{\varrho_0} + \varrho_0 e_0 - (\varrho\mathbf{u})_0\cdot\mathbf{V}(0)\right)\mathrm{d}x$$
$$- \int_0^\tau \int_{\Omega_t} \left(\varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathbf{V} + p\,\mathrm{div}_x\mathbf{V} - \mathbb{S} : \nabla_x\mathbf{V} + \varrho\mathbf{u}\cdot\partial_t\mathbf{V}\right)\mathrm{d}x\mathrm{d}t + \int_{\Omega_t} \varrho\mathbf{u}\cdot\mathbf{V}(\tau,\cdot)\,\mathrm{d}x \tag{2.29}$$

holds for a.a. $\tau \in (0,T)$.

Definition 2.1. *We say that the trio* $(\varrho, \mathbf{u}, \vartheta)$ *is a* variational solution *of problem* (1.1)–(1.4) *with boundary conditions* (1.7)–(1.9) *and initial conditions* (2.20)–(2.23) *if*

- $\varrho \in L^\infty(0,T;L^{\frac{5}{3}}(\mathbb{R}^3))$, $\varrho \geq 0$, $\varrho \in L^q(Q_T)$ *for certain* $q > \frac{5}{3}$,
- $\mathbf{u}$, $\nabla \mathbf{u} \in L^2(Q_T)$, $\varrho \mathbf{u} \in L^\infty(0,T;L^1(\mathbb{R}^3))$,
- $\vartheta > 0$ *a.a. on* Q_T, $\vartheta \in L^\infty((0,T);L^4(\mathbb{R}^3))$, ϑ, $\nabla \vartheta \in L^2(Q_T)$, *and* $\log \vartheta$, $\nabla \log \vartheta \in L^2(Q_T)$,
- ϱs, $\varrho s \mathbf{u}$, $\frac{\mathbf{q}}{\vartheta} \in L^1(Q_T)$,
- *relations* (2.24)–(2.29) *are satisfied.*

Remark 2.1. *In contrast to [10] we consider energy inequality rather than energy equation. Although it seems that we are losing a lot of information our definition of weak solution is still sufficient. Namely, if the above defined weak solution is smooth enough it will be a strong one. For a justification of this fact we refer to [27, Section 1.2] .*

At this stage, we are ready to state the main result of the present paper:

Theorem 2.1. *Let* $\Omega_0 \subset \mathbb{R}^3$ *be a bounded domain of class* $C^{2+\nu}$ *with some* $\nu > 0$, *and let* $\mathbf{V} \in C^1([0,T];C^3_c(\mathbb{R}^3;\mathbb{R}^3))$ *be given and satisfy* (1.5), (1.6). *Assume that hypothesis* (2.1)–(2.19) *are satisfied.*

Then the problem (1.1)–(1.4) *with boundary conditions* (1.8), (1.9) *and initial conditions* (2.20)–(2.23) *admits a variational solution in the sense of Definition* 2.1 *on any finite time interval* $(0,T)$.

The rest of the paper is devoted to the proof of Theorem 2.1.

3 Approximate problem

3.1 Penalized problem - weak formulation

Choosing $R > 0$ such that

$$\mathbf{V}|_{[0,T]\times\{|x|>R\}} = 0, \quad \overline{\Omega}_0 \subset \{|x| < R\}$$

we take the reference domain $B = \{|x| < 2R\}$.

The shear viscosity coefficients μ_ω and η_ω are taken such that

$$\mu_\omega(\vartheta,\cdot) \in C^\infty_c([0,T]\times\mathbb{R}^3),\ 0 < \omega\mu(\vartheta) \leq \mu_\omega(\vartheta,t,x) \leq \mu(\vartheta) \text{ in } [0,T]\times B,\ \mu_\omega(\vartheta,\tau,\cdot)|_{\Omega_\tau} = \mu(\vartheta) \text{ for any } \tau \in [0,T] \tag{3.1}$$

$$\eta_\omega(\vartheta,\cdot) \in C^\infty_c([0,T]\times\mathbb{R}^3),\ 0 < \omega\eta(\vartheta) \leq \eta_\omega(\vartheta,t,x) \leq \eta(\vartheta) \text{ in } [0,T]\times B,\ \eta_\omega(\vartheta,\tau,\cdot)|_{\Omega_\tau} = \eta(\vartheta) \text{ for any } \tau \in [0,T] \tag{3.2}$$

and

$$\mu_\omega, \eta_\omega \to 0 \text{ a.e. in } ((0,T)\times B)\setminus Q_T \qquad \text{as } \omega \to 0. \tag{3.3}$$

We introduce a variable heat conductivity coefficient as follows:

$$\kappa_\nu(\vartheta,t,x) = \chi_\nu(t,x)\kappa(\vartheta), \text{ where } \chi_\nu = 1 \text{ in } Q_T \text{ and } \chi_\nu = \nu \text{ in } ((0,T)\times B)\setminus Q_T. \tag{3.4}$$

Similarly we introduce a variable coefficient $a := a_\eta(t,x)$ which represents the radiative part of pressure, internal energy and entropy (see (2.10), (2.11), and (2.12)). Namely, we assume that

$$a_\eta(t,x) = \chi_\eta(t,x)a, \text{ where } a > 0 \text{ and } \chi_\eta = 1 \text{ in } Q_T \text{ and } \chi_\eta = \eta \text{ in } ((0,T)\times B\setminus Q_T. \tag{3.5}$$

We use a letter a for both coefficient and constant. Anyway, it is always clear which meaning is considered.

Moreover being motivated by the approximation for the existence theory we follow [10] and set:

$$p_{\eta,\delta}(\varrho,\vartheta) = p_M(\varrho,\vartheta) + \frac{a_\eta}{3}\vartheta^4 + \delta\varrho^\beta,\ \beta \geq 4,\ \delta > 0, \tag{3.6}$$

$$e_\eta(\varrho,\vartheta) = e_M(\varrho,\vartheta) + a_\eta\frac{\vartheta^4}{\varrho}, \qquad s_\eta(\varrho,\vartheta) = s_M(\varrho,\vartheta) + \frac{4}{3}a_\eta\frac{\vartheta^3}{\varrho}. \tag{3.7}$$

Finally, let ϱ_0, $(\varrho\mathbf{u})_0$ and ϑ_0 be initial conditions as specified in Theorem 2.1. We define modified initial data $\varrho_{0,\delta}$, $(\varrho\mathbf{u})_{0,\delta}$ and $\vartheta_{0,\delta}$ so that

$$\varrho_{0,\delta} \geq 0,\ \varrho_{0,\delta} \not\equiv 0,\ \varrho_{0,\delta}|_{\mathbb{R}^3\setminus\Omega_0} = 0,\ \int_B \left(\varrho_{0,\delta}^{\frac{5}{3}} + \delta\varrho_{0,\delta}^\beta\right)\,\mathrm{d}x \leq c,\ \varrho_{0,\delta} \to \varrho_0 \text{ in } L^{\frac{5}{3}}(B),\ |\{\varrho_{0,\delta} < \varrho_0\}| \to 0, \tag{3.8}$$

$$(\varrho\mathbf{u})_{0,\delta} = \begin{cases} (\varrho\mathbf{u})_0 & \text{if } \varrho_{0,\delta} \geq \varrho_0, \\ 0 & \text{otherwise}, \end{cases} \tag{3.9}$$

$$0 < \underline{\vartheta} \leq \vartheta_{0,\delta} \text{ and } \vartheta_{0,\delta} \in L^\infty(B) \cap C^{2+\nu}(B). \tag{3.10}$$

Moreover

$$\int_{\Omega_0} \varrho_{0,\delta} e(\varrho_{0,\delta},\vartheta_{0,\delta})\,\mathrm{d}x \to \int_{\Omega_0} \varrho_0 e(\varrho_0,\vartheta_0)\,\mathrm{d}x$$

and

$$\varrho_{0,\delta} s(\varrho_{0,\delta},\vartheta_{0,\delta}) \to \varrho_0 s(\varrho_0,\vartheta_0) \text{ weakly in } L^1(\Omega_0).$$

Now we are ready to state the weak formulation of the *penalized problem.*

Again, we consider $\varrho, \mathbf{u}$ to be zero outside of $(0,T) \times B$. The weak (renormalized) formulation of the continuity equation reads as

$$\int_0^T \int_B \varrho B(\varrho)(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t = \int_0^T \int_B b(\varrho)\mathrm{div}_x\mathbf{u}\varphi\,\mathrm{d}x\mathrm{d}t - \int_B \varrho_{0,\delta}B(\varrho_{0,\delta})\varphi(0)\,\mathrm{d}x \tag{3.11}$$

for any $\varphi \in C^1_c([0,T)\times\mathbb{R}^3)$, and any $b \in L^\infty \cap C[0,\infty)$ such that $b(0) = 0$ and $B(\varrho) = B(1) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z$. The momentum equation is represented by the family of integral identities

$$\int_0^T \int_B \left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p_{\eta,\delta}(\varrho,\vartheta)\mathrm{div}_x\boldsymbol{\varphi} - \mathbb{S}_\omega : \nabla_x\boldsymbol{\varphi}\right)\,\mathrm{d}x\mathrm{d}t - \frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} ((\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\ \boldsymbol{\varphi}\cdot\mathbf{n})\,\mathrm{dS}_x\,\mathrm{d}t \tag{3.12}$$

$$= -\int_B (\varrho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x$$

$$\text{with} \quad \mathbb{S}_\omega = \mu_\omega(\vartheta,t,x)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta_\omega(\vartheta,t,x)\mathrm{div}_x\mathbf{u}\mathbb{I}$$

for any test function $\boldsymbol{\varphi} \in C_c^\infty([0,T)\times B;\mathbb{R}^3)$.

$$\int_0^T \int_B \varrho s_\eta(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_B \frac{\kappa_\nu(\vartheta,t,x)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t \tag{3.13}$$

$$+\int_0^T\int_B \frac{\varphi}{\vartheta}\left(\mathbb{S}_\omega:\nabla_x\mathbf{u}+\frac{\kappa_\nu(\vartheta,t,x)|\nabla_x\vartheta|^2}{\vartheta}\right)\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_B \lambda\vartheta^4\,\mathrm{d}x\mathrm{d}t\le -\int_B(\varrho s)_{0,\delta,\eta}\varphi(0)\,\mathrm{d}x$$

for all $\varphi\in C_c^1([0,T)\times\overline{B})$, $\varphi\ge 0$, where $(\varrho s)_{0,\delta,\eta}:=\varrho_{0,\delta}s_\eta(\varrho_{0,\delta},\vartheta_{0,\delta})$ and

$$\sigma_{\omega,\nu}\ge\frac{1}{\vartheta}\left(\mathbb{S}_\omega:\nabla_x\mathbf{u}+\frac{\kappa_\nu(\vartheta,t,x)|\nabla_x\vartheta|^2}{\vartheta}\right).$$

The solution to penalized problem should satisfy the energy equation

$$\int_0^T\int_B\left(\frac12\varrho|\mathbf{u}|^2+\varrho e_\eta+\frac{\delta}{\beta-1}\varrho^\beta\right)\partial_t\psi-\lambda\vartheta^5\psi\,\mathrm{d}x\mathrm{d}t=\frac{1}{\varepsilon}\int_0^T\int_{\Gamma_t}(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\;\mathbf{u}\cdot\mathbf{n}\psi\,\mathrm{dS}_x\;\mathrm{d}t$$
$$-\int_B\left(\frac12\frac{(\varrho\mathbf{u})_{0,\delta}^2}{\varrho_{0,\delta}}+\varrho_{0,\delta}e_{0,\delta,\eta}+\frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta\right)\psi(0)\,\mathrm{d}x \quad (3.14)$$

for all $\psi\in C_c^1([0,T))$. Here we denoted $e_{0,\delta,\eta}:=e_\eta(\varrho_{0,\delta},\vartheta_{0,\delta})$. However, we will rather work with the following modification which can be obtained from (3.12) and (3.14):

$$\int_0^T\int_B\left(\frac12\varrho|\mathbf{u}|^2+\varrho e_\eta+\frac{\delta}{\beta-1}\varrho^\beta\right)\partial_t\psi-\lambda\vartheta^5\psi\,\mathrm{d}x\mathrm{d}t-\frac1\varepsilon\int_0^T\int_{\Gamma_t}|(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2\,\psi\,\mathrm{dS}_x\,\mathrm{d}t$$
$$=-\int_B\left(\frac12\frac{(\varrho\mathbf{u})_{0,\delta}^2}{\varrho_{0,\delta}}+\varrho_{0,\delta}e_{0,\delta,\eta}+\frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta-(\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0)\right)\psi(0)\,\mathrm{d}x \quad (3.15)$$
$$-\int_0^T\int_B(\mathbb{S}_\omega:\nabla_x\mathbf{V}\psi-\varrho\mathbf{u}\cdot\partial_t(\mathbf{V}\psi)-\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\mathbf{V}\psi-p_{\eta,\delta}\mathrm{div}_x\mathbf{V}\psi)\,\mathrm{d}x\mathrm{d}t$$

for all $\psi\in C_c^1([0,T))$.

