Weak Solutions for the Compressible Navier-Stokes Equations: Existence, Stability, and Longtime Behavior

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Abstract

This double-sized chapter contains two related themes that were supposed to be covered by two independent chapters of the handbook in the original project: (1) weak solutions of the Navier-Stokes equations in the barotropic regime and (2) weak solutions of the Navier-Stokes-Fourier system.

We shall discuss for both systems:

(1) Various notions of weak solutions, their relevance, and their mutual relations.
(2) Global existence of weak solutions.
(3) Notions of relative energy functional, dissipative solutions and relative energy inequality and its impact on the investigation of the stability analysis of compressible flows.
(4) Weak strong uniqueness principle.
(5) Longtime behavior of weak solutions.

For physical reasons, we shall limit ourselves to the three-dimensional physical space.

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1 Introduction

1.1 Weak Solutions

The Navier-Stokes-Fourier system is a system of partial differential equations describing evolution of density $\varrho = \varrho(t,x)$, absolute temperature $\vartheta = \vartheta(t,x)$, and velocity $\mathbf{u} = \mathbf{u}(t,x)$ of a viscous compressible and heat-conducting fluid filling domain Ω ($x \in \Omega$) within the time interval $t \in [0,T)$. There are several ways to define weak solutions for the complete Navier-Stokes-Fourier system. Here, we shall mention three of them: the convenience of each definition depends on the mathematical assumptions that one imposes on the constitutive laws for pressure (internal energy) on one hand and on the transport coefficients on the other hand. Indeed, the weak formulation of the momentum and continuity equations is standard, while for the weak formulation of the energy conservation, one has at least three reasonable options that are not equivalent within the class of irregular solutions: (1) formulation in terms of the internal energy, (2) formulation in terms of the specific entropy, and (3) formulation in terms of the total energy.

The first and second one are continuations of the theories based on the so-called effective viscous flux identity started by P.L. Lions [77], and the third one, due to Bresch and Desjardins [7], can be considered as a continuation of theories based on new a priori estimates in the line started by Kazhikov [72].

The first approach due to Feireisl [30] based on a weak formulation of the continuity, momentum, and internal energy equations is convenient for the pressure p and internal energy e of type

$$p(\varrho,\vartheta) = p_c(\varrho) + \vartheta\, p_{\rm th}(\varrho),\;\; p_c(\varrho) \approx_\infty \varrho^\gamma,\;\; \gamma>3/2 \tag{1}$$

$$e(\varrho,\vartheta) = e_c(\varrho) + e_{\rm th}(\vartheta),\;\; e_c(\varrho) = \int_1^\varrho \frac{p(z)}{z^2}{\rm d}z,\;\; e_{\rm th}(\vartheta) \approx \vartheta^{\omega+1},\;\; \omega \geq 0,$$

where $p_{\rm th}$ must be monotone and dominated by a certain power of $p_c(\varrho)$ for large ϱ's (more precisely by $p_c^{1/3}(\varrho)$). Here, ϑ denotes the temperature, ϱ denotes the density, and γ is the adiabatic coefficient of the fluid. The heat conductivity in this approach has to be temperature dependent (with a convenient power growth), and the viscosity coefficients have to be constant.

The second approach was developed in [33] preceded by a compactness result in [32]. It exploits the observation of Ducomet and Feireisl [20, 21] on the regularizing effect of the radiative pressure on the weak solutions of the magnetohydrodynamic equations. It involves, besides the standard weak formulation of continuity and momentum equations, the weak formulation of the conservation of energy in terms of the specific entropy that includes explicitly the second law of thermodynamics via the entropy production rate being a nonnegative measure. This approach is applicable for the pressure and internal energy laws $p(\varrho,\vartheta)$, $e(\varrho,\vartheta)$, exhibiting the coercivity of types ϱ^γ and ϑ^4 for large densities and temperatures; a prototype example is

$$p(\varrho,\vartheta) \approx \varrho^\gamma + \vartheta p_{\rm th}(\varrho) + \vartheta^4, \tag{2}$$
$$e(\varrho,\vartheta) \approx \varrho^\gamma + \vartheta^{\omega+1} + \frac{\vartheta^4}{\varrho},\ \omega \geq 0,$$

