



INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

**Hairy AdS black holes with a toroidal
horizon in 4D Einstein–nonlinear
 σ -model system**

Marco Astorino

Fabrizio Canfora

Alex Giacomini

Marcello Ortaggio

Preprint No. 87-2017

PRAHA 2017

Hairy AdS black holes with a toroidal horizon in 4D Einstein–nonlinear σ -model system

Marco Astorino^{1*}, Fabrizio Canfora^{2†}, Alex Giacomini^{3‡}, Marcello Ortaggio^{4,3§}

¹*Universidad Adolfo Ibanez, Viña del Mar, Chile.*

²*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.*

³*Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile
Edificio Emilio Pugín, cuarto piso, Campus Isla Teja, Valdivia, Chile.*

⁴*Institute of Mathematics of the Czech Academy of Sciences
Žitná 25, 115 67 Prague 1, Czech Republic.*

Abstract

An exact hairy asymptotically locally AdS black hole solution with a flat horizon in the Einstein–nonlinear sigma model system in (3+1) dimensions is constructed. The ansatz for the nonlinear $SU(2)$ field is regular everywhere and depends explicitly on Killing coordinates, but in such a way that its energy-momentum tensor is compatible with a metric with Killing fields. The solution is characterized by a discrete parameter which has neither topological nor Noether charge associated with it and therefore represents a hair. A $U(1)$ gauge field interacting with Einstein gravity can also be included. The thermodynamics is analyzed. Interestingly, the hairy black hole is always thermodynamically favored with respect to the corresponding black hole with vanishing Pionic field.

*marco.astorino@gmail.com

†canfora@cecs.cl

‡alexgiacomini@uach.cl

§ortaggio@math.cas.cz

1 Introduction

The nonlinear sigma model is a useful theoretical tool with applications ranging from quantum field theory to statistical mechanics systems like quantum magnetism, the quantum Hall effect, meson interactions, super fluid ^3He , and string theory [1]. The most relevant application of the $SU(2)$ nonlinear sigma model in particle physics is the description of the dynamics of Pions at low energy in 3+1 dimensions (see for instance [2]; for a detailed review [3]). Consequently, the analysis of the coupling of the nonlinear sigma model to General Relativity is extremely important both from the theoretical and from the phenomenological point of view. On the other hand, due to the complexity of the field equations (which usually reduce to a non-linear system of coupled PDEs), the Einstein-nonlinear sigma model system has been analyzed mostly relying on numerical analyses (classic references are [7–11]).

However, the recent generalization of the boson star ansatz to $SU(2)$ -valued scalar fields (introduced in [12–22] and [23–25]) allows to construct also non-trivial gravitating soliton solutions (see in particular [13] and [19]) as well as to analyse explicitly thermodynamic properties of solitons (see [25]). These are necessary ingredients to build Pionic black holes with non-trivial thermodynamics.

Following the strategy devised in those references, we will construct a class of analytic black hole solutions of the Einstein- Λ -nonlinear sigma model system with intriguing geometrical properties. This family of black holes possesses a flat horizon⁵ and a discrete hairy parameter. The thermodynamics can be analysed explicitly. The first law is satisfied and an interesting feature of these black holes is that the hairy solution has always less free energy than the corresponding vacuum solution: thus, the present analysis suggests that the coupling of Pions with the gravitational field can act as a sort of catalysis for the Pions themselves. This family of black holes with flat horizons can also be generalized to the case in which there is a $U(1)$ gauge field coupled to Einstein gravity. The interesting thermodynamical features of these black holes remain in this case as well. This is qualitatively similar (in a phenomenological setting such as the Einstein-Pions system) to the recent findings of [26] in which a family of asymptotically flat black holes with non-Abelian hair has been presented which are thermodynamically favoured over the Reissner-Nordström solution.

The paper is organized as follows. In section 2, the Einstein-Nonlinear σ -model is introduced and a convenient parametrization is described. The exact hairy black hole solutions are constructed in section 3. In section 4, the thermodynamic behavior of the solutions is analyzed. In the last section 5, some conclusions are drawn.

2 The action

We consider the Einstein-non-linear sigma model system in four dimensions, with a possible cosmological constant. This describes the low-energy dynamics of pions, whose degrees of freedom are encoded

⁵Black holes with planar horizons have recently attracted a lot of attention due to their applications in holography, see, e.g., [27–29].

in an $SU(2)$ group-valued scalar field U [3]. The action of the system is

$$S = S_G + S_{\text{Pions}}, \quad (1)$$

where the gravitational action S_G and the nonlinear sigma model action S_{Pions} are given by

$$S_G = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (\mathcal{R} - 2\Lambda), \quad (2)$$

$$S_{\text{Pions}} = \frac{K}{4} \int d^4x \sqrt{-g} \text{Tr} (R^\mu R_\mu). \quad (3)$$

Here we have defined

$$R_\mu = U^{-1} \nabla_\mu U, \quad U \in SU(2), \quad (4)$$

while \mathcal{R} is the Ricci scalar, G is Newton's constant, Λ is the cosmological constant and the parameter K is positive. In our conventions $c = \hbar = 1$, the space-time signature is $(-, +, +, +)$ and Greek indices run over space-time. In the case of the non-linear sigma model on flat space-times, the coupling constant K is fixed experimentally. On the other hand, its true meaning in the context of AdS physics is to introduce a new length scale. Indeed, it is natural to expect that the ratio of this new length scale with the AdS radius will be a relevant quantity in the thermodynamics of the black holes which will be analyzed in the following sections. Our results confirm this expectation (we thank the anonymous referee for this comment).