Definition 3.1. *Let ε, η, ω, ν, λ, and δ be positive parameters and let $\beta>4$. We say that a trio $(\varrho,\mathbf{u},\vartheta)$ is a variational solution to the penalized problem with initial data* (3.8)–(3.10) *if*

- $\varrho\in L^\infty((0,T);L^{\frac53}(\mathbb{R}^3))\cap L^\infty((0,T);L^\beta(\mathbb{R}^3))$, $\varrho\ge 0$, $\varrho\in L^q((0,T)\times B)$ *for certain* $q>\beta$,
- $\mathbf{u}$, $\nabla\mathbf{u}\in L^2((0,T)\times B)$, $\varrho\mathbf{u}\in L^\infty(0,T;L^1(\mathbb{R}^3))$,
- $\vartheta>0$ *a.a. on* $(0,T)\times B$, $\vartheta\in L^\infty((0,T);L^4(B))$, ϑ, $\nabla\vartheta\in L^2((0,T)\times B)$, *and* $\log\vartheta$, $\nabla\log\vartheta\in L^2((0,T)\times B)$,
- ϱs, $\varrho s\mathbf{u}$, $\frac{\mathbf{q}}{\vartheta}\in L^1((0,T)\times B)$,
- *relations* (3.11)–(3.13), (3.15) *are satisfied.*

The choice of the no-slip boundary condition $\mathbf{u}|_{\partial B}=0$ is not essential here as the proof follows the one presented in [10]. We have the following existence theorem concerning weak solutions to the penalized problem.

Theorem 3.1. *Let $\mathbf{V}\in C^1([0,T];C_c^3(\mathbb{R}^3;\mathbb{R}^3))$ be given. Let $\beta>4$. Let thermodynamical functions and coefficients satisfy* (3.1)–(3.7) *with assumptions on p, e, s, μ, η, κ as in Theorem* 2.1. *Let initial data satisfy* (3.8)–(3.10). *Finally, let ε, η, ω, ν, λ, $\delta>0$.*

Then the penalized problem admits a variational solution on any time interval $(0,T)$ in the sense specified by Definition 3.1.

Proof. As mentioned above, the proof itself does not differ from [10, Section 3] so we present just a short description. It is necessary to regularize the continuity equation by a viscous term $\Delta\varrho$ and to add appropriate terms to the momentum and energy equation. The starting point of considerations at the level of Galerkin approximations is then the internal energy equation [10, Equation (3.55)] instead of the entropy balance. However we need to accommodate two additional difficulties, namely

1. The term $\frac{1}{\varepsilon}\int_0^\tau\int_{\Gamma_t}((\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\,\boldsymbol{\varphi}\cdot\mathbf{n})\,\mathrm{dS}_x\,\mathrm{d}t$ in (3.12) and $\frac{1}{\varepsilon}\int_0^T\int_{\Gamma_t}(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\;\mathbf{u}\cdot\mathbf{n}\psi \mathrm{d}S_x\;\mathrm{d}t$ in (3.14). These terms do not cause much trouble as each can be treated as a "compact" perturbation.

2. The jumps in functions $\kappa_\nu(\vartheta,t,x)$ and $a_\eta(t,x)$. To this end we first introduce mollifications of the jump function $\chi_A(t,x)$ where stands for ν or η (see (3.4), (3.5)). By $\chi_A^\alpha(t,x)$ we denote smooth function with value 1 on Q_T and with value A on $B\setminus Q_T^\alpha$, where Q_T^α denotes the α-neighborhood of Q_T in spacetime. The terms related to the parameter η are treated in the straightforward way. The terms related to the parameter ν requires a little more attention.

   Since we want the leading term in the internal energy equation [10, (3.55)] to be Laplacian, we need to add to this equation a term of the form $\mathrm{div}_x(\mathcal{K}(\vartheta)\nabla_x\chi_\nu^\alpha)$, where $\mathcal{K}(\vartheta)=\int_1^\vartheta\kappa(z)dz$. This way we obtain

$$\begin{aligned}\partial_t(\varrho e_\eta(\varrho,\vartheta))+\mathrm{div}_x(\varrho e_\eta(\varrho,\vartheta)\mathbf{u})-\mathrm{div}_x\nabla_x(\chi_\nu^\alpha(t,x)\mathcal{K}(\vartheta))\\ =\mathbb{S}_\omega:\nabla_x\mathbf{u}-p_{\eta,\delta}(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}-\mathrm{div}_x(\mathcal{K}(\vartheta)\nabla_x\chi_\nu^\alpha(t,x))-\lambda\vartheta^5.\end{aligned}\tag{3.16}$$

   We can then follow the theory in [10, Section 3.4.2] to deduce existence of strong solutions to equation (3.16) and when passing to the limit in the Galerkin approximations switch to the entropy balance and global total energy balance. Then we pass to the limit with the artificial viscosity parameter as in [10, Section 3].

   As a final step in the proof of Theorem 3.1 we need to pass with α to zero. Note that

$$\chi_A^\alpha\to\chi_A\qquad\text{strongly in } L^p(B)\text{ for any } p<\infty.\tag{3.17}$$

   The limit passage with α is straightforward in the terms related to parameter η, in particular due to the presence of term $\lambda\vartheta^5$ providing uniform bounds of high power of the temperature on B.

   Concerning the terms related to ν, there are two of them in the entropy balance (3.13), which we need to take care of. First, there is the nonnegative term $\int_0^T\int_B\chi_\nu^\alpha\kappa(\vartheta)\left|\nabla_x\vartheta\right|^2\vartheta^{-2}\varphi\,\mathrm{d}x\mathrm{d}t$. Due to the nonnegativity of the functions appearing in this term we can use the inequality $\chi_\nu\le\chi_\nu^\alpha$ and weak lower semicontinuity to pass to the limit in the same way as we later explain in (4.25).

   Since we have at hand the same apriori estimates as we work with in the first limit passage in the proof of Theorem 2.1, we know in particular (3.49) which together with (3.17) is enough to pass to the limit with α in the term $\int_0^T\int_B\chi_\nu^\alpha\kappa(\vartheta)\vartheta^{-1}\nabla_x\vartheta\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t$.

□

3.2 Modified energy inequality and uniform bounds

We use $\varphi(t,x)\equiv 1\cdot\psi_\xi(t)$ where $\psi_\xi\in C_c^1([0,T))$ is non-increasing function which fulfills

$$\psi_\xi(t)=\left\{\begin{array}{l}1\text{ for } t<\tau-\xi\\ 0\text{ for } t\ge\tau\end{array}\right.\qquad\text{for some }\tau\in(0,T)\text{ and arbitrary }\xi>0\tag{3.18}$$

as a test function in (3.13) in order to derive (after passing with $\xi \to 0$)

$$-\int_B \varrho s_\eta(\varrho,\vartheta)(\tau,\cdot)\,\mathrm{d}x + \int_0^\tau \int_B \frac{1}{\vartheta}\left(\mathbb{S}_\omega : \nabla_x \mathbf{u} + \frac{\kappa_\nu(\vartheta)|\nabla_x \vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_B \lambda \vartheta^4 \,\mathrm{d}x\mathrm{d}t \leq -\int_B (\varrho s)_{0,\delta,\eta}\,\mathrm{d}x.$$

The same test function applied in (3.15) implies

$$\begin{aligned}\int_B \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e_\eta + \frac{\delta}{\beta-1}\varrho^\beta\right)(\tau,\cdot)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} |(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2 \,\mathrm{d}S_x \mathrm{d}t + \int_0^\tau \int_B \lambda\vartheta^5 \,\mathrm{d}x\mathrm{d}t \\ = \int_0^\tau \int_B (\mathbb{S}_\omega : \nabla_x \mathbf{V} - \varrho \mathbf{u}\cdot \mathbf{V}_t - \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x \mathbf{V} - p_{\eta,\delta}(\varrho,\vartheta)\mathrm{div}_x \mathbf{V})\,\mathrm{d}x\mathrm{d}t \\ + \int_B \varrho\mathbf{u}\cdot\mathbf{V}(\tau,\cdot)\,\mathrm{d}x + \int_B \left(\frac{1}{2}\frac{(\varrho\mathbf{u})^2_{0,\delta}}{\varrho_{0,\delta}} + \varrho_{0,\delta}e_{0,\delta,\eta} + \frac{\delta}{\beta-1}\varrho^\beta_{0,\delta} - (\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0)\right)\mathrm{d}x\end{aligned}$$

for almost all $\tau \in (0,T)$ and we deduce that

$$\begin{aligned}&\int_B \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{1,\eta}(\varrho,\vartheta) + \frac{\delta}{\beta-1}\varrho^\beta\right)(\tau,\cdot)\,\mathrm{d}x \\ &+ \frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} |(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2\,\mathrm{d}S_x\mathrm{d}t + \int_0^\tau \int_B \frac{1}{\vartheta}\left(\mathbb{S}_\omega : \nabla_x\mathbf{u} + \frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_B \lambda\vartheta^5\,\mathrm{d}x\mathrm{d}t \\ &\leq \int_0^\tau \int_B \left(\mathbb{S}_\omega : \nabla_x\mathbf{V} - \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathbf{V} - \varrho\mathbf{u}\cdot\mathbf{V}_t - p_{\eta,\delta}(\varrho,\vartheta)\mathrm{div}_x\mathbf{V} + \lambda\vartheta^4\right)\mathrm{d}x\mathrm{d}t \\ &+ \int_B \varrho\mathbf{u}\cdot\mathbf{V}(\tau,\cdot)\,\mathrm{d}x + \int_B \left(\frac{1}{2}\frac{(\varrho\mathbf{u})^2_{0,\delta}}{\varrho_{0,\delta}} + H_{1,\eta}(\varrho_{0,\delta},\vartheta_{0,\delta}) + \frac{\delta}{\beta-1}\varrho^\beta_{0,\delta} - (\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0,\cdot)\right)\mathrm{d}x,\end{aligned} \tag{3.19}$$

for almost all $\tau \in (0,T)$, where

$$H_{1,\eta}(\varrho,\vartheta) = \varrho(e_\eta(\varrho,\vartheta) - s_\eta(\varrho,\vartheta))$$

(see [10, Chapter 2.2.3]) is a Helmholtz function.