where $p_{\rm th}$ is the same as in (1). The viscosity coefficients in this theory are in general temperature dependent and have to behave like $(1+\vartheta)^\beta$; the heat conductivity has to behave like $(1+\vartheta)^\alpha$, where loosely speaking $\alpha > 0$ has to be larger when $\beta \geq 0$ becomes smaller. For example, for the pressure law of monoatomic gas with radiation that behaves like $\varrho^{5/3}$ (for large ϱ's and ϑ fixed) and like ϑ^4 (for large ϑ's and ϱ fixed) – see Sect. 2.4, in particular (47)–(49) – the theory gives $\beta \in [2/5, 1]$ within physically reasonable value $\alpha = 3$ (see [33]), while for the pressure law of type (2) obeying the above asymptotic condition for $p_{\rm th}$, one can achieve values $\beta \in [0, 4/3]$ provided $\alpha = \frac{16}{3} - \beta$ (see [32]). If $\gamma > 3$ one can achieve values $-4 \leq \beta \leq 0$ provided $\alpha \geq \frac{16}{3} + |\beta|$. The latter situation corresponds rather to compressible fluids than to gasses (see [58]).

Both above formulations are sufficiently weak to allow *existence of variational solutions for large data* and reasonable in the sense that *any sufficiently regular weak solution is a classical solution*.

The second formulation including balance of entropy as the pointwise conservation of energy in the weak formulation has an important advantage in comparison with the first formulation based on the pointwise energy conservation in terms of the internal energy balance. Indeed, in the second approach, the thermodynamic stability conditions can be reformulated in terms of an evolution variational inequality (called *relative energy inequality*) governing a specific functional called *relative energy functional*, which is able to measure a "distance" between a weak solution $(\varrho,\vartheta,\mathbf{u})$ and any other (sufficiently regular) state of the fluid $(r,\Theta,\mathbf{U})$. This inequality is automatically satisfied by *any weak solution* based on the *balance of entropy* (see [34] and [49, 50] for the barotropic case). It appears that the *relative* *energy inequality* encodes most of the stability properties of compressible fluids and is, in fact, responsible for the robustness of this type of weak solutions with respect to perturbations of initial conditions and external forces as well as with respect to singular limits involving various physically reasonable small parameters appearing in the nondimensional formulation of the Navier-Stokes-Fourier system.

The third approach due to Bresch and Desjardins [7, 8] (see also Mellet, Vasseur [83]) is convenient in the case when the shear viscosity μ and the bulk viscosity η depend on the density and satisfy the differential identity

$$(\eta - \frac{2}{3}\mu)'(\varrho) = 2\varrho\mu'(\varrho) - 2\mu(\varrho),$$

and pressure is in the form (1), where however $p_c(\varrho)$ is singular at $\varrho \to 0$. The main ingredient in the proof in this situation is the fact that the particular relation between viscosities stated above makes possible to establish a new mathematical entropy identity, which provides estimates for the gradient of density. This estimate implies compactness of the sequence of approximating densities. In spite of the compactness, the construction of the solutions in this situation is a tough problem. It was so far possible under the additional nonphysical assumption that p_c explodes at the vacua. Only recently, two preprints [75, 105] appeared suggesting an explicit construction of the global solutions in the "simple" barotropic case in the physically reasonable situation when the cold pressure p_c is not singular at zero.

In this chapter we shall concentrate to the first two formulations; the third formulation is investigated in a separate chapter of the handbook.

In the mathematical literature, there is another notion of weak solutions to the compressible Navier-Stokes equations due to D. Hoff [64–68] and references quoted there. Hoff's solutions must have essentially bounded density, but discontinuities are allowed. Solutions in Hoff's class are almost unique (see [68]). A drawback is that their existence is guaranteed only for small initial data. They will be treated in a separate chapter of the handbook.

1.1.1 Lions' Approach and Feireisl's Approach

The concept of weak solutions in fluid dynamics was introduced in 1934 by Leray [74] in the context of incompressible Newtonian fluids. It has been extended more than 60 years later to the Newtonian compressible fluids in barotropic regime (meaning that $p = p(\varrho) \approx \varrho^\gamma$) by Lions [77].

The Lions theory relies on two crucial observations:

(1) A discovery of a certain weak continuity property of the quantity

$$p(\varrho) + (\frac{4}{3}\mu + \eta)\mathrm{div}\mathbf{u}$$

called effective viscous flux. This part is essential for the existence proof; it employs certain cancelation properties that are available due to the structure of the equations that are mathematically expressed through a commutator involving density, momentum, and the Riesz operator. The main ideas related to the effective viscous flux identity will be explained in Sect. 8 (namely, in Sect. 8.4).