The resulting Einstein equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (5)$$

where $G_{\mu\nu}$ is the Einstein tensor and the energy-momentum tensor is

$$T_{\mu\nu} = -\frac{K}{2} \text{Tr} \left(R_\mu R_\nu - \frac{1}{2} g_{\mu\nu} R^\alpha R_\alpha \right). \quad (6)$$

For the nonlinear $SU(2)$ sigma model, $T_{\mu\nu}$ can be seen to satisfy the dominant and strong energy conditions [30]. Finally, the matter field equations are

$$\nabla^\mu R_\mu = 0. \quad (7)$$

In the following, it will be useful to write R_μ as

$$R_\mu = i R_\mu^j \sigma_j, \quad (8)$$

where σ_j are the Pauli matrices. Furthermore, we adopt the standard parametrization of the $SU(2)$ -valued scalar $U(x^\mu)$

$$U^{\pm 1}(x^\mu) = Y^0(x^\mu) \mathbf{I} \pm i Y^j(x^\mu) \sigma_j, \quad (Y^0)^2 + Y^i Y_i = 1, \quad (9)$$

where \mathbf{I} is the 2×2 identity matrix. From (8) one thus finds

$$R_\mu^i = \varepsilon^{ijk} Y_j \nabla_\mu Y_k + Y^0 \nabla_\mu Y^i - Y^i \nabla_\mu Y^0. \quad (10)$$

Defining the quadratic combination

$$\mathcal{S}_{\mu\nu} := \delta_{ij} R_\mu^i R_\nu^j = \nabla_\mu Y^0 \nabla_\nu Y^0 + \nabla_\mu Y^i \nabla_\nu Y^i = G_{ij}(Y) \nabla_\mu Y^i \nabla_\nu Y^j, \quad (11)$$

with

$$G_{ij} := \delta_{ij} + \frac{Y_i Y_j}{1 - Y^k Y_k} \quad (12)$$

being the S^3 metric of the target space,⁶ we obtain that the action (3) reads

$$S_{\text{Pions}} = -K \int d^4x \sqrt{-g} \left[\frac{1}{2} G_{ij} (\nabla_\mu Y^i) (\nabla^\mu Y^j) \right], \quad (13)$$

while the energy-momentum tensor (6) takes the form

$$T_{\mu\nu} = K \left(\mathcal{S}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{S} \right). \quad (14)$$

The second of (9) means that $Y^I := (Y^0, Y^i)$ define a round unit 3-sphere in the internal space. A useful set of coordinates (H, A, G) in the internal space is defined by

$$\begin{aligned} Y^0 &= \cos H \sin A, & Y^1 &= \sin H \cos G, \\ Y^3 &= \cos H \cos A, & Y^2 &= \sin H \sin G, \end{aligned} \quad (15)$$

where $H \in [0, \pi/2]$, while $A \in [0, 2\pi k_1]$, $G \in [0, 2\pi k_2]$ are both periodic Killing coordinates of S^3 , with k_1 and k_2 positive integers (there is clearly some redundancy in this choice of the periodicities – the standard choice $k_1 = k_2 = 1$ already covers the whole S^3 – but this will become physically meaningful later on). In [19] and [25] it has been shown that similar parametrizations are extremely useful both in curved and in flat spaces. The tensor $\mathcal{S}_{\mu\nu}$ (11) takes the form

$$\mathcal{S}_{\mu\nu} = (\nabla_\mu H)(\nabla_\nu H) + \cos^2 H (\nabla_\mu A)(\nabla_\nu A) + \sin^2 H (\nabla_\mu G)(\nabla_\nu G). \quad (16)$$

If one defines the following combinations of the Killing coordinates

$$\Phi_+ = G + A, \quad \Phi_- = G - A, \quad (17)$$

⁶For the special field configuration $Y^0 = 0$, eq. (12) should be replaced by $G_{ij} := \delta_{ij}$. This case will not be considered in this paper.

the field equations (7) can be written in the compact form⁷

$$\square H - \frac{1}{2} \sin(2H) \nabla \Phi_+ \cdot \nabla \Phi_- = 0, \quad (18)$$

$$\sin(2H) \square \Phi_+ + 2 \nabla H \cdot [\nabla \Phi_- + \cos(2H) \nabla \Phi_+] = 0, \quad (19)$$

$$\sin(2H) \square \Phi_- + 2 \nabla H \cdot [\nabla \Phi_+ + \cos(2H) \nabla \Phi_-] = 0. \quad (20)$$

$SU(2)$ -valued scalar fields may possess a non-trivial topological charge which, mathematically, is a suitable homotopy class or winding number W . Its explicit expression as an integral over a suitable three-dimensional hypersurface Σ can be found, e.g., in [1]. However, in the present paper we will consider only configurations with $W = 0$.