Since the vector field $\mathbf{V}$ is regular suitable manipulations with the Hölder, Young and Poincar—' e inequalities and thermodynamical hypothesis yield[1]

$$\begin{aligned}\int_B \varrho\mathbf{u}\cdot\mathbf{V}(\tau,\cdot)\,\mathrm{d}x \leq c(\mathbf{V})\int_B \sqrt{\varrho}\sqrt{\varrho}|\mathbf{u}|(\tau,\cdot)\,\mathrm{d}x \leq c + \frac{1}{4}\int_B \varrho|\mathbf{u}|^2(\tau,\cdot)\,\mathrm{d}x, \\ \int_0^\tau \int_B \mathbb{S}_\omega : \nabla_x\mathbf{V}\,\mathrm{d}x\mathrm{d}t \leq c(\mathbf{V},\lambda) + \int_0^\tau \int_B \frac{1}{6}\lambda\vartheta^5\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^\tau \int_B \frac{1}{\vartheta}\mathbb{S}_\omega : \nabla_x\mathbf{u}\,\mathrm{d}x\mathrm{d}t, \\ \left|\int_0^\tau \int_B \varrho\mathbf{u}\cdot\partial_t\mathbf{V}\,\mathrm{d}x\mathrm{d}t\right| \leq c\int_0^\tau \int_B \sqrt{\varrho}\sqrt{\varrho}|\mathbf{u}|\,\mathrm{d}x\mathrm{d}t \leq c(\mathbf{V},\varrho_0) + c\int_0^\tau \int_B \varrho|\mathbf{u}|^2\,\mathrm{d}x\mathrm{d}t, \\ \left|\int_0^\tau \int_B \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathbf{V}\,\mathrm{d}x\mathrm{d}t\right| \leq c\int_0^\tau \int_B \varrho|\mathbf{u}|^2\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{3.20}$$

[1]Hereinafter, c is a constant which is independent of solution. It depends on data, right hand side and it may vary from line to line. It may also depend on penalization parameters, however we emphasise this particular dependence as it play a role in further computations.

In order to deal with pressure term let us notice that

$$P'(Z) > 0 \quad \text{ for all } Z \geq 0. \tag{3.21}$$

Indeed, (2.13) provides that $P'(Z) > 0$ if $0 < Z < \underline{Z}$ or $Z > \overline{Z}$. This together with (2.17) gives (3.21). Next by (2.14), (2.18), (2.19) one can infer that

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{3.22}$$

Therefore by (2.17), (2.18), (2.19) we obtain the following bound on the molecular pressure p_M

$$\underline{c}\varrho\vartheta \leq p_M \leq \overline{c}\varrho\vartheta \quad \text{ if } \varrho < \overline{Z}\vartheta^{\frac{3}{2}},$$

$$\underline{c}\varrho^{\frac{5}{3}} \leq p_M \leq \overline{c} \begin{cases} \vartheta^{\frac{5}{2}} & \text{ if } \varrho < \overline{Z}\vartheta^{\frac{3}{2}} \\ \varrho^{\frac{5}{3}} & \text{ if } \varrho > \overline{Z}\vartheta^{\frac{3}{2}}, \end{cases} \tag{3.23}$$

where we use also monotonicity of p_M in ϱ to control it on the set $\underline{Z}\vartheta^{\frac{3}{2}} \leq \varrho \leq \overline{Z}\vartheta^{\frac{3}{2}}$. With the above informations at hand we deduce

$$\begin{aligned} \left| \int_0^\tau \int_B p_{\eta,\delta}(\varrho,\vartheta)\mathrm{div}_x \mathbf{V} \,\mathrm{d}x\mathrm{d}t \right| \leq c(\mathbf{V},\lambda) + c(\mathbf{V}) \int_0^\tau \int_B a_\eta \vartheta^4 \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_B \frac{1}{6}\lambda\vartheta^5 \,\mathrm{d}x\mathrm{d}t \\ + c(\mathbf{V}) \int_0^\tau \int_B \varrho^{\frac{5}{3}} \,\mathrm{d}x\mathrm{d}t + c(\mathbf{V}) \int_0^\tau \int_B \frac{\delta}{\beta - 1}\varrho^\beta \,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.24}$$

Finally, as we may assume that $\lambda \leq 1$, we deduce by the Young inequality

$$\int_0^\tau \int_B \lambda\vartheta^4 \,\mathrm{d}x\mathrm{d}t \leq \lambda^{1/5} \int_0^\tau \int_B \lambda^{4/5}\vartheta^4 \,\mathrm{d}x\mathrm{d}t \leq c + \frac{1}{6} \int_0^\tau \int_B \lambda\vartheta^5 \,\mathrm{d}x\mathrm{d}t.$$

Moreover let us recall that

$$\varrho e_\eta = \varrho e_M + a_\eta \vartheta^4$$

and one can prove that

$$\varrho e_\eta \geq a_\eta\vartheta^4 + \frac{3p_\infty}{2}\varrho^{\frac{5}{3}}. \tag{3.25}$$

Indeed, by (2.18), (2.19), and (3.22)

$$\lim_{\vartheta\to 0^+} e_M(\varrho,\vartheta) = \frac{3}{2}\varrho^{\frac{2}{3}} p_\infty. \tag{3.26}$$

By (2.14) e_M is strictly increasing function of ϑ on $(0,\infty)$ for any fixed ϱ. This together with (2.11) and (3.26) justify that

$$\varrho e_\eta(\varrho,\vartheta) \geq \frac{3p_\infty}{2}\varrho^{\frac{5}{3}} + a_\eta\vartheta^4 \tag{3.27}$$

and consequently (3.25) holds. Furthermore, see (2.15), $e_M(\varrho,\vartheta) = \underline{e}_M(\varrho) + \int_0^\vartheta \frac{\partial e_M}{\partial\vartheta}(\vartheta,\tau)\,\mathrm{d}\tau$. This together with (2.14) and (3.26) provides

$$0 \leq e_M(\varrho,\vartheta) \leq c(\varrho^{\frac{2}{3}} + \vartheta). \tag{3.28}$$

Finally, summarising all above considerations by Gronwall inequality we obtain the following

$$
\begin{aligned}
&\int_B \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{1,\eta}(\varrho,\vartheta) + \frac{\delta}{\beta-1}\varrho^\beta\right)(\tau,\cdot)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^\tau\int_{\Gamma_t}|(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2\,\mathrm{d}S_x\mathrm{d}t\\
&+\int_0^\tau\int_B\frac{1}{\vartheta}\left(\frac{1}{2}\mathbb{S}_\omega:\nabla_x\mathbf{u} + \frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_B\frac{1}{2}\lambda\vartheta^5\,\mathrm{d}x\mathrm{d}t\\
&\leq c(\mathbf{V},\lambda)\int_B\left(\frac{1}{2}\frac{(\varrho\mathbf{u})_{0,\delta}^2}{\varrho_{0,\delta}} + H_{1,\eta}(\varrho_{0,\delta},\vartheta_{0,\delta}) + \frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta - (\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0,\cdot)+1\right)\mathrm{d}x.
\end{aligned}
\tag{3.29}
$$

Let us notice that the continuity equations provides that

$$
\int_B\varrho(\tau) = \int_B\varrho(0) \text{ for all } t\in(0,T). \tag{3.30}
$$

Setting $\bar{\varrho}$ constant such that $\int_B(\varrho-\bar{\varrho})\,\mathrm{d}x = 0$ for a.a. $\tau\in[0,T)$ we may rewrite (3.29) as the following total dissipation inequality

$$
\begin{aligned}
&\int_B\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{1,\eta}(\varrho,\vartheta) - (\varrho-\bar{\varrho})\frac{\partial H_{1,\eta}(\bar{\varrho},1)}{\partial\varrho} - H_{1,\eta}(\bar{\varrho},1) + \frac{\delta}{\beta-1}\varrho^\beta\right)(\tau,\cdot)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_0^\tau\int_{\Gamma_t}|(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}|^2\,\mathrm{d}S_x\mathrm{d}t\\
&\qquad+\int_0^\tau\int_B\frac{1}{\vartheta}\left(\frac{1}{2}\mathbb{S}_\omega:\nabla_x\mathbf{u} + \frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_B\frac{1}{2}\lambda\vartheta^5\,\mathrm{d}x\mathrm{d}t\\
&\leq c(\mathbf{V},\lambda)\int_B\left(\frac{1}{2}\frac{(\varrho\mathbf{u})_{0,\delta}^2}{\varrho_{0,\delta}} + H_{1,\eta}(\varrho_{0,\delta},\vartheta_{0,\delta}) + \frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta - (\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0,\cdot)+1\right)\mathrm{d}x\\
&\qquad\qquad - \int_B\left((\varrho_{0,\delta}-\bar{\varrho})\frac{\partial H_{1,\eta}(\bar{\varrho},1)}{\partial\varrho} - H_{1,\eta}(\bar{\varrho},1)\right)\mathrm{d}x.
\end{aligned}
\tag{3.31}
$$

In such form the left hand side of (3.31) is nonnegative due to the hypothesis of thermodynamic stability (2.13), (2.14).

Directly from (3.31) we obtain that

$$
\int_0^T\int_{\Gamma_t}|(\mathbf{u}-\mathbf{V})\mathbf{n}|^2\,\mathrm{dS}_x\,\mathrm{d}t \leq \varepsilon c(\lambda), \tag{3.32}
$$

$$
\operatorname*{ess\,sup}_{\tau\in(0,T)}\|\delta\varrho^\beta(\tau,\cdot)\|_{L^1(B)} \leq c(\lambda), \tag{3.33}
$$

$$
\operatorname*{ess\,sup}_{\tau\in(0,T)}\|\sqrt{\varrho}\mathbf{u}(\tau,\cdot)\|_{L^2(B)} \leq c(\lambda), \tag{3.34}
$$

$$
\left\|\lambda\vartheta^5\right\|_{L^1((0,T)\times B)} \leq c(\lambda). \tag{3.35}
$$

By (3.31) we get also

$$
\int_0^T\int_B\omega|\nabla_x\mathbf{u}|^2\,\mathrm{d}x\mathrm{d}t \leq c(\lambda)
$$

and then (2.1), (2.2), the generalized Korn-Poincaré inequality ([10, Proposition 2.1]) and (1.5) give rise to

$$
\|\mathbf{u}\|_{L^2((0,T)\times B)} + \|\nabla_x\mathbf{u}\|_{L^2((0,T)\times B)} \leq c(\lambda,\omega). \tag{3.36}
$$

Next by (3.31), (2.6), (2.7) we get

$$\int_0^T \int_B \chi_\nu \left(|\nabla_x \log(\vartheta)|^2 + |\nabla_x \vartheta^{\frac{3}{2}}|^2 \right) \,\mathrm{d}x\mathrm{d}t \le c(\lambda). \tag{3.37}$$

Since H_1 is coercive and bounded from below by [10, Proposition 3.2] we get the following

$$\operatorname*{ess\ sup}_{\tau\in(0,T)} \|\varrho e_\eta\|_{L^1(B)} \le c(\lambda), \tag{3.38}$$

and consequently

$$\operatorname*{ess\ sup}_{\tau\in(0,T)} \|a_\eta \vartheta^4(\tau,\cdot)\|_{L^1(B)} \le c(\lambda), \tag{3.39}$$

$$\operatorname*{ess\ sup}_{\tau\in(0,T)} \|\varrho(\tau,\cdot)\|_{L^{\frac{5}{3}}(B)} \le c(\lambda). \tag{3.40}$$

Then by (3.37), (3.39), and by Poincaré inequality (see [10, Proposition 2.2])

$$\|\vartheta^\gamma\|_{L^2(0,T;W^{1,2}(B))} \le c(\lambda,\nu) \quad \text{for any } 1 \le \gamma \le \frac{3}{2}. \tag{3.41}$$

Moreover by (3.37), (3.39), (3.40)

$$\|\log\vartheta\|_{L^2(0,T;W^{1,2}(B))} \le c(\lambda,\nu) \tag{3.42}$$

(for more details see [10, Chapter 2.2.4]). From (3.40) and (3.34) we deduce

$$\|\varrho\mathbf{u}\|_{L^\infty(0,T;L^{\frac{5}{4}}(B))} \le c(\lambda). \tag{3.43}$$

Moreover, we use the technique based on the Bogovskii operator in order to derive the existence of $\pi > 0$ fulfilling

$$\int\int_K \left(p(\varrho,\vartheta)\varrho^\pi + \delta\varrho^{\beta+\pi} \right) \,\mathrm{d}x\mathrm{d}t \le c(K), \tag{3.44}$$

for any compact $K \subset (0,T)\times B$ such that

$$K \cap \left(\cup_{\tau\in[0,T]} \left(\{\tau\} \times \Gamma_\tau \right) \right) = \emptyset. \tag{3.45}$$

It is worth pointing out that π in (3.44) can be chosen independently of ε, ν, ω, η, λ and δ. For details we refer reader to [6, Section 4.2] or [13].