(2) Theory of renormalized solutions to the transport equation that P.L. Lions introduced together with DiPerna in [18]. In the context of compressible Navier-Stokes equations, the DiPerna-Lions transport theory applies to the continuity equation. The theory asserts among others that the limiting density is a renormalized solution to the continuity equation provided it is square integrable. This hypothesis is satisfied only provided $\gamma \geq 9/5$. The condition on the squared integrability of the density is the principal obstacle to the improvement of the Lions result.

Notice that some indications on the particular importance of the effective viscous flux were known at about the same time to several authors and used in different problems dealing with small data (see Hoff [64] and Padula [86]) and that the suggestion to use the continuity equations to evaluate the oscillations in the sequence of approximating densities has been formulated and performed in the one-dimensional case by D. Serre [97].

All physically reasonable adiabatic coefficients γ for gases belong to the interval $(1, 5/3)$, the value $\gamma = 5/3$ being reserved for the monoatomic gas. This is the reason why it is interesting and important to relax the condition on the adiabatic coefficient in the Lions theory. This has been done by Feireisl et al. in [47]. The new additional aspects of this extension are based on the previous observations by Feireisl in [27] and are the following:

(1) As suggested in [27], the authors have used the oscillations defect measure to evaluate the oscillations in the sequence of approximating densities and proved that it is bounded provided $\gamma > 3/2$. This part of the proof will be discussed in detail in Sect. 8 (namely, Sect. 8.5).
(2) The boundedness of the oscillations defect measure is a criterion that replaces the condition of the squared integrability of the density in the DiPerna-Lions transport theory. Consequently if any term of the sequence of approximating densities satisfies the renormalized continuity equation, and if the oscillations defect measure of this sequence is bounded, then the weak limit of the sequence is again a renormalized solution of the continuity equation. This property is discussed in detail in Sect. 8.6.

Recently the authors of so far unpublished paper [9] suggested an alternative way to the Lions' approach of measuring of oscillations in the density sequence, which promises to be slightly more robust than the Lions' approach.

1.1.2 Weak Solutions for the Complete Navier-Stokes-Fourier System

The existence theory for the complete Navier-Stokes-Fourier system (with possibly temperature-dependent viscosities) employs both Lions' and Feireisl's techniques.

Most of its additional difficulties dwell in the possible existence of vacuum regions in relation to the temperature approximations. In what follows, we describe general ideas on how these problems can be solved:

First approach.

(1) The procedure to prove strong convergence of the approximated density sequence ϱ_n via the Lions-Feireisl approach involves solely continuity and momentum equations. The weak limit of the sequence $\vartheta_n p_{\rm th}(\varrho_n)$ can be identified with expression $\vartheta \overline{p_{\rm th}(\varrho)}$, where ϑ is a weak limit of the approximated temperature sequence ϑ_n and $\overline{p_{\rm th}(\varrho)}$ is the weak limit of the sequence $p_{\rm th}(\varrho_n)$. After this observation, the Lions-Feireisl method including effective viscous flux identity bound for the oscillations defect measure and renormalized continuity equation can be performed exactly as in the barotropic case, leading to the almost everywhere convergence of ϱ_n to a function $\varrho \geq 0$. After this observation, the problem is reduced to the limit passage in the internal energy balance. The details are described in Sections 8.2, 8.3, 8.4, 8.5, and 8.6.

(2) In this case the internal energy balance provides an estimate of $\partial_t(\varrho\vartheta^{\omega+1})$ (and not of $\partial_t\vartheta^{\omega+1}$). Loosely speaking, this estimate eliminates possible oscillations outside vacua in the set $\{(t,x) \in Q_T | \varrho(t,x) > 0\}$ ($Q_T = (0,T)\times\Omega$), but, unfortunately, does not discard oscillations on the vacuum set $\{(t,x) \in Q_T | \varrho(t,x) = 0\}$ which can be of nonzero measure. Consequently we can reasonably hope to obtain almost everywhere convergence of the approximated temperature sequence ϑ_n to $\tilde{\vartheta}$ on the set $\{(t,x)|\varrho(t,x) > 0\} \subset Q_T$. This observation in combination with the almost everywhere convergence of density established in *item* (1) allows to pass to the limit in all terms of the weak formulation of the internal energy balance containing multiples of ϱ.