3 Toroidal black hole solutions

3.1 Uncharged black hole

We now analyze configuration which are a natural non-topological generalization of the ansatz of [19, 25].

We will consider a static spacetime with a flat base manifold and the following diagonal metric

$$ds^2 = -f(r) dt^2 + h(r) dr^2 + r^2 (d\theta^2 + d\varphi^2). \quad (21)$$

In this geometry, for the matter field (15) we choose the “adapted” configuration

$$H = \frac{\pi}{4}, \quad A = \theta, \quad G = \varphi, \quad (22)$$

which is consistent with (15) if we set the periodicity of the angular spacetime coordinates as

$$\theta \in [0, 2\pi k_1], \quad \varphi \in [0, 2\pi k_2], \quad (23)$$

making the base manifold in (21) a (flat) 2-torus.⁸ It is easy to see that, thanks to (21) and (22), the field equations (18)–(20) are satisfied identically. In addition, by construction we have $W = 0$ (since the field (22) is 2-dimensional), as we required. We further note that this Pionic configuration has also zero Noether charge associated with the global isospin symmetry of the model. Indeed, the Noether charge is proportional to the spatial integral of the time-component of the Noether current associated with the global Isospin symmetry of the model. In the present case, due to the fact that the $SU(2)$ -valued field does not depend on time, its Noether charge vanishes identically.

⁷It is useful to recall the simple identity $(g^{\mu\nu} h f_{,\nu})_{;\mu} = \nabla h \cdot \nabla f + h \square f$.

⁸Here the integers k_1 and k_2 are used to define the identifications of points in the physical spacetime and are therefore not redundant anymore (as opposed to (15)). Also note that the cylindrical [and planar] topology is included in the formal limit $k_1 \rightarrow +\infty$ [and $k_2 \rightarrow +\infty$].

One still needs to solve Einstein's equations (5) with (14), which here becomes

$$T_{\mu\nu} = \frac{K}{2r^2}(f\delta_\mu^t\delta_\nu^t - h\delta_\mu^r\delta_\nu^r). \quad (24)$$

Eq. (5) is thus solved by (21) (up to a constant rescaling of t) with

$$\frac{1}{h(r)} = f(r) = -4\pi GK - \frac{\mu}{r} - \frac{\Lambda}{3}r^2, \quad (25)$$

where μ is an integration constant.

Geometric features of the spacetime are described more transparently if we introduce two “normalized” coordinates $x, y \in [0, 1]$ such that

$$\theta = 2\pi k_1 x, \quad \varphi = 2\pi k_2 y, \quad (26)$$

and additionally rescale

$$t \mapsto 2\pi t, \quad r \mapsto \frac{r}{2\pi}, \quad \mu \mapsto \frac{\mu}{(2\pi)^3}. \quad (27)$$

Defining the convenient parameters⁹

$$b^2 = 16\pi^3 GK, \quad \ell^2 = -\frac{3}{\Lambda}, \quad (28)$$

the final form of the solution is given by the field (22), (26) in the spacetime

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (k_1^2 dx^2 + k_2^2 dy^2), \quad (29)$$

$$f(r) = -b^2 - \frac{\mu}{r} + \frac{r^2}{\ell^2}. \quad (30)$$

Note that b^2 is not an integration constant, but is fixed by the coupling constants of the theory.

Let us mention a possible alternative parametrization of this solution. Rescaling

$$k_1 = \epsilon \tilde{k}_1, \quad k_2 = \epsilon \tilde{k}_2, \quad \mu = \tilde{\mu}/\epsilon^3, \quad r = \tilde{r}/\epsilon, \quad t = \epsilon \tilde{t}, \quad (31)$$

one obtains

$$ds^2 = -f(\tilde{r}) d\tilde{t}^2 + \frac{d\tilde{r}^2}{f(\tilde{r})} + \tilde{r}^2 (\tilde{k}_1^2 dx^2 + \tilde{k}_2^2 dy^2), \quad (32)$$

$$f(\tilde{r}) = -\epsilon^2 b^2 - \frac{\tilde{\mu}}{\tilde{r}} + \frac{\tilde{r}^2}{\ell^2}. \quad (33)$$

This allows one to set $\epsilon = 0$, if desired, thus recovering the known toroidal vacuum solution [31–35] – this limit is however only “formal”, since the Pionic field (22), (26) degenerates discontinuously to a single point in the internal space.

⁹Here we are interested in solutions in AdS, but in (25) Λ can take any sign or vanish.