By hypothesis (2.13)–(2.19) and Gibbs' relation one can deduce that

$$|s_M(\varrho,\vartheta)| \le c(1 + |\log\varrho| + |\log\vartheta|) \quad \text{for all } \varrho,\ \vartheta > 0 \text{ and some } c > 0,$$

see [10, Section 3.2] for details. Therefore, there exists $c > 0$ such that

$$\varrho s_\eta(\varrho,\vartheta) \le c(\varrho + \varrho|\log\varrho| + \varrho|\log\vartheta| + a_\eta\vartheta^3). \tag{3.46}$$

Relation (3.46) together with (3.35), (3.40), (3.42) give rise to

$$\|\varrho s_\eta(\varrho,\vartheta)\|_{L^p((0,T)\times B)} \le c(\lambda,\nu) \quad \text{with some } p > 1. \tag{3.47}$$

Moreover, the above combined with (3.43), (3.36) and the Sobolev embedding theorem yields

$$\|\varrho s_\eta(\varrho,\vartheta)\mathbf{u}\|_{L^q((0,T)\times B)} \le c(\lambda,\nu,\omega) \quad \text{with some } q>1. \tag{3.48}$$

By (3.35), (3.41) and by observation that (due to (2.6), (2.7))

$$\frac{\kappa_\nu(\vartheta)}{\vartheta}|\nabla_x\vartheta| \le c\chi_\nu\left(|\nabla_x\log(\vartheta)| + \vartheta^{\frac{3}{2}}|\nabla_x\vartheta^{\frac{3}{2}}|\right),$$

we infer that

$$\left\|\frac{\kappa_\nu(\vartheta)}{\vartheta}\nabla_x\vartheta\right\|_{L^r((0,T)\times B)} \le c(\lambda,\nu) \quad \text{with certain } r>1. \tag{3.49}$$

Next, according to (3.28), (3.40), (3.35) we have that

$$\|\varrho e_\eta(\varrho,\vartheta)\|_{L^1((0,T)\times B)} \le c(\lambda) \tag{3.50}$$

and due to (3.44) higher integrability independent of δ can be obtained only away of Γ_τ interface, namely on sets as in (3.45).

4 Singular limits

In this section, we perform successively the singular limits $\varepsilon\to 0$, $\eta\to 0$, $\omega\to 0$, $\nu\to 0$, $\lambda\to 0$ and $\delta\to 0$.

4.1 Penalization limit. Passing with $\varepsilon\to 0$

4.1.1 Direct consequences of uniform bounds

Firstly, we proceed with $\varepsilon\to 0$ in (3.11), (3.12), (3.13), (3.14) and (3.15) while other parameters ν, ω, η, λ and δ remain fixed. Let $\{\varrho_\varepsilon,\mathbf{u}_\varepsilon,\vartheta_\varepsilon\}_{\varepsilon>0}$ be the corresponding sequence of weak solutions of the penalized problem given by Theorem 3.1.

First of all, directly from (3.32), we derive that

$$(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}(\tau,\cdot)|_{\Gamma_\tau} = 0 \text{ for a.a. } \tau\in[0,T]. \tag{4.1}$$

in the limit as $\varepsilon\to 0$.

By (3.40) we have

$$\varrho_\varepsilon \to \varrho \quad \text{weakly-(*) in } L^\infty(0,T;L^{\frac{5}{3}}(B)), \tag{4.2}$$

by (3.39)

$$\vartheta_\varepsilon \to \vartheta \quad \text{weakly-(*) in } L^\infty(0,T;L^4(B)), \tag{4.3}$$

and due to (3.41) we get

$$\vartheta_\varepsilon \to \vartheta \quad \text{weakly in } L^2(0,T;W^{1,2}(B)). \tag{4.4}$$

Due to (3.35), (3.39), and (3.41) we have also that

$$\vartheta_\varepsilon^4 \to \overline{\vartheta^4} \quad \text{weakly in } L^1((0,T)\times B), \tag{4.5}$$

$$\vartheta_\varepsilon^5 \to \overline{\vartheta^5} \quad \text{weakly in } L^1((0,T)\times B). \tag{4.6}$$

Here and in the rest of the paper the bar denotes a weak limit of a composed or nonlinear function.

Then, (3.40) together with the equation of continuity (3.11), imply that

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T]; L^{\frac{5}{3}}(B)). \tag{4.7}$$

Next by (3.36), up to a subsequence, we get

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(B,\mathbb{R}^3)). \tag{4.8}$$

Then since $L^{\frac{5}{3}}(B) \hookrightarrow\hookrightarrow W^{-1,2}(B)$, by (4.2), (4.8), (4.7), (3.34), we obtain

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho\mathbf{u} \text{ weakly-(*) in } L^\infty(0,T; L^{\frac{5}{4}}(B;\mathbb{R}^3)). \tag{4.9}$$

Due to the embedding $W_0^{1,2}(B) \hookrightarrow L^6(B)$ we infer

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho\mathbf{u}\otimes\mathbf{u}} \text{ weakly in } L^2(0,T; L^{\frac{30}{29}}(B;\mathbb{R}^3)). \tag{4.10}$$

From (3.12) we deduce that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho\mathbf{u} \text{ in } C_{\rm weak}([T_1,T_2]; L^{\frac{5}{4}}(O;\mathbb{R}^3)) \tag{4.11}$$

for any space-time cylinder such that

$$(T_1,T_2)\times O \subset [0,T]\times B,\ [T_1,T_2]\times\overline{O}\cap\cup_{\tau\in[0,T]}(\{\tau\}\times\Gamma_\tau) = \emptyset. \tag{4.12}$$

Since $L^{\frac{5}{4}}(B) \hookrightarrow\hookrightarrow W^{-1,2}(B)$, we conclude that

$$\overline{\varrho\mathbf{u}\otimes\mathbf{u}} = \varrho\mathbf{u}\otimes\mathbf{u} \text{ a.a. in } (0,T)\times B. \tag{4.13}$$

Next by (3.35), (3.40), (3.44) and asymptotic behaviour of p_M we obtian

$$p_{\eta,\delta}(\varrho_\varepsilon,\vartheta_\varepsilon) = p_M(\varrho_\varepsilon,\vartheta_\varepsilon) + \frac{a_\eta}{3}\vartheta_\varepsilon^4 + \delta\varrho^\beta \to \overline{p_M(\varrho,\vartheta)} + \frac{a_\eta}{3}\overline{\vartheta^4} + \delta\overline{\varrho^\beta} \quad \text{weakly in } L^1(K) \text{ with } K \text{ as in (3.45).} \tag{4.14}$$

Due to (3.36), (3.35), (2.2)

$$\mathbb{S}_\omega(\vartheta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon) \to \overline{\mathbb{S}_\omega(\vartheta,\nabla_x\mathbf{u})} \quad \text{weakly in } L^{\frac{4}{3}}((0,T)\times B). \tag{4.15}$$

According to (3.47)

$$\varrho_\varepsilon s_\eta(\varrho_\varepsilon,\vartheta_\varepsilon) \to \overline{\varrho s_\eta(\varrho,\vartheta)} \quad \text{weakly in } L^q((0,T)\times B) \text{ with some } q>1, \tag{4.16}$$

and by (3.48)

$$\varrho_\varepsilon s_\eta(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon \to \overline{\varrho s_\eta(\varrho,\vartheta)\mathbf{u}} \quad \text{weakly in } L^p((0,T)\times B) \text{ with some } p>1. \tag{4.17}$$

More details for the above considerations can be found in [5, 10].

4.1.2 Pointwise convergence of the temperature and the density

In order to show a.e. convergence of the temperature we follow [10]. The proof is based on the Div-Curl Lemma (see [10, Section 3.6.2]) and Young measures methods. To this end we set

$$\mathbf{U}_\varepsilon = \left[\varrho_\varepsilon s_\eta(\varrho_\varepsilon,\vartheta_\varepsilon), \varrho_\varepsilon s_\eta(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon + \frac{\kappa_\nu(\vartheta_\varepsilon)\nabla\vartheta_\varepsilon}{\vartheta_\varepsilon}\right]$$
$$\mathbf{W}_\varepsilon = [G(\vartheta_\varepsilon), 0, 0, 0],$$

where G is a bounded and Lipschitz function on $[0,\infty)$. Then due to estimates obtained in previous section $\mathrm{Div}_{t,x}\mathbf{U}_\varepsilon$ is precompact in $W^{-1,s}((0,T)\times B)$ and $\mathrm{Curl}_{t,x}\mathbf{W}_\varepsilon$ is precompact in $W^{-1,s}((0,T)\times B)^{4\times 4}$ with certain $s>1$. Therefore we can deduce that assumptions of Div-Curl Lemma for $\mathbf{U}_\varepsilon$ and $\mathbf{W}_\varepsilon$ are satisfied and we may derive

$$\overline{\varrho s_\eta(\varrho,\vartheta)G(\vartheta)} = \overline{\varrho s_\eta(\varrho,\vartheta)}\ \overline{G(\vartheta)}. \tag{4.18}$$

Next step is to show that

$$\overline{\varrho s_M(\varrho,\vartheta)G(\vartheta)} \geq \overline{\varrho s_M(\varrho,\vartheta)}\ \overline{G(\vartheta)}, \qquad \overline{\vartheta^3 G(\vartheta)} \geq \overline{\vartheta^3}\ \overline{G(\vartheta)}. \tag{4.19}$$

for any continuous and increasing function G. It can be derived by application of the theory of parametrized Young measures. Details can be found in [10, Section 3.6.2]. Combining (4.18), (4.19) and taking $G(\vartheta)=\vartheta$ we deduce

$$\overline{\vartheta^4} = \overline{\vartheta^3}\ \vartheta$$

which yields

$$\vartheta_\varepsilon \to \vartheta \ \text{ a.a. in } (0,T)\times B. \tag{4.20}$$

Moreover due to (3.42) the limit temperature ϑ is positive a.e. on the set $(0,T)\times B$. Similarly as in [19, Section 4.1.2] we deduce

$$\varrho_\varepsilon \to \varrho \text{ a.e. in } (0,T)\times B. \tag{4.21}$$

4.1.3 The limit system as $\varepsilon \to 0$

Let us summarise our considerations from Section 4.1.1, 4.1.2. Passing to the limit in (3.11) we obtain by (4.7), (4.9) that

$$\int_B \varrho\varphi(\tau,\cdot)\ \mathrm{d}x - \int_B \varrho_{0,\delta}\varphi(0,\cdot)\ \mathrm{d}x = \int_0^\tau\int_B (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi)\ \mathrm{d}x\mathrm{d}t \tag{4.22}$$

for any $\tau\in[0,T]$ and any $\varphi\in C_c^\infty([0,T]\times\mathbb{R}^3)$. Moreover, the limit solutions satisfies also the renormalized equation in the same for as (3.11).