(3) The term corresponding to the heat flux $\mathrm{div}\mathbf{q}(\vartheta, \nabla_x\vartheta)$ can be written in the form $-\Delta\mathcal{K}(\vartheta)$ with convenient strictly monotone function $\mathcal{K}$ provided the heat flux $\mathbf{q}$ is given by the Fourier law with the coefficient of heat conductivity dependent only on temperature. The available estimates provide a weak limit $\overline{\mathcal{K}(\vartheta)}$ of the sequence $\mathcal{K}(\vartheta_n)$ in $L^1(Q_T)$. One can now define a new temperature $\vartheta = \mathcal{K}^{-1}(\overline{\mathcal{K}(\vartheta)})$ that is equal to the almost everywhere limit $\tilde{\vartheta}$ of the approximated temperature sequence on the set $\{(t,x)|\varrho(t,x) > 0\}$ established in *item* (2). (In the real proof, the sequence $\mathcal{K}(\vartheta_n)$ is bounded only in $L^1(Q_T)$ which does not prevent concentrations. One can however perform the proof by using convenient truncations of $\mathcal{K}$ using a procedure reminiscent to Chacon's biting limit see [11]).

(4) Fortunately, the above term is the only term in the internal energy balance (except the term involving $\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}$, whose limit passage can be treated by the lower weak semi-continuity provided the stress tensor obeys, e.g., the Navier-Stokes law for the Newtonian fluids) which is not a multiple of ϱ. Therefore, we can replace in all remaining terms the temperature $\tilde{\vartheta}$ by

the new temperature ϑ. A detailed development of ideas described in *items* (2), (3), and (4) is available in Sect. 8.8.

Second approach.

(1) In order to reduce the investigation to a situation similar to the barotropic case, one has to prove first the almost everywhere convergence of the approximated temperature sequence. In contrast with the previous case, this seems to be possible, thanks to the presence of radiation energy. Indeed the energy conservation allows to estimate $\partial_t(\vartheta^4)$ and thus to discard the possible time oscillations in the approximated temperature sequence. Since in this setting we are dealing with entropy balance rather than with the energy balance, this point involves the treatment of the entropy production rate as a Radon measure and a convenient use of the compensated compactness, namely, of the Div-Curl lemma in combination with the theory of parametrized Young measures. The crucial condition allowing to conclude is the monotonicity of the entropy with respect to temperature.

(2) Even after the strong convergence of temperature is known, the weak continuity of the effective viscous flux is not an obvious issue. It requires to use another cancelation property that is mathematically expressed through another commutator including shear viscosity, symmetric velocity gradient, and the Riesz operator. The ideas described in *items* (1) and (2) are treated in Sects. 10.1 and 10.2.

(3) Once the weak continuity property of the effective viscous flux is known, the proof follows the lines of Lions' and Feireisl's approaches: (a) one proves first the boundedness of the oscillations defect measure for the sequence of densities; (b) the boundedness of oscillations defect measure implies that the limiting density is a renormalized solution to the continuity equation; and (c) the renormalized continuity equation is used to show that the oscillations in the density sequence do not increase in time. This means the strong convergence of density. The details to this part of the proof are available in Sect. 10.3.

Weak solutions for the compressible barotropic equations are introduced in Sect. 4 along with the main existence results and their qualitative properties, while those for the complete Navier-Stokes-Fourier system are introduced in Sects. 7 and 9. We provide the detailed description of the main ideas of the existence proofs of weak solutions in Sects. 8 and 10.

Weak solutions in the theory of compressible Navier-Stokes equations are usually constructed via several levels of approximations including small parameters via suitable approximating system of PDEs. Construction of weak solutions through numerical schemes is a very recent topic which goes out of scope of this chapter. The reader can consult [55, 71], or monograph [56] for the recent development in this subject.

1.2 Relative Energy and Robustness of the Class of Weak Solutions

Weak solutions are not known to be uniquely determined (cf., e.g., exposition of Fefferman [26] dealing with three-dimensional incompressible Navier-Stokes equations) and may exhibit rather pathological properties (see, e.g., Hoff and Serre [69]). So far, the best property that one may expect in the direction of a unique result is the weak-strong uniqueness, meaning that any weak solution coincides with the strong solution emanating from the same initial data as long as the latter exists. The weak-strong uniqueness principle is known for the incompressible Navier-Stokes equations since the works of Prodi [95] and Serrin [98] (see [25] for the later development). About 50 years later, the weak-strong uniqueness problem has been revisited by Desjardins [17] and Germain [61] for the compressible Navier-Stokes equations. They obtained some partial and conditional results. Finally, the unconditional weak-strong uniqueness principle has been proved in [50] (see also related paper [49]).