From now on we stick to the parametrization (29), (30). The base space is a flat 2-torus with Teichmüller parameter ik_1/k_2 (cf., e.g., [35]). Metric (29) is similar to the well-known vacuum topological black holes, for which the base manifold can be spherical, hyperbolic or toroidal [31–35]. However, while in the absence of Pions the constant term in the lapse function is fixed by the curvature of the base manifold (i.e., $+1, -1, 0$ in the spherical, hyperbolic and toroidal case, respectively), in (30) it takes a negative value despite the fact that the base is flat. As a consequence, the base space area

$$\sigma = k_1 k_2, \quad (34)$$

plays the role of an extra (discrete) parameter that cannot be rescaled away by a redefinition of μ (as opposed to toroidal black holes in vacuum [31–35]). Recalling that the present Pionic configuration has neither topological nor Noether charge, σ can be considered as an integer Pionic hair of the black hole.

Because of the form of the lapse function (30), one can easily adapt to the present context the discussion of the causal structure and horizons of hyperbolic vacuum black holes given in [34, 35] (to which we refer for more details). First, since $r = 0$ is a curvature singularity, we restrict ourselves to the range $r > 0$. The discriminant of $f(r) = 0$ reads (up to an overall positive factor) $\Delta = 4\ell^2 b^6 - 27\mu^2$, while the three roots satisfy $r_1 r_2 r_3 = \mu \ell^2 > 0$ and $r_1 + r_2 + r_3 = 0$. It follows that there cannot be three real roots with the same sign, and that there are no real positive roots (only) when Δ and μ are both negative. To be more precise, let us define the (positive) critical value of μ (such that $\Delta = 0$ for $\mu = \pm\mu_c$, cf. [34, 35])

$$\mu_c = \frac{2\sqrt{3}}{9} \ell b^3. \quad (35)$$

There exists a unique, positive simple root r_+ for $\mu \geq 0$, two distinct positive roots r_+ and r_- (with $r_+ > r_-$) for $-\mu_c < \mu < 0$, a double positive root for $\mu = -\mu_c$, and no positive roots for $\mu < -\mu_c$, the latter case thus describing a naked singularity. The critical value $\mu = -\mu_c$ gives rise to a degenerate Killing horizon at $r = r_e \equiv \ell b / \sqrt{3}$, however this spacetime does not describe a black hole [34, 35]. In order to have a black hole solution, one thus needs to take $\mu > -\mu_c$. Therefore r_+ is a monotonically increasing function of μ and $r_+ > r_e$.

3.2 Charged black hole

The generalization of the above model to the presence of a $U(1)$ gauge field minimally coupled to General Relativity is direct. The main motivation for this extension consists in analyzing whether or not the thermodynamic behaviour disclosed in the Einstein-nonlinear sigma model system (discussed in section 4) resists after the inclusion of further reasonable matter fields.

More precisely, we add to the action (1) the electromagnetic term

$$S_{\text{Max}} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} (F_{\mu\nu} F^{\mu\nu}) , \quad (36)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which adds to the energy-momentum tensor (14) the usual Maxwell term

$$\frac{1}{4\pi G} \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right). \quad (37)$$

We will focus on static electric fields defined by the potential

$$A_\mu = \left(-\frac{q}{r}, 0, 0, 0 \right). \quad (38)$$

This form of A_μ has the property of being compatible with the metric (29), i.e., the Maxwell equations

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0$$

are automatically satisfied. Solving Einstein's equations with the ansatz (21), (22) and (38) shows that the contribution of the electric field consists just in an upgrading of the the lapse function (30) to

$$f(r) = -b^2 - \frac{\mu}{r} + \frac{r^2}{\ell^2} + \frac{q^2}{r^2}. \quad (39)$$

Of course, the presence of an electric field modifies the structure of the black hole horizons. In general in this charged case we have both an inner horizon r_- and an outer horizon r_+ , see [34] for a related discussion.

4 Thermodynamic behavior of the asymptotically locally AdS black hole solution

4.1 Uncharged black hole

Let us begin by analyzing the electrically uncharged case. We employ the Euclidean approach [37] and thus replace $t \mapsto it$ in (29). The temperature is given by the inverse of the Euclidean time period, i.e.,

$$T \equiv \beta^{-1} = \frac{1}{4\pi} f'(r_+) = \frac{1}{4\pi} \left(\frac{\mu}{r_+^2} + \frac{2r_+}{\ell^2} \right), \quad (40)$$

or, expressing $\mu = r_+(r_+^2/\ell^2 - b^2)$ as a function of r_+ , by

$$T = \frac{1}{4\pi} \left(\frac{-b^2}{r_+} + \frac{3r_+}{\ell^2} \right). \quad (41)$$

This is a monotonically increasing function of r_+ (and thus of μ), with $T \in [0, +\infty)$ vanishing for the extremal solution with $\mu = -\mu_c$. Inverting this relation admits only one positive root for r_+ , which expresses the horizon radius as a function of T

$$r_+ = \frac{\ell^2}{3} \left(2\pi T + \sqrt{4\pi^2 T^2 + \frac{3b^2}{\ell^2}} \right), \quad (42)$$

therefore no bifurcations of the free energy similar to the Hawking-Page [36] phase transition are possible here.