Next we proceed to a limit in (3.12). Since we have at hand only the *local estimates* on the pressure, see (3.44), (3.45), we have to restrict ourselves to the class of test functions

$$\boldsymbol{\varphi}\in C^1([0,T);W_0^{1,\infty}(B;\mathbb{R}^3)),\ \mathrm{supp}[\mathrm{div}_x\boldsymbol{\varphi}(\tau,\cdot)]\cap\Gamma_\tau=\emptyset,\ \boldsymbol{\varphi}\cdot\mathbf{n}|_{\Gamma_\tau}=0 \text{ for all } \tau\in[0,T]. \tag{4.23}$$

In accordance with (4.2), (4.5), (4.8), (4.9), (4.10), (4.13), (4.14), (4.15), (4.20), (4.21), assumptions on (2.2), (2.3), the momentum equation reads

$$\int_0^\tau\int_B (\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\boldsymbol{\varphi} + p_{\eta,\delta}(\varrho,\vartheta)\mathrm{div}_x\boldsymbol{\varphi} - \mathbb{S}_\omega:\nabla_x\boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t \tag{4.24}$$

$$= -\int_B (\varrho\mathbf{u})_{0,\delta} \cdot \boldsymbol{\varphi}(0,\cdot)\, \mathrm{d}x$$

for any test function $\boldsymbol{\varphi}$ as in (4.23).

Further, due to (3.41), (3.49), and (4.20) we get $\frac{\kappa_\nu(\vartheta_\varepsilon)}{\vartheta_\varepsilon}|\nabla_x\vartheta_\varepsilon| \to \frac{\kappa_\nu(\vartheta)}{\vartheta}|\nabla_x\vartheta|$ weakly in $L^1((0,T)\times B)$. The terms $\frac{1}{\vartheta}\mathbb{S}_\omega(\vartheta_\varepsilon,\nabla_x\mathbf{u}_\varepsilon) : \nabla_x\mathbf{u}_\varepsilon$ and $\frac{\kappa_\nu(\vartheta_\varepsilon)|\nabla_x\vartheta_\varepsilon|^2}{\vartheta}$ for $\vartheta \geq 0$ are lower weakly semicontinuous. Then this together with (4.5), (4.6), (4.16), (4.17), (4.20), (4.21) allows to conclude that

$$\begin{aligned}\int_0^T\int_B \varrho s_\eta(\varrho,\vartheta)(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_B \frac{\kappa_\nu(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t \\ + \int_0^T\int_B \frac{\varphi}{\vartheta}\left(\mathbb{S}_\omega : \nabla_x\mathbf{u} + \frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t - \int_0^T\int_B \lambda\vartheta^4\,\mathrm{d}x\mathrm{d}t \leq -\int_B (\varrho s)_{0,\delta,\eta}\varphi(0)\,\mathrm{d}x\end{aligned} \tag{4.25}$$

as $\varepsilon \to 0$.

Finally, we can proceed to a limit with $\varepsilon \to 0$ in (3.15). Since the sequence $\{\varrho_\varepsilon e_\eta(\varrho_\varepsilon,\vartheta_\varepsilon)\}_\varepsilon$ is nonnegative and (4.20), (4.21), (3.50), by the Fatou lemma we deduce

$$\limsup \int_0^T\int_B \varrho_\varepsilon e_\eta(\varrho_\varepsilon,\vartheta_\varepsilon)\partial_t\psi\,\mathrm{d}x\mathrm{d}t \leq \int_0^T\int_B \varrho e_\eta(\varrho,\vartheta)\partial_t\psi\,\mathrm{d}x\mathrm{d}t$$

as far as $\partial_t\psi \leq 0$. Using (1.6), (4.14) and arguments used also above we obtain

$$\begin{aligned}\int_0^T\int_B\left(\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e_\eta(\varrho,\vartheta) + \frac{\delta}{\beta-1}\varrho^\beta\right)\partial_t\psi - \lambda\vartheta^5\psi\right)\mathrm{d}x\mathrm{d}t \\ \geq -\int_B\left(\frac{1}{2}\frac{(\varrho\mathbf{u})_0^2}{\varrho_{0,\delta}} + \varrho_{0,\delta}e_{0,\delta,\eta} + \frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta - (\varrho\mathbf{u})_0\cdot\mathbf{V}(0,\cdot)\right)\psi(0)\,\mathrm{d}x \\ -\int_0^T\int_B (\mathbb{S}_\omega : \nabla_x\mathbf{V}\psi - \varrho\mathbf{u}\cdot\partial_t(\mathbf{V}\psi) - \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathbf{V}\psi - p_{\eta,\delta}(\varrho,\vartheta)\mathrm{div}_x\mathbf{V}\psi)\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{4.26}$$

for all $\psi \in C_c^1([0,T))$, $\partial_t\psi \leq 0$.

4.2 Fundamental lemma and extending the class of test functions

In order to get rid of the density dependent terms supported by the "solid" part $((0,T)\times B)\setminus Q_T$ we use [9, Lemma 4.1] which reads as

Lemma 4.1. *Let $\varrho \in L^\infty(0,T;L^2(B))$, $\varrho \geq 0$, $\mathbf{u} \in L^2(0,T;W_0^{1,2}(B;\mathbb{R}^3))$ be a weak solution of the equation of continuity, specifically,*

$$\int_B\left(\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0\varphi(0,\cdot)\right)\mathrm{d}x = \int_0^\tau\int_B\left(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right)\mathrm{d}x\,\mathrm{d}t \tag{4.27}$$

for any $\tau \in [0,T]$ and any test function $\varphi \in C_c^1([0,T]\times\mathbb{R}^3)$.

In addition, assume that

$$(\mathbf{u}-\mathbf{V})(\tau,\cdot)\cdot\mathbf{n}|_{\Gamma_\tau} = 0 \text{ for a.a. } \tau \in (0,T), \tag{4.28}$$

and that

$$\varrho_0 \in L^2(\mathbb{R}^3),\ \varrho_0 \geq 0,\ \varrho_0|_{B\setminus\Omega_0} = 0.$$

Then
$$\varrho(\tau,\cdot)|_{B\setminus\Omega_\tau}=0 \textit{ for any } \tau\in[0,T].$$

Since we have set our initial density $\varrho_{0,\delta}$ to be zero on $B\setminus\Omega_0$ (see (3.8)), by virtue of Lemma 4.1, the continuity equation (4.22) reads

$$\int_{\Omega_\tau}\varrho\varphi(\tau,\cdot)\,\mathrm{d}x-\int_{\Omega_0}\varrho_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x=\int_0^\tau\int_{\Omega_t}\left(\varrho\partial_t\varphi+\varrho\mathbf{u}\cdot\nabla_x\varphi\right)\,\mathrm{d}x\mathrm{d}t \tag{4.29}$$

for any $\tau\in[0,T]$ and any $\varphi\in C_c^\infty([0,T]\times\mathbb{R}^3)$. Moreover the following renormalized formulation is satisfied

$$\int_0^T\int_{\Omega_t}\varrho B(\varrho)(\partial_t\varphi+\mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t=\int_0^T\int_{\Omega_t}b(\varrho)\mathrm{div}_x\mathbf{u}\varphi\,\mathrm{d}x\mathrm{d}t-\int_{\Omega_0}\varrho_{0,\delta}B(\varrho_{0,\delta})\varphi(0)\,\mathrm{d}x \tag{4.30}$$

for any $\varphi\in C_c^1([0,T)\times\mathbb{R}^3)$, and any $b\in L^\infty\cap C[0,\infty)$ such that $b(0)=0$ and $B(\varrho)=B(1)+\int_1^\varrho\frac{b(z)}{z^2}\mathrm{d}z$. Next the momentum equation (4.24) reduces to

$$\begin{aligned}&\int_0^T\int_{\Omega_t}\left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}+\varrho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\boldsymbol{\varphi}+p_{\eta,\delta}(\varrho,\vartheta)\mathrm{div}_x\boldsymbol{\varphi}-\mathbb{S}_\omega(\vartheta,\nabla_x\mathbf{u}):\nabla_x\boldsymbol{\varphi}\right)\,\mathrm{d}x\mathrm{d}t \qquad (4.31)\\ &=-\int_{\Omega_0}(\varrho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x+\int_0^T\int_{B\setminus\Omega_t}\mathbb{S}_\omega(\vartheta,\nabla_x\mathbf{u}):\nabla\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_{B\setminus\Omega_t}\frac{a_\eta}{3}\vartheta^4\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

for any test function $\boldsymbol{\varphi}$ as in (4.23). We remark that in this step we crucially need the extra pressure term $\delta\varrho^\beta$ ensuring the density ϱ to be square integrable (see (3.33)).

Next, we argue the same way as in [9, Section 4.3.1] to conclude, that the momentum equation (4.31) holds in fact for any test function $\boldsymbol{\varphi}$ such that

$$\boldsymbol{\varphi}\in C_c^\infty([0,T]\times B;\mathbb{R}^3),\qquad \boldsymbol{\varphi}(\tau,\cdot)\cdot\mathbf{n}|_{\Gamma_\tau}=0\quad\text{ for any }\tau\in[0,T]. \tag{4.32}$$

Moreover, by Lemma 4.1 and the choice of initial data the balance of entropy (4.25) takes the following form

$$\begin{aligned}&\int_0^T\int_{\Omega_t}\varrho s(\varrho,\vartheta)(\partial_t\varphi+\mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{B\setminus\Omega_t}\frac{4}{3}a_\eta\vartheta^3(\partial_t\varphi+\mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t\\ &\quad-\int_0^T\int_{\Omega_t}\frac{\kappa_\nu(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_{B\setminus\Omega_t}\frac{\kappa_\nu(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t\\ &\quad+\int_0^T\int_{\Omega_t}\frac{\varphi}{\vartheta}\left(\mathbb{S}_\omega:\nabla_x\mathbf{u}+\frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{B\setminus\Omega_t}\frac{\varphi}{\vartheta}\left(\mathbb{S}_\omega:\nabla_x\mathbf{u}+\frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\,\mathrm{d}x\mathrm{d}t\\ &\quad-\int_0^T\int_B\lambda\vartheta^4\,\mathrm{d}x\mathrm{d}t\le-\int_{\Omega_0}(\varrho s)_{0,\delta}\varphi(0)\,\mathrm{d}x-\int_{B\setminus\Omega_0}\frac{4}{3}a_\eta\vartheta_{0,\delta}^3\varphi(0)\,\mathrm{d}x\end{aligned} \tag{4.33}$$

for all $\varphi \in C^1_c([0,T)\times\overline{B})$, $\varphi \geq 0$. Finally, total energy balance (4.26) reads

$$
\begin{aligned}
&\int_0^T\int_{\Omega_t}\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)+\frac{\delta}{\beta-1}\varrho^\beta\right)\partial_t\psi\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{B\setminus\Omega_t}a_\eta\vartheta^4\partial_t\psi\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_B\lambda\vartheta^5\psi\,\mathrm{d}x\mathrm{d}t\\
&\geq-\int_{\Omega_0}\left(\frac{1}{2}\frac{(\varrho\mathbf{u})^2_{0,\delta}}{\varrho_{0,\delta}}+\varrho_{0,\delta}e_{0,\delta}+\frac{\delta}{\beta-1}\varrho^\beta_{0,\delta}-(\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0,\cdot)\right)\psi(0)\,\mathrm{d}x-\int_{B\setminus\Omega_0}a_\eta\vartheta^4_{0,\delta}\psi(0)\,\mathrm{d}x\\
&-\int_0^T\int_{\Omega_t}\left(\mathbb{S}_\omega:\nabla_x\mathbf{V}\psi-\varrho\mathbf{u}\cdot\partial_t(\mathbf{V}\psi)-\varrho(\mathbf{u}\otimes\mathbf{u}):\nabla_x\mathbf{V}\psi-p_\delta(\varrho,\vartheta)\mathrm{div}_x\mathbf{V}\psi\right)\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_{B\setminus\Omega_t}\left(\mathbb{S}_\omega:\nabla_x\mathbf{V}-\frac{1}{3}a_\eta\vartheta^4\mathrm{div}_x\mathbf{V}\right)\psi\,\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{4.34}
$$

for all $\psi \in C^1_c([0,T))$, $\partial_t\psi \leq 0$.