Only very recently the weak-strong uniqueness property has been proved in [34] for weak solutions of the complete Navier-Stokes-Fourier system in the entropy formulation introduced in [33].

In all cases cited above, the weak-strong uniqueness principle has been achieved by the method of *relative energy* that is reminiscent to the *relative entropy method.* Relative entropy method was brought to the mathematical fluid mechanics by C. Dafermeos [16] and has been used later in various contexts by different authors (see [77], Saint-Raymond [96], Grenier [63], Masmoudi [80], Ukai [103], Wang and Jiang [107], among others). The notion of dissipative solutions introduced in Lions [76] for the incompressible Euler equations is very much related to the concept of relative entropies.

Regardless the fact that [16] is about conservation laws (disregarding the dissipation) while [33] includes dissipative effects, the main difference between the relative energy and relative entropy methods is the following: the starting point of [16] (in the case of complete Euler system) is the *balance of internal energy*, and the output is the *relative entropy inequality*, while the starting point in [33] is the balance of entropy and the output is the *relative energy inequality*. The procedure suggested in [16] cannot be repeated in the context of *weak solutions* to the Navier-Stokes-Fourier system unless one supposes additionally that the density and temperature are bounded from below by positive constants. It is however not known whether the latter condition is satisfied globally in time for any weak solution.

The relative energy method is introduced in Sects. 5 and 11.

We have already mentioned that the relative energy inequality encodes most of the robustness properties of the weak solutions to the compressible Navier-Stokes-equations and to the Navier-Stokes-Fourier system. Let us mention a few applications:

(1) If one takes for the test state $(r, \Theta, \mathbf{U})$ a strong solution in the relative energy inequality, one obtains a stability estimate of a strong solution (emanating from initial data $(r_0, \Theta_0, \mathbf{U}_0)$ and external force $\mathbf{g}$) within the class of weak solutions (emanating from initial data $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ and external force $\mathbf{f}$), in terms of difference of the external forces and relative energy of the initial data. This statement yields, in particular, the weak-strong uniqueness principle saying that the weak solution coincides with the strong solution as long as the strong solution exists, provided both solutions emanate from the same initial data and external forces (see again [34,49,50] for the barotropic case). These applications will be investigated in Sects. 5 and 11.
(2) The large time behavior of weak solutions, namely, convergence to the equilibrium states in the case of conservative forces, energy blow up in the case of nonconservative forces, and questions related to the bounded absorbing sets and attractors can be treated on the basis of the relative energy inequality (see [44] and references quoted there). These applications are investigated in Sects. 6 and 12.
(3) There is a bunch of applications of the relative energy inequality related to the investigation of singular limits in the nondimensional version of the compressible Navier-Stokes equations and the Navier-Stokes-Fourier system involving various combinations of low Mach, Froude, Rossby, Péclet numbers, and large Reynolds number toward reduced target systems as long as we know that the target system admits a regular solution (at least locally in time). Practically all so far rigorously obtained singular limits within the complete Navier-Stokes-Fourier system have been obtained by the relative energy method. Another family of problems, where the relative energy inequality appeared to be a crucial tool, are limits connected to dimension reduction. We refer to [3, 5, 35–38, 52, 79, 100] for a few examples to some of these applications. The problem of the singular limits in the compressible Navier-Stokes equations will be discussed in another two independent chapters of the handbook.
(4) The numerical version of the relative energy inequality is employed in [60] to investigate the error estimates of numerical schemes solving the compressible Navier-Stokes equations. The reader can consult also, e.g., [54, 57] among others, for the recent developments of this subject. These applications go far beyond the scope of this handbook.

The chapter is organized as follows. We start with a short introduction to the thermodynamics of viscous fluids (Sect. 2) followed by a review section collecting the most important specific mathematical tools for the treatment of compressible Navier-Stokes equations (Sect. 3). Sections 4, 4, and 6 are devoted to the compressible Navier-Stokes equations in barotropic regime (treating the notions of weak solutions, finite and bounded energy weak solutions, renormalized weak solutions, dissipative solutions, relative energy inequality, weak-strong uniqueness, and longtime behavior). The same issue is then revisited for the full Navier-Stokes-Fourier system through Sects. 7, 8, 9, 10, 11, and 12.