The Euclidean action (which we denote by I to distinguish it from its Lorentzian counterpart) is a sum of three contributions, namely

$$I = I_{bulk} + I_{surf} + I_{ct}, \quad (43)$$

with (recall (11)–(13))

$$I_{bulk} = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{g} \left(\mathcal{R} + \frac{6}{\ell^2} \right) + \frac{K}{2} \int_{\mathcal{M}} d^4x \sqrt{g} G_{ij} g^{\mu\nu} \partial_\mu Y^i \partial_\nu Y^j, \quad (44)$$

$$I_{surf} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{h} K_B, \quad (45)$$

where h_{ab} is the metric induced on the boundary $\partial\mathcal{M}$ at $r = r_B$, and K_B is the trace of the extrinsic curvature of $\partial\mathcal{M}$ as embedded in \mathcal{M} . For $r_B \rightarrow +\infty$, both I_{bulk} and I_{surf} are divergent when evaluated on our solutions, nevertheless (43) can be made finite by an appropriate definition of the counterterm action I_{ct} . The latter consists of boundary integrals involving scalars constructed only from boundary quantities. At this purpose, taking inspiration from [29], where similar matter field was studied, we naturally generalise the counterterm action to the more general collection of scalar fields, we are considering, in the following way

$$I_{ct} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^3x \sqrt{h} \frac{\ell}{2} \left(\mathcal{R}_B + \frac{4}{\ell^2} \right) - \frac{K}{2} \int_{\partial\mathcal{M}} d^3x \sqrt{h} \ell G_{ij} h^{ab} \partial_a Y^i \partial_b Y^j, \quad (46)$$

where \mathcal{R}_B is the Ricci scalar of the boundary metric. When the sigma model coupling of the scalar fields is trivialised to become a kinetic interaction, i.e. $G_{ij} = \delta_{ij}$, the axion counterterm of [29] is, indeed, recovered from (46). The AdS gravitational part of the above counterterms is well-known [38] (see also [39, 40]). Taking the limit $r_B \rightarrow +\infty$ gives the renormalized action

$$I = -\frac{\beta\sigma r_+}{16\pi G \ell^2} (r_+^2 + b^2 \ell^2). \quad (47)$$

It is easy to see that $I \sim T^2$ for a large T (similarly as for hyperbolic vacuum black holes [35]).

The energy $E = -\partial_\beta \log Z$, the entropy $S = (1 - \beta \partial_\beta) \log Z$ and the free energy $F = -T \log Z$ of the solutions in the semiclassical approximation $\log Z \approx -I$ thus read

$$E = \frac{\sigma r_+}{8\pi G} \left(\frac{r_+^2}{\ell^2} - b^2 \right) = \frac{\sigma \mu}{8\pi G}, \quad S = \frac{\sigma r_+^2}{4G}, \quad F = -\frac{\sigma r_+}{16\pi G \ell^2} (r_+^2 + b^2 \ell^2). \quad (48)$$

The area law is thus clearly obeyed. By defining the the mass as

$$M = E, \quad (49)$$

it is easy to check with (41) that the first law of black hole thermodynamics is satisfied, i.e.,

$$T\delta S = \frac{\sigma}{8\pi G} \left(-b^2 + \frac{3r_+^2}{\ell^2} \right) \delta r_+ = \delta M . \quad (50)$$

The local thermodynamic stability with respect to thermal fluctuations is given by the positivity of the heat capacity function C , i.e.,

$$C := T \left(\frac{\partial S}{\partial T} \right) = T \left(\frac{\partial S}{\partial r_+} \right) \left(\frac{\partial T}{\partial r_+} \right)^{-1} = \frac{r_+^2 \tau (3r_+^2 - b^2 \ell^2)}{2G(b^2 \ell^2 + 3r_+^2)} . \quad (51)$$

Since black hole solutions satisfy $r_+ > \frac{b\ell}{\sqrt{3}}$ (section 3.1), they are stable under temperature fluctuation (this is also true in the absence of the Pionic matter, cf. [34,35]). In other words, M is a monotonically increasing function of T (see also (52) below).