4.3 Limit in radiation $\eta \to 0$

Let us denote by $\{\varrho_\eta,\mathbf{u}_\eta,\vartheta_\eta\}_{\eta>0}$ solutions to the system (4.29), (4.30), (4.31), (4.33), (4.34). In this section we pass to the limit with $\eta \to 0$ and get rid of radiative components of the pressure, internal entropy and internal energy functions outside of the fluid domain. Let us notice that estimates obtained in Section 3.2 are independent of parameter η (if not emphasised). Therefore by (3.35) and as $a_\eta = \eta a$ on $B\setminus\Omega_\tau$ for $\tau\in[0,T]$

$$
\left|\int_0^T\int_{B\setminus\Omega_t}\frac{1}{3}a\eta\vartheta^4_\eta\mathrm{div}_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t\right|\leq\eta c\|\vartheta^5_\eta\|^{\frac{4}{5}}_{L^1((0,T)\times B)}\|\mathrm{div}_x\boldsymbol{\varphi}\|_{L^\infty((0,T)\times(B\setminus\Omega_t))}\to 0 \text{ as } \eta\to 0, \tag{4.35}
$$

where $\boldsymbol{\varphi}$ is as in (4.32). In a similar way

$$
\int_0^T\int_{B\setminus\Omega_t}\frac{1}{3}a_\eta\vartheta^4_\eta\mathrm{div}_x\mathbf{V}\,\mathrm{d}x\mathrm{d}t\,\mathrm{d}x\mathrm{d}t\to 0 \text{ as } \eta\to 0, \tag{4.36}
$$

$$
\int_0^T\int_{B\setminus\Omega_t}\frac{4}{3}a_\eta\vartheta^3_\eta\partial_t\varphi\,\mathrm{d}x\mathrm{d}t\to 0 \text{ as } \eta\to 0 \quad \text{for any } \varphi\in C^1_c([0,T)\times\overline{B}). \tag{4.37}
$$

Next by (3.35), (3.36)

$$
\left|\int_0^T\int_{B\setminus\Omega_t}\frac{4}{3}a_\eta\vartheta^3_\eta\mathbf{u}_\eta\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t\right|\leq\eta c(T,B)\|\vartheta^5_\eta\|^{\frac{3}{10}}_{L^1((0,T)\times B)}\|\mathbf{u}_\eta\|^{\frac{1}{2}}_{L^2(0,T;L^6(B))}\|\nabla_x\varphi\|_{L^\infty((0,T)\times(B\setminus\Omega_t))}\to 0 \text{ as } \eta\to 0 \tag{4.38}
$$

for any $\varphi\in C^1_c([0,T)\times\overline{B})$. Since $\vartheta_\eta\to\vartheta$ weakly in $L^1((0,T)\times B)$, due to [5, Corollary 2.2], we obtain

$$
\int_0^T\int_B\lambda\vartheta^5\,\mathrm{d}x\mathrm{d}t\leq\liminf_{\eta\to 0}\int_0^T\int_B\lambda\vartheta^5_\eta\,\mathrm{d}x\mathrm{d}t.
$$

The initial condition terms involving a_η obviously converge to zero. To pass to the limit in remaining terms outside of the fluid part and in all terms in the fluid part we use the same arguments as for passing with $\varepsilon\to 0$. The continuity equation in the limit $\eta\to 0$ takes the same form as in (4.29), (4.30). The momentum equation (4.31) as $\eta\to 0$ satisfies

$$
\int_0^T\int_{\Omega_t}\left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}+\varrho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\boldsymbol{\varphi}+p_\delta(\varrho,\vartheta)\mathrm{div}_x\boldsymbol{\varphi}-\mathbb{S}_\omega(\vartheta,\nabla_x\mathbf{u}):\nabla_x\boldsymbol{\varphi}\right)\mathrm{d}x\mathrm{d}t \tag{4.39}
$$

$$= -\int_{\Omega_0} (\varrho\mathbf{u})_{0,\delta} \cdot \boldsymbol{\varphi}(0,\cdot)\ \mathrm{d}x - \int_0^T \int_{B\setminus\Omega_t} \mathbb{S}_\omega(\vartheta, \nabla_x \mathbf{u}) : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t$$

for any test function $\boldsymbol{\varphi}$ as in (4.32). The entropy inequality (4.33) in the limit reads as

$$\begin{aligned}
&\int_0^T \int_{\Omega_t} \varrho s(\varrho,\vartheta)(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\Omega_t} \frac{\kappa_\nu(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{B\setminus\Omega_t} \frac{\kappa_\nu(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega_t} \frac{\varphi}{\vartheta}\left(\mathbb{S}_\omega : \nabla_x\mathbf{u} + \frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{B\setminus\Omega_t} \frac{\varphi}{\vartheta}\left(\mathbb{S}_\omega : \nabla_x\mathbf{u} + \frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t \qquad (4.40)\\
&\quad - \int_0^T \int_B \lambda\vartheta^4\,\mathrm{d}x\mathrm{d}t \le -\int_{\Omega_0} (\varrho s)_{0,\delta}\varphi(0)\,\mathrm{d}x
\end{aligned}$$

for all $\varphi \in C_c^1([0,T)\times\overline{B})$, $\varphi \ge 0$. Finally, total energy balance (4.34) in the limit $\eta \to 0$ satisfies

$$\begin{aligned}
&\int_0^T \int_{\Omega_t} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e + \frac{\delta}{\beta-1}\varrho^\beta\right)\partial_t\psi\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_B \lambda\vartheta^5\psi\,\mathrm{d}x\mathrm{d}t \\
&\ge -\int_{\Omega_0} \left(\frac{1}{2}\frac{(\varrho\mathbf{u})_{0,\delta}^2}{\varrho_{0,\delta}} + \varrho_{0,\delta}e_{0,\delta} + \frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta - (\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0,\cdot)\right)\psi(0)\,\mathrm{d}x \\
&- \int_0^T \int_{\Omega_t} \left(\mathbb{S}_\omega : \nabla_x\mathbf{V}\psi - \varrho\mathbf{u}\cdot\partial_t(\mathbf{V}\psi) - \varrho(\mathbf{u}\otimes\mathbf{u}) : \nabla_x\mathbf{V}\psi - p_\delta(\varrho,\vartheta)\mathrm{div}_x\mathbf{V}\psi\right)\mathrm{d}x\mathrm{d}t \qquad (4.41)\\
&- \int_0^\tau \int_{B\setminus\Omega_t} \mathbb{S}_\omega(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{V}\psi\,\mathrm{d}x\mathrm{d}t
\end{aligned}$$

for all $\psi \in C_c^1([0,T))$, $\psi \ge 0$, $\partial_t\psi \le 0$.

4.4 Vanishing viscosity $\omega \to 0$

Let us denote by $\{\varrho_\omega, \mathbf{u}_\omega, \vartheta_\omega\}_{\omega>0}$ solutions to (4.29), (4.30), (4.39), (4.40), and (4.41). Now our aim is to pass with $\omega \to 0$ in order to get rid of terms related to the viscous stress tensor outside of the fluid domain. Let us notice that again we may obtain analogous estimates as in Section 3.2 which are independent of ω if not emphasised. For the viscous term in the momentum equation (4.39) we observe that by (3.31), (3.35) for any $\boldsymbol{\varphi}$ as in (4.32) we have

$$\begin{aligned}
&\int_0^T \int_{B\setminus\Omega_t} \mathbb{S}_\omega(\vartheta_\omega,\nabla_x\mathbf{u}_\omega) : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \le \sqrt{\omega}\int_0^T \int_{B\setminus\Omega_t} \frac{1}{\sqrt{\vartheta_\omega}}\sqrt{\omega}\mathbb{S}(\vartheta_\omega,\nabla_x\mathbf{u}_\omega)\sqrt{\vartheta_\omega} : \nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t \\
&\quad \le \sqrt{\omega}\frac{1}{2}\int_0^T \int_{B\setminus\Omega_t} \frac{1}{\vartheta_\omega}\omega|\mathbb{S}(\vartheta_\omega,\nabla_x\mathbf{u}_\omega)|^2\,\mathrm{d}x\mathrm{d}t + \sqrt{\omega}\frac{1}{2}\int_0^T \int_{B\setminus\Omega_t} \vartheta_\omega|\nabla_x\boldsymbol{\varphi}|^2\,\mathrm{d}x\mathrm{d}t \qquad (4.42)\\
&\quad \le \sqrt{\omega}c\int_0^T \int_{B\setminus\Omega_t} \frac{1}{\vartheta_\omega}\mathbb{S}_\omega(\vartheta_\omega,\nabla_x\mathbf{u}_\omega) : \nabla_x\mathbf{u}_\omega\,\mathrm{d}x\mathrm{d}t + \sqrt{\omega}c(\boldsymbol{\varphi},\lambda)\int_0^T \int_{B\setminus\Omega_t} \lambda\vartheta_\omega^5\,\mathrm{d}x\mathrm{d}t \le \sqrt{\omega}c \to 0 \text{ as } \omega \to 0.
\end{aligned}$$

In a similar way we show that in the total energy inequality (4.41)

$$\int_0^T \int_{B\setminus\Omega_t} \mathbb{S}_\omega(\vartheta_\omega,\nabla_x\mathbf{u}_\omega) : \nabla_x\mathbf{V}\,\mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } \omega \to 0. \qquad (4.43)$$

When passing with $\omega \to 0$ in entropy inequality (4.40) we skip the term $\int_0^T \int_{B\setminus\Omega_t} \frac{\varphi}{\vartheta}\mathbb{S}_\omega : \nabla_x \mathbf{u} \, \mathrm{d}x\mathrm{d}t$ since it is positive for all $\varphi \in C_c^1([0,T)\times \overline{B})$, $\varphi \geq 0$. Since on the fluid domain all estimates obtained in Section 3.2 holds true, we can pass tho the limit with $\omega \to 0$ in all terms on the fluid domain and on the remaining one outside of the fluid domain in the same way as in previous Section 4.3.