2 Thermodynamics of Viscous Compressible Fluids

We shall describe the motion of a compressible, *viscous and heat-conducting fluid* sometimes called also *a viscous gas*. For simplicity, we suppose that the fluid fills a fixed domain $\Omega \subset \mathbb{R}^3$, and we shall investigate its evolution through an (arbitrary) large time interval $(0, T)$. We denote by $Q_T = (0, T) \times \Omega$ the space-time cylinder. The motion will be described by means of three basic state variables: the mass density $\varrho = \varrho(t, x)$, the velocity field $\mathbf{u} = \mathbf{u}(t, x)$, and the absolute temperature $\vartheta = \vartheta(t, x)$, where $t \in (0, T)$ is the time variable and $x \in \Omega \subset \mathbb{R}^3$ is the space variable in the Eulerian coordinate system. The physical nature of density and temperature requires that the density is nonnegative function on Q_T, and the absolute temperature is positive function on Q_T. We shall investigate the time evolution of these quantities. It is described by the balance laws of physics expressed through the following partial differential equations:

(i) *Conservation of mass*

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0. \tag{3}$$

(ii) *Conservation of linear momentum*

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) + \varrho \mathbf{f}. \tag{4}$$

(iii) *Conservation of internal energy – first law of thermodynamics*

$$\begin{aligned} \partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta) + p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u} \\ = \mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}. \end{aligned} \tag{5}$$

In these equations $p = p(\varrho, \vartheta)$ is the *pressure*, $e = e(\varrho, \vartheta)$ is the *(specific) internal energy*, $\mathbb{S} = \mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u})$ is the *viscous stress tensor*, and $\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta)$ is the *heat flux*. They are given functions characterizing the gas. The quantity $\mathbf{f} = \mathbf{f}(t, x)$ is a given function expressing the specific external forces. For the sake of simplicity, we do not consider the external heat sources.

In physics, there are at least two another ways of writing the conservation of energy (5): in terms of the *specific total energy* and in terms of the *specific entropy*.

Formulation of the first law in terms of the kinetic energy. The *specific total energy* is the sum of *specific kinetic energy* $e_{\mathrm{kin}} = \frac{1}{2}\mathbf{u}^2$ and the specific internal energy $e(\varrho, \vartheta)$

$$e_{\mathrm{tot}}(\varrho, \mathbf{u}) = \frac{1}{2}\mathbf{u}^2 + e(\varrho, \vartheta). \tag{6}$$

Due to (3)–(5), it must obey equation

$$\partial_t(\varrho e_{\rm tot}(\varrho,\vartheta)) + {\rm div}_x\Big(\Big(\varrho e_{\rm tot}(\varrho,\vartheta) + p(\varrho,\vartheta)\Big)\mathbf{u}\Big) + {\rm div}_x\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta) \tag{7}$$

$$= {\rm div}_x\Big(\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u})\cdot\mathbf{u}\Big) + \varrho\mathbf{f}\cdot\mathbf{u}.$$

Formulation of the first law in terms of the specific entropy. The *second law of thermodynamics* postulates existence of the *specific entropy* $s = s(\varrho,\vartheta)$ defined by the *Gibbs relation*

$$\vartheta {\rm d}s(\varrho,\vartheta) = {\rm d}e(\varrho,\vartheta) - \frac{p(\varrho,\vartheta)}{\varrho^2}{\rm d}\varrho \tag{8}$$

that must obey the *balance of entropy* equation

$$\partial_t(\varrho s(\varrho,\vartheta)) + {\rm div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + {\rm div}_x\left(\frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)}{\vartheta}\right) = \sigma, \tag{9}$$

where the quantity σ must be nonnegative. It is called the *entropy production rate*. In the present situation,

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right). \tag{10}$$

If p, e, $\mathbb{S}$, $\mathbf{q}$ are differentiable functions of their respective arguments, if density ϱ and temperature ϑ are positive and sufficiently smooth on Q_T, and if the velocity field $\mathbf{u}$ is sufficiently smooth on Q_T, then equations (5), (7), and (9)–(10) are equivalent. This equivalence does not need to be necessarily true if the functions above do not possess enough regularity.

Therefore, in spite of the fact that weak formulation of the balance of energy based on each of equations (5), (7), and (9), respectively, is equally physically justifiable, it may lead to weak solutions with different properties. It may happen that some of the possible definitions of weak solutions may be more advantageous in some situations and may even lead to global in time existence results, while other definition will fail to have this property, depending on the flow regimes and constitutive laws characterizing the gas.