We further note that (49) gives a partly negative mass spectrum (similarly as obtained in [39] for hyperbolic vacuum black holes – cf. [34,35] for a different approach). Note also that M and S depend not only on the radial length r_+ (i.e., on the integration constant μ), but also on the angular periods via σ . It can thus be seen that, for a given M , larger entropies are attained for larger (discrete) values of σ . In addition, for a given T (i.e., for a fixed r_+) there exists a discrete infinity of black holes parametrized by σ , for which the functions (48) take different values, namely

$$M = \frac{\ell\sigma}{108\pi G} \left(2\pi T\ell + \sqrt{4\pi^2 T^2 \ell^2 + 3b^2} \right) \left(4\pi^2 T^2 \ell^2 + 2\pi T\ell \sqrt{4\pi^2 T^2 \ell^2 + 3b^2} - 3b^2 \right) , \quad (52)$$

$$S = \frac{\ell^4 \sigma}{9G} \left(2\pi^2 T^2 + \pi T \sqrt{4\pi^2 T^2 + \frac{3b^2}{\ell^2}} + \frac{3b^2}{4\ell^2} \right) , \quad (53)$$

$$F = -\frac{\ell\sigma}{108\pi G} \left(2\pi T\ell + \sqrt{4\pi^2 T^2 \ell^2 + 3b^2} \right) \left(3b^2 + 2\pi^2 T^2 \ell^2 + \pi T\ell \sqrt{4\pi^2 T^2 \ell^2 + 3b^2} \right) . \quad (54)$$

At high temperatures, $M \sim \sigma T^3$, $S \sim \sigma T^2$ and $F \sim -\sigma T^3$.

It is interesting to compare this black hole solution dressed with a matter field with the corresponding vacuum solution (i.e., with $b = 0$). The entropy of the toroidal vacuum black hole is [34,35]

$$S_0 = \frac{\ell^4 \sigma}{9G} 4\pi^2 T^2 . \quad (55)$$

Clearly $S_0 < S$, i.e., at equal temperature the vacuum black hole has always lower entropy than the dressed solution. In order to see which black hole is thermodynamically favored at a given temperature, it is necessary to compare the respective free energies. From (54), it is obvious that the free energy is more negative for $b^2 \neq 0$, i.e., the solution with Pionic matter fields is thermodynamically favored at any temperature.

This result is remarkable, since usually black holes dressed with matter fields in asymptotically locally AdS spacetimes are thermodynamically favored over their matter-free counterparts only for low temperatures (see, e.g., [41,42]). Recently, results qualitatively similar to ours have been found in [26].

However, this reference obtained numerically solutions of a higher order version of Yang-Mills theory minimally coupled to General Relativity which are asymptotically flat – here we have constructed analytic results for Pions minimally coupled with AdS gravity.

4.2 Charged black hole

The extension to the electrically charged case is straightforward. Now the lapse function is given by (39). The electric charge reads $Q = \frac{q\sigma}{4\pi}$ and the mass is the same as in the uncharged case. Expressing $M = M(r_+, Q)$ as a function of the horizon radius and of the electric charge, one readily finds that the first law of thermodynamics

$$\delta M = T\delta S + \Phi\delta Q \quad (56)$$

is again fulfilled. The Coulomb electric potential Φ at the horizon is defined, as usual, as

$$\Phi = \chi^\mu A_\mu \Big|_{r_\infty} - \chi^\mu A_\mu \Big|_{r_+} = \frac{q}{r_+} \quad , \quad (57)$$

where χ^μ represents the Killing vector ∂_t , while the temperature of the charged solution is

$$T = \frac{1}{4\pi} \left(\frac{\mu}{r_+^2} + \frac{2r_+}{\ell^2} - \frac{q^2}{r_+^3} \right) = \frac{1}{4\pi r_+^3} \left(\frac{3r_+^4}{\ell^2} - b^2 r_+^2 - q^2 \right). \quad (58)$$

In order to check the thermodynamic stability of this solution, we compare the free energy of the electrically charged black hole with Pionic field with the electrically charged black hole without Pionic field. The free energy in the grand canonical ensemble, i.e. when both the temperature T and the Coulomb potential Φ are considered fixed, for the charged pionic solution is given by the Gibbs free energy

$$F = M - TS - \Phi Q \quad . \quad (59)$$

When μ is expressed in terms of r_+ , the free energy becomes

$$F(r_+, q) = -\frac{r_+^4 + \ell^2(q^2 + b^2 r_+^2)}{16\pi r_+ \ell^2} \sigma \quad . \quad (60)$$

In the electrically charged case, it is convenient to write the free energy as a function of the intensive parameters such as the temperature T and the Coulomb potential Φ . This can be done inverting eqs. (57) and (58). Again, there is only one positive root of the radial position of the event horizon as a function of the temperature $r_+(T)$

$$r_+(T) = \frac{2}{3}\pi\ell^2 T + \frac{\ell}{3}\sqrt{4\pi\ell^2 T^2 + 3(b^2 + \Phi^2)}. \quad (61)$$

Finally the Gibbs free energy reads as follows

$$F(T, \Phi) = \frac{-\sigma\ell}{108\pi} \left(2\pi\ell T + \sqrt{4\pi^2\ell^2 T^2 + 3b^2 + 3\Phi^2} \right) \left[3b^2 + \pi\ell T \left(2\pi\ell T + \sqrt{4\pi^2\ell^2 T^2 + 3b^2 + 3\Phi^2} \right) \right]. \quad (62)$$

This expression for the free energy is the generalisation of the uncharged one, given in eq. (54), which can be straightly recovered in the vanishing electric potential limit.