All the above arguments allow as to pass with $\omega \to 0$. Then the continuity equation takes the same form as in (4.29), (4.30). The momentum equation (4.39) in the limit $\omega \to 0$ reads

$$\int_0^T \int_{\Omega_t} (\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p_\delta(\varrho,\vartheta)\mathrm{div}_x\boldsymbol{\varphi} - \mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi}) \,\mathrm{d}x\mathrm{d}t = -\int_{\Omega_0} (\varrho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x \tag{4.44}$$

for any test function $\boldsymbol{\varphi}$ as in (4.32). The entropy inequality (4.40) as $\omega \to 0$ satisfies

$$\begin{aligned}
&\int_0^T \int_{\Omega_t} \varrho s(\varrho,\vartheta)(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\Omega_t} \frac{\kappa_\nu(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{B\setminus\Omega_t} \frac{\kappa_\nu(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T\int_{\Omega_t} \frac{\varphi}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u} + \frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t + \int_0^T\int_{B\setminus\Omega_t} \frac{\varphi}{\vartheta}\left(\frac{\kappa_\nu(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^T\int_B \lambda\vartheta^4\,\mathrm{d}x\mathrm{d}t \leq -\int_{\Omega_0} (\varrho s)_{0,\delta}\varphi(0)\,\mathrm{d}x
\end{aligned} \tag{4.45}$$

for all $\varphi \in C_c^1([0,T)\times\overline{B})$, $\varphi \geq 0$. Finally, for the total energy balance (4.41) we get for $\omega \to 0$ that

$$\begin{aligned}
&\int_0^T\int_{\Omega_\tau} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e + \frac{\delta}{\beta-1}\varrho^\beta\right)\partial_t\psi\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_B \lambda\vartheta^5\psi\,\mathrm{d}x\mathrm{d}t \\
&\geq -\int_{\Omega_0}\left(\frac{1}{2}\frac{(\varrho\mathbf{u})_{0,\delta}^2}{\varrho_{0,\delta}} + \varrho_{0,\delta}e_{0,\delta} + \frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta - (\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0,\cdot)\right)\psi(0)\,\mathrm{d}x \\
&\quad - \int_0^T\int_{\Omega_t} (\mathbb{S}:\nabla_x\mathbf{V}\psi - \varrho\mathbf{u}\cdot\partial_t(\mathbf{V}\psi) - \varrho(\mathbf{u}\otimes\mathbf{u}):\nabla_x\mathbf{V}\psi - p_\delta(\varrho,\vartheta)\mathrm{div}_x\mathbf{V}\psi)\,\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{4.46}$$

for all $\psi \in C_c^1([0,T))$, $\psi \geq 0$, $\partial_t\psi \leq 0$.

4.5 Vanishing conductivity $\nu \to 0$

Let $\{\varrho_\nu, \mathbf{u}_\nu, \vartheta_\nu\}_{\nu>0}$ denote solutions to the system obtained in the previous Section 4.4, namely satisfying (4.29), (4.30), (4.44), (4.45), and (4.46) for each fixed $\nu > 0$. In this section we pass with $\nu \to 0$ what allow us to show that the term $\int_0^T\int_{B\setminus\Omega_t} \frac{\kappa_\nu(\vartheta_\nu)\nabla_x\vartheta}{\vartheta}\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t$ vanishes in the limit for any $\varphi \in C_c^1([0,T)\times\overline{B})$.

Let us notice that for each fixed $\nu > 0$ we still keep that

$$\left\|\frac{\chi_\nu\kappa(\vartheta_\nu)|\nabla_x\vartheta_\nu|^2}{\vartheta_\nu^2}\right\|_{L^1((0,T)\times B)} \leq c(\lambda) \quad \text{and} \quad \|\lambda\vartheta_\nu^5\|_{L^1((0,T)\times B)} \leq c(\lambda)$$

independent w.r.t. ν. Therefore due to (2.5)–(2.7) we deduce that

$$\begin{aligned}
\int_0^T\int_{B\setminus\Omega_t} \frac{\kappa_\nu(\vartheta_\nu)\nabla_x\vartheta_\nu}{\vartheta_\nu}\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t &\leq \sqrt{\nu}\left(\int_0^T\int_{B\setminus\Omega_t}\frac{\nu\kappa(\vartheta_\nu)|\nabla_x\vartheta_\nu|^2}{\vartheta_\nu^2}\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T\int_{B\setminus\Omega_t}|\nabla_x\varphi|^2\kappa(\vartheta_\nu)\,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}} \\
&\to 0 \quad \text{as } \nu \to 0.
\end{aligned}$$

Since $\int_0^T \int_{B\setminus\Omega_t} \frac{\varphi}{\vartheta_\nu}\left(\frac{\kappa_\nu(\vartheta_\nu)|\nabla_x\vartheta_\nu|^2}{\vartheta_\nu}\right) \mathrm{d}x\mathrm{d}t$ is positive for any $\varphi \in C_c^1([0,T)\times\overline{B})$, $\varphi \geq 0$ and any $\nu > 0$, we can skip this term in the vanishing conductivity limit of the internal entropy inequality. Next proceeding in the analogous way as in Section 4.3 we may pass with $\nu \to 0$ in the remaining terms of (4.29), (4.30), (4.44), (4.45), (4.46). Therefore we obtain that the continuity equation in the limit satisfies again (4.29), (4.30). The momentum equation takes the same for as in (4.44) as $\nu \to 0$. The entropy inequality (4.45) in the limit $\nu \to 0$ reads

$$\begin{aligned}
&\int_0^T \int_{\Omega_t} \varrho s(\varrho,\vartheta)(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\Omega_t} \frac{\kappa(\vartheta)\nabla_x\vartheta\cdot\nabla_x\varphi}{\vartheta}\,\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_{\Omega_t} \frac{\varphi}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u} + \frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t - \int_0^T \int_B \lambda\vartheta^4\,\mathrm{d}x\mathrm{d}t \leq -\int_{\Omega_0} (\varrho s)_{0,\delta}\varphi(0)\,\mathrm{d}x
\end{aligned} \tag{4.47}$$

for all $\varphi \in C_c^1([0,T)\times\overline{B})$, $\varphi \geq 0$. The energy inequality in the limit $\nu \to 0$ have the same form as in (4.46).

4.6 Vanishing additional temperature term, $\lambda \to 0$

In this section we get rid the term related to coefficient λ - the only terms which are left also outside of a fluid domain. Let $\{\varrho_\lambda, \mathbf{u}_\lambda, \vartheta_\lambda\}_{\lambda\in(0,1)}$ be solution to the limit system obtained in previous Section 4.5. In order to pass with $\lambda \to 0$ in (4.29), (4.30), (4.44), (4.47), and (4.46) we need to provide uniform estimates analogous to these obtained in Section 3.2, but independent of λ. To this end we proceed in a similar way as therein, we only need to modify (3.20) and (3.24) as follows (for convenience we skip the subscript λ in below notation)

$$\int_0^\tau \int_{\Omega_t} \mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{V}\,\mathrm{d}x\mathrm{d}t \leq c(\mathbf{V}) + \int_0^\tau \int_{\Omega_t} a\vartheta^4\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^\tau \int_{\Omega_t} \frac{1}{\vartheta}\mathbb{S}:\nabla_x\mathbf{u}\,\mathrm{d}x\mathrm{d}t,$$

$$\left|\int_0^\tau \int_{\Omega_t} p_\delta(\varrho,\vartheta)\mathrm{div}_x\mathbf{V}\,\mathrm{d}x\mathrm{d}t\right| \leq c(\mathbf{V}) + c(\mathbf{V})\int_0^\tau \int_{\Omega_t} a\vartheta^4\,\mathrm{d}x\mathrm{d}t + c(\mathbf{V})\int_0^\tau \int_{\Omega_t} \varrho^{\frac{5}{3}}\,\mathrm{d}x\mathrm{d}t + c(\mathbf{V})\int_0^\tau \int_{\Omega_t} \frac{\delta}{\beta-1}\varrho^\beta\,\mathrm{d}x\mathrm{d}t.$$

Consequently we may easy obtain that

$$\begin{aligned}
&\int_{\Omega_\tau} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_1(\varrho,\vartheta) - (\varrho-\bar{\varrho})\frac{\partial H_1(\bar{\varrho},1)}{\partial\varrho} - H_1(\bar{\varrho},1) + \frac{\delta}{\beta-1}\varrho^\beta\right)(\tau,\cdot)\,\mathrm{d}x \\
&\quad + \int_0^\tau \int_{\Omega_t} \frac{1}{\vartheta}\left(\frac{1}{2}\mathbb{S}:\nabla_x\mathbf{u} + \frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_B \frac{1}{2}\lambda\vartheta^5\,\mathrm{d}x\mathrm{d}t \\
&\leq c(\mathbf{V})\int_{\Omega_0} \left(\frac{1}{2}\frac{(\varrho\mathbf{u})_{0,\delta}^2}{\varrho_{0,\delta}} + H_1(\varrho_{0,\delta},\vartheta_{0,\delta}) + \frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta - (\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0,\cdot) + 1\right)\mathrm{d}x \\
&\quad - \int_{\Omega_0} \left((\varrho_{0,\delta}-\bar{\varrho})\frac{\partial H_1(\bar{\varrho},1)}{\partial\varrho} - H_1(\bar{\varrho},1)\right)\mathrm{d}x.
\end{aligned} \tag{4.48}$$

Hence the following estimates hold

$$\operatorname*{ess\ sup}_{\tau\in(0,T)} \|\delta\varrho^\beta(\tau,\cdot)\|_{L^1(\Omega_\tau)} + \operatorname*{ess\ sup}_{\tau\in(0,T)} \|\sqrt{\varrho}\mathbf{u}(\tau,\cdot)\|_{L^2(\Omega_\tau)} \leq c, \tag{4.49}$$

$$\left\|\lambda\vartheta^5\right\|_{L^1((0,T)\times B)} \leq c, \tag{4.50}$$

$$\|\mathbf{u}\|_{L^2(Q_T)} + \|\nabla_x \mathbf{u}\|_{L^2(Q_T)} \leq c, \tag{4.51}$$

$$\operatorname{ess} \sup_{\tau \in (0,T)} \|a\vartheta^4(\tau,\cdot)\|_{L^1(\Omega_\tau)} + \operatorname{ess} \sup_{\tau \in (0,T)} \|\varrho(\tau,\cdot)\|_{L^{\frac{5}{3}}(\Omega_\tau)} \leq c. \tag{4.52}$$

$$\int_0^T \int_{\Omega_t} \left(|\nabla_x \log(\vartheta)|^2 + |\nabla_x \vartheta^{\frac{3}{2}}|^2 \right) \mathrm{d}x \mathrm{d}t \leq c, \tag{4.53}$$

$$\int \int_K \left(p(\varrho,\vartheta)\varrho^\pi + \delta \varrho^{\beta+\pi} \right) \mathrm{d}x \mathrm{d}t \leq c(K) \text{ for certain } \pi > 0 \tag{4.54}$$

$$\text{and for any compact } K \subset Q_T \text{ such that } K \cap \left(\cup_{\tau \in [0,T]} \left(\{\tau\} \times \Gamma_\tau \right) \right) = \emptyset, \tag{4.55}$$

and

$$\|\varrho s(\varrho,\vartheta)\|_{L^q(Q_T)} + \|\varrho s(\varrho,\vartheta)\mathbf{u}\|_{L^q(Q_T)} + \left\| \frac{\kappa_\nu(\vartheta)}{\vartheta} \nabla_x \vartheta \right\|_{L^q(Q_T)} \leq c \quad \text{with certain } q > 1, \tag{4.56}$$

$$\|\varrho e(\varrho,\vartheta)\|_{L^1(Q_T)} \leq c \quad \text{and} \quad \|\varrho e(\varrho,\vartheta)\|_{L^p(K)} \leq c \quad \text{with some } p > 1 \text{ and } K \text{ as in (4.55)}. \tag{4.57}$$