Again, the free energy of the dressed black hole turns out to be more more negative at the same temperature and electric potential, than the one of the black hole without Pionic matter field, as can be seen from eq. (62) and in Figure 1. Therefore the charged pionic solution is always preferred.

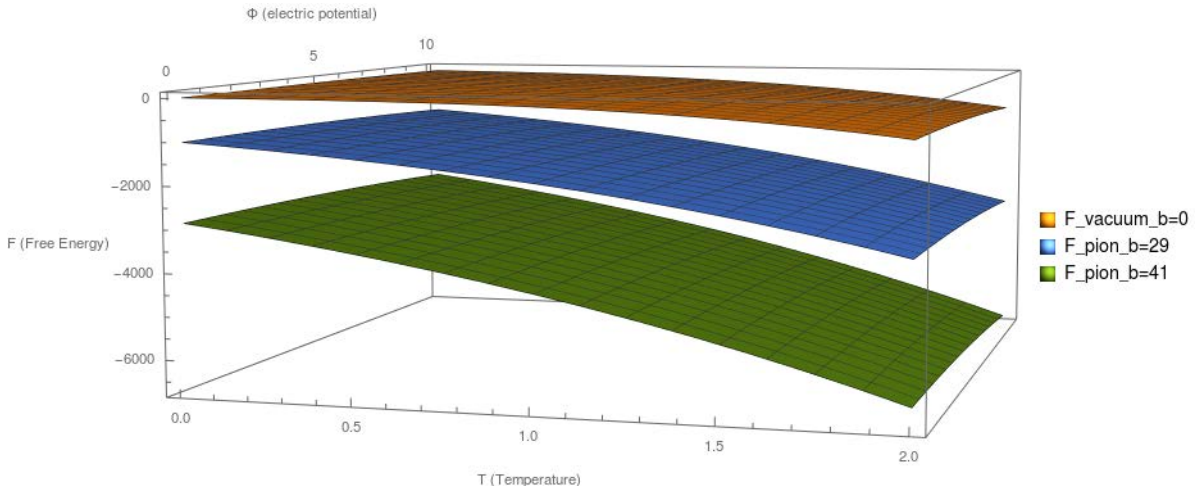


Figure 1: Gibbs free energy $F(T, \Phi)$ comparison between the electrovacuum (yellow) and the pionic (blue and green) black holes as function of the temperature T and electric potential Φ for some fixed numerical values of the cosmological constant and the base manifold area $\ell = \pi, \sigma = 1$. The picture does not change qualitatively for different values of the chosen pionic parameter b (chosen as 29 and 41 respectively in blue and the green surfaces): the hairy configuration is always preferred at a given temperature and electric potential since its free energy is lower.

5 Conclusions

We constructed exact static solutions of the four-dimensional Einstein theory, minimally coupled to the nonlinear sigma model. A $U(1)$ gauge field interacting with Einstein gravity can also be included. The ansatz for the nonlinear sigma model depends explicitly on the Killing coordinates, but in such a way that its energy-momentum tensor is compatible with the Killing fields of the metric. The most interesting result is the construction of an analytic hairy black hole sourced by a clouds of Pions in the Einstein non-linear sigma model system with a negative cosmological constant. Such an asymptotically locally AdS black hole has a flat horizon. The hairy parameter is discrete due to the $SU(2)$ structure of the matter field – this is thus an explicit example of a discrete primary hair. The thermodynamic analysis reveals that the hairy black hole is always favoured with respect to the black hole without the Pions cloud. The inclusion of a $U(1)$ gauge field minimally coupled to General Relativity does not break this picture. This is a highly phenomenological realization in the Einstein-Pions system of

a behaviour similar to the recent findings of [26] in a different context.

Acknowledgements

M.O. is grateful to Roberto Emparan for a useful discussion. This work has been partially funded by the FONDECYT grants 1160137 (F.C.), 1150246 (A.G.), 11160945 (M.A.), the CONICYT grant DPI20140053 (A.G.) and research plan RVO: 67985840 and research grant GAČR 14-37086G (M.O.). M.A. is supported by Conicyt - PAI grant n° 79150061. The stay of M.O. at Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile has been supported by CONICYT PAI ATRACCIÓN DE CAPITAL HUMANO AVANZADO DEL EXTRANJERO Folio 80150028. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

References

- [1] N. Manton and P. Sutcliffe, *Topological Solitons*, Cambridge University Press, Cambridge, 2007.
- [2] J. Gasser, H. Leutwyler, *Nucl. Phys. B* **250**, 465 (1985).
- [3] V. P. Nair, “*Quantum Field Theory: A Modern Perspective*”, Springer (2005).
- [4] G.H. Derrick, *J. Math. Phys.* **5**: 1252–1254 (1964).
- [5] D. J. Kaup, *Phys. Rev.***172**, 1331–1342 (1968).
- [6] S. L. Liebling, C. Palenzuela, *Living Rev. Relativity* **15**, (2012), 6.
- [7] S. Droz, M. Heusler, and N. Straumann, *Phys. Lett. B* 268, 371 (1991).
- [8] H. Luckock and I. Moss, *Phys. Lett. B* 176, 341 (1986).
- [9] S. Droz, M. Heusler, and N. Straumann, *Phys. Lett. B* 271, 61 (1991).
- [10] N. K. Glendenning, T. Kodama, and F. R. Klinkhamer, *Phys. Rev. D* 38, 3226 (1988); B.M.A.G. Piette and G. I. Probert, *Phys. Rev. D* 75, 125023 (2007); G.W. Gibbons, C. M. Warnick, and W.W. Wong, *J. Math. Phys. (N.Y.)* 52, 012905 (2011); S. Nelmes and B. M. A. G. Piette, *Phys. Rev. D* 84, 085017 (2011).
- [11] P. Bizon and T. Chmaj, *Phys. Rev. D* 58, 041501 (1998); P. Bizon, T. Chmaj, and A. Rostworowski, *Phys. Rev. D* 75, 121702 (2007); S. Zajac, *Acta Phys. Pol. B* 40, 1617 (2009); 42, 249 (2011).
- [12] F. Canfora, P. Salgado-Rebolledo, *Phys. Rev. D* **87**, 045023 (2013).
- [13] F. Canfora, H. Maeda, *Phys. Rev. D* **87**, 084049 (2013).
- [14] F. Canfora, *Phys. Rev. D* **88**, 065028 (2013).

- [15] F. Canfora, F. Correa, J. Zanelli, *Phys. Rev. D* **90**, 085002 (2014).
- [16] S. Chen, Y. Li, Y. Yang, *Phys. Rev. D* **89** (2014), 025007.
- [17] F. Canfora, M. Kurkov, M. Di Mauro, A. Naddeo, *Eur.Phys.J.* **C75** (2015) 9, 443.
- [18] S. Chen, Y. Yang, *Nucl. Phys. B* **904** (2016) 470.
- [19] E. Ayon-Beato, F. Canfora, J. Zanelli, *Phys. Lett. B* **752**, (2016) 201-205.
- [20] F. Canfora, G. Tallarita, *Phys. Rev. D* **94** (2016), 025037.
- [21] F. Canfora, G. Tallarita, *Phys. Rev. D* **91** (2015), 085033.
- [22] F. Canfora, G. Tallarita, *JHEP* 1409 (2014) 136.
- [23] F. Canfora, A. Paliathanasis, T. Taves, J. Zanelli, *Phys. Rev. D* **95** (2017), 065032.
- [24] F. Canfora, G. Tallarita, *Nucl. Phys. B* **921** (2017) 394.
- [25] P. Alvarez, F. Canfora, N. Dimakis, A. Paliathanasis, arXiv:1707.07421, *Phys. Lett. B* **773** (2017) 401-407.
- [26] C. Herdeiro, V. Paturyan, E. Radu, D. H. Tchraikian, *Phys. Lett. B* **772**, (2017), 63-69
- [27] T. Andrade and B. Withers, *JHEP*05 (2014) 101
- [28] B. Gouteraux, *JHEP*04 (2014) 181
- [29] M. M. Caldarelli, A. Christodoulou, T. Papadimitriou and K. Skenderis, *JHEP*1704 (2017) 001
- [30] G.W. Gibbons, *Phys. Lett. B***566**, 171 (2003)
- [31] J. P. S. Lemos, *Class. Quantum Grav.*, 12 (1995) 1081
- [32] C.-g. Huang and C.-b. Liang, *Phys. Lett. A*, 201 (1995) 27
- [33] R. B. Mann, *Class. Quantum Grav.* 14 (1997) L109
- [34] D. Brill, J. Louko and P. Peldán, *Phys. Rev D*56 (1997) 3600
- [35] L. Vanzo, *Phys. Rev. D*56 (1997) 6475
- [36] S. W. Hawking and D. N. Page, *Commun. Math. Phys.* 87 (1083) 577
- [37] G. W. Gibbons and S. W. Hawking, *Phys. Rev. D* 15 (1977) 2738
- [38] V. Balasubramanian and P. Kraus, *Commun. Math. Phys.* 208 (1999),413
- [39] R. Emparan, C. V. Johnson, and R. C. Myers, *Phys. Rev. D* 60 (1999) 104001

- [40] R. B. Mann, Phys. Rev. D, 60 (1999) 104047
- [41] S. S. Gubser, Phys. Rev. Lett. 101 (2008) 191601
- [42] C. Martínez, R. Troncoso, and J. Zanelli, Phys. Rev. D 70 (2004) 084035
- [43] H. A. Gonzalez, M. Hassaine and C. Martinez, Phys. Rev. D80 (2009) 104008