Then due to (4.50)

$$\int_0^T \int_B \lambda \vartheta_\lambda^4 \, \mathrm{d}x \mathrm{d}t \to 0 \quad \text{as } \lambda \to 0.$$

Next notice that the term $\int_0^\tau \int_B \lambda \vartheta_\lambda^5 \psi \, \mathrm{d}x \mathrm{d}t$ is non-negative for all $\lambda > 0$ and all $\psi \in C_c^1([0,T))$, $\psi \geq 0$ and therefore can be skipped in the total energy inequality (4.46). In all other terms we pass in the same way as in Section 4.1. Not that (4.52) provides enough informations in steps where (3.35) is used therein. Consequently in the limit $\lambda \to 0$ the continuity equation and the momentum equation take the same fore as in (4.29), (4.30), (4.44), respectevly. The entropy inequality satisfies

$$\begin{aligned} &\int_0^T \int_{\Omega_t} \varrho s(\varrho,\vartheta)(\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi) \, \mathrm{d}x \mathrm{d}t - \int_0^T \int_{\Omega_t} \frac{\kappa(\vartheta) \nabla_x \vartheta \cdot \nabla_x \varphi}{\vartheta} \, \mathrm{d}x \mathrm{d}t + \int_0^T \int_{\Omega_t} \frac{\varphi}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} + \frac{\kappa(\vartheta)|\nabla_x \vartheta|^2}{\vartheta} \right) \mathrm{d}x \mathrm{d}t \\ &\qquad \leq - \int_{\Omega_0} (\varrho s)_{0,\delta} \varphi(0) \, \mathrm{d}x \end{aligned} \tag{4.58}$$

for all $\varphi \in C_c^1([0,T) \times \overline{B})$, $\varphi \geq 0$ as $\lambda \to 0$. And for the total energy inequality we get for $\lambda \to 0$ that

$$\begin{aligned} \int_0^T \int_{\Omega_\tau} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e + \frac{\delta}{\beta - 1} \varrho^\beta \right) \partial_t \psi \, \mathrm{d}x \mathrm{d}t \geq &- \int_{\Omega_0} \left(\frac{1}{2} \frac{(\varrho \mathbf{u})_{0,\delta}^2}{\varrho_{0,\delta}} + \varrho_{0,\delta} e_{0,\delta} + \frac{\delta}{\beta - 1} \varrho_{0,\delta}^\beta - (\varrho \mathbf{u})_{0,\delta} \cdot \mathbf{V}(0,\cdot) \right) \psi(0) \, \mathrm{d}x \\ &- \int_0^T \int_{\Omega_t} \left(\mathbb{S} : \nabla_x \mathbf{V} \psi - \varrho \mathbf{u} \cdot \partial_t (\mathbf{V} \psi) - \varrho (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \mathbf{V} \psi - p_\delta(\varrho,\vartheta) \mathrm{div}_x \mathbf{V} \psi \right) \mathrm{d}x \mathrm{d}t \end{aligned} \tag{4.59}$$

for all $\psi \in C_c^1([0,T))$, $\psi \geq 0$, $\partial_t \psi \leq 0$.

4.7 Conclusion of the proof – artificial pressure and temperature term

We proceed with δ to 0 similarly as in [10]. Finally in the energy inequality we choose a test function in a form of (3.18) and pass with $\xi \to 0$. Consequently obtain that our system satisfies the form required in the Theorem 2.1.

5 Discussion

- Let is point out that we considered the full slip boundary condition for the velocity field. We can consider the Dirichlet condition for the velocity field. In that case it will be examined by means of Birkman's penalized method. For more details see [8].
- We can also consider the Navier type of boundary conditions.
- The regularity of domain Ω follows from applications of level set method in the Fundamental Lemma 4.1.
- The essential in our paper was considering energy inequality instead of energy equation together with introducing the term $\lambda\vartheta^5$ in the energy balance and the term $\lambda\vartheta^4$ into the entropy balance.

Remark 5.1. *The condition* (1.6) *is not restrictive. Indeed, for a general* $\mathbf{V} \in C^1([0,T]; C^3_c(\mathbb{R}^3, \mathbb{R}^3))$ *we can find* $\mathbf{w} \in W^{1,\infty}(Q_T)$ *such that* $(\mathbf{V} - \mathbf{w})|_{\Gamma_\tau} = 0$ *for all* $\tau \in [0,T]$ *such that* $\mathrm{div}_x \mathbf{w} = 0$ *on some neighborhood of* Γ_τ *– see [9, Section 4.3.1]. This function can be used in place of* $\mathbf{V}$ *in the definition of weak solutions and later also in the approximate problem. Due to the fact that* $\boldsymbol{\varphi} = \mathbf{V} - \mathbf{w}$ *is a suitable test function in* (2.25) *and* (3.12)*, the appropriate energy balances remain valid.*

References

[1] S. S. Antman, J. P. Wilber. The asymptotic problem for the springlike motion of a heavy piston in a viscous gas. *Quart. Appl. Math.*,**65**, 3: 471-498, 2007.

[2] J. Březina, O. Kreml, and V. Mácha Dimension reduction for the full Navier-Stokes-Fourier system *J. Math. Fluid Mech.*, **19**:659–683, 2017.

[3] M. Bulíček, J. Málek, and K.R. Rajagopal. Navier's slip and evolutionary Navier-Stokes-like systems with pressure and shear-rate dependent viscosity. *Indiana Univ. Math. J.*, **56**:51–86, 2007.

[4] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**:511–547, 1989.

[5] E. Feireisl. *Dynamics of viscous compressible fluids.* Oxford University Press, Oxford, 2004.

[6] E. Feireisl. On the motion of a viscous, compressible, and heat conducting fluid. *Indiana Univ. Math. J.*, **53**:1707–1740, 2004.

[7] E. Feireisl, B. J. Jin, and A. Novotný. Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system. *J. Math. Fluid Mech. 14*, no. 4, 717–730, 2012.

[8] E. Feireisl, J. Neustupa, and J. Stebel. Convergence of a Brinkman-type penalization for compressible fluid flows. *J. Differential Equations*, **250**(1):596–606, 2011.

[9] E. Feireisl, O. Kreml, Š. Nečasová, J. Neustupa, J. Stebel. *Weak solutions to the barotropic Navier-Stokes system with slip boundary conditions in time dependent domains.* J. Differential Equations **254** no. 1, 125–140, 2013.

[10] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids.* Birkhäuser-Verlag, Basel, 2009.

[11] E. Feireisl and A. Novotný Weak-strong uniqueness property for the full Navier-Stokes-Fourier System *Arch. Rat. Mech. Anal.*, **204**:683–706, 2012.

[12] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids. *J. Math. Fluid Mech.*, **3**:358–392, 2001.

[13] E. Feireisl and H. Petzeltová. On integrability up to the boundary of the weak solutions of the Navier-Stokes equations of compressible flow. *Commun. Partial Differential Equations*, **25**(3-4):755–767, 2000.

[14] E. Feireisl, V. Mácha, Š. Nečasová, M. Tucsnak. Analysis of the adiabatic piston problem via methods of continuum mechanics. *Accepted to ANHP*

[15] H. Fujita, N. Sauer. On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries. *J. Fac. Sci. Univ. Tokyo*, Sect. I **17** : 403–420, 1970.

[16] C. Gruber and G. P. Morriss. A Boltzmann equation approach to the dynamics of the simple piston. *J. Statist. Phys.*, **113**(1-2):297–333, 2003.

[17] C. Gruber, S. Pache, and A. Lesne. Two-time-scale relaxation towards thermal equilibrium of the enigmatic piston. *J. Statist. Phys.*, **112**(5-6):1177–1206, 2003.

[18] C. Gruber, S. Pache, and A. Lesne. On the second law of thermodynamics and the piston problem. *J. Statist. Phys.*, **117**(3-4):739–772, 2004.

[19] O. Kreml, V. Mácha, Š. Nečasová and A. Wróblewska-Kamińska. Weak solutions to the full Navier-Stokes-Fourier system with slip boundary conditions in time dependent domains. *Journal de Mathématiques Pures et Appliquées.* available online September 2017.

[20] O. A. Ladyzhenskaja. An initial-boundary value problem for the Navier-Stokes equations in domains with boundary changing in time. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **11**:97–128, 1968.

[21] E. H. Lieb. Some problems in statistical mechanics that I would like to see solved. *Phys. A*, **263**(1-4):491–499, 1999. STATPHYS 20 (Paris, 1998).

[22] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models.* Oxford Science Publication, Oxford, 1998.

[23] D. Maity, T. Takahashi, and M. Tucsnak. Analysis of a system modelling the motion of a piston in a viscous gas. *J. Math. Fluid Mech.* **19** 3: 551–579, 2017.

[24] J. Neustupa. Existence of a weak solution to the Navier-Stokes equation in a general time-varying domain by the Rothe method. *Math. Methods Appl. Sci.*, **32**(6):653–683, 2009.

[25] J. Neustupa and P. Penel. A weak solvability of the Navier-Stokes equation with Navier's boundary condition around a ball striking the wall. In *Advances in mathematical fluid mechanics*, pages 385–407. Springer, Berlin, 2010.

[26] J. Neustupa and P. Penel. A weak solvability of the Navier-Stokes system with Navier's boundary condition around moving and striking bodies. *Recent Developments of Mathematical Fluid Mechanics*, Editors: Amann, H., Giga, Y., Kozono, H., Okamoto, H., Yamazaki, M. To appear in Birkhauser, series: Advances in Mathematical Fluid Mechanics.

[27] L. Poul. On dynamics of fluids in astrophysics. *J. Evol. Equ.* 9, 37–66, 2009.

[28] N. V. Priezjev and S.M. Troian. Influence of periodic wall roughness on the slip behaviour at liquid/solid interfaces: molecular versus continuum predictions. *J. Fluid Mech.*, **554**:25–46, 2006.

[29] J. Saal. Maximal regularity for the Stokes system on noncylindrical space-time domains. *J. Math. Soc. Japan*, **58**, 3: 617-641, 2006.

[30] V. V. Shelukhin. The unique solvability of the problem of motion of a piston in a viscous gas. *Dinamika Sploshn. Sredy*, (**31**):132–150, 1977.

[31] V. V. Shelukhin. Motion with a contact discontinuity in a viscous heat conducting gas. *Dinamika Sploshn. Sredy*, (**57**):131–152, 1982.

[32] Y. Stokes and G. Carrey. On generalised penalty approaches for slip, free surface and related boundary conditions in viscous flow simulation. *Inter. J. Numer. Meth. Heat Fluid Flow*, **21**:668–702, 2011.

[33] P. Wright. A simple piston problem in one dimension. *Nonlinearity*, **19**(10):2365–2389, 2006.

[34] P. Wright. The periodic oscillation of an adiabatic piston in two or three dimensions. *Comm. Math. Phys.*, **275**(2):553–580, 2007.

[35] P. Wright. *Rigorous results for the periodic oscillation of an adiabatic piston.* ProQuest LLC, Ann Arbor, MI, 2007. Thesis (Ph.D.)–New York University.