If $\varrho > 0$ on Q_T and ϱ, $\mathbf{u}$ belong to $C^1(Q_T)$, then the continuity equation is equivalent to the family of so-called *renormalized continuity equations*:

$$\partial_t b(\varrho) + {\rm div}_x\Big(b(\varrho)\mathbf{u}\Big) + \Big(\varrho b'(\varrho) - b(\varrho)\Big){\rm div}_x\mathbf{u} = 0 \text{ for all } b \in C^1(0,\infty). \tag{11}$$

Again, if the couple $(\varrho,\mathbf{u})$ does not possess enough regularity, this property does not need to be true, in general.

2.1 Navier-Stokes-Fourier System

We suppose that the viscous stress $\mathbb{S}$ is described by Newton's law

$$\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u})=\mu(\varrho,\vartheta)\mathbb{T}(\nabla_x\mathbf{u})+\eta(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}\mathbb{I},\ \mathbb{T}(\nabla_x\mathbf{u})=\nabla_x\mathbf{u}+(\nabla_x\mathbf{u})^T-\frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{12}$$

where $\mathbb{I}$ is the identity tensor, while $\mathbf{q}$ is the heat flux satisfying Fourier's law

$$\mathbf{q} = -\kappa(\varrho,\vartheta)\nabla_x\vartheta. \tag{13}$$

The quantities μ, η, and κ are called transport coefficients, more specifically, shear and bulk viscosities, and heat conductivity, respectively. According to the second thermodynamical law, they have to be all nonnegative. We are however dealing with viscous and heat conducting fluids; we shall therefore always suppose that the transport coefficients satisfy at least

$$\mu(\varrho,\vartheta) > 0,\ \ \eta(\varrho,\vartheta) \geq 0,\ \ \kappa(\varrho,\vartheta) > 0, \tag{14}$$

and we shall assume the following minimal regularity,

$$(\mu,\eta,\kappa) \in C^1([0,\infty)^2). \tag{15}$$

The system of equations (3)–(5) (where (5) may be replaced by (7) or by (9)–(10)) with the constitutive relations (12) and (13) is called *Navier-Stokes-Fourier system*.

Physical considerations suggest that the heat conductivity behaves

$$\kappa(\vartheta) \approx \vartheta^\alpha,\ \ \alpha \geq 3\ \ \text{for large values of } \vartheta \tag{16}$$

due to the radiation effects. The approximation of viscosity coefficients by constants

$$\mu > 0, \quad \eta \geq 0 \tag{17}$$

is considered in many situations as satisfactory. The kinetic theory predicts

$$\mu(\vartheta) \approx \sqrt{\vartheta},\ \text{for large values of } \vartheta \tag{18}$$

(see [108]).

2.2 Domain, Conservative Boundary Conditions and Initial Data

2.2.1 Initial Data

Equations (3)–(5) are supplemented with initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ \varrho\mathbf{u}(0,\cdot) = \varrho_0\mathbf{u}_0,\ \varrho e(\varrho,\vartheta)(0,\cdot) = \varrho_0 e(\varrho_0,\vartheta_0),\ \varrho_0 \geq 0, \vartheta_0 > 0, \tag{19}$$

where ϱ_0, ϑ_0, and $\mathbf{u}_0$ are given functions.

2.2.2 Boundary Conditions

We shall always assume that Ω has globally uniformly Lipschitz boundary. If Ω is bounded, we will deal with *no-slip* boundary conditions for velocity

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{20}$$

and zero heat transfer conditions through the boundary

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{21}$$

where $\mathbf{n}$ denotes the external normal to the boundary $\partial\Omega$ of Ω. The *no-slip* boundary conditions can be replaced in many cases by the *complete slip* boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \mathbb{S}\mathbf{n}\times\mathbf{n}|_{\partial\Omega} = 0, \tag{22}$$

or with *Navier's slip* boundary conditions

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \Big(\mathbb{S}\mathbf{n}\times\mathbf{n}+\Lambda(\mathbf{u}\times\mathbf{n})\Big)|_{\partial\Omega} = 0, \text{ where } \Lambda \geq 0 \text{ is the friction coefficient.} \tag{23}$$

If Ω is an unbounded domain, one has to prescribe in addition to boundary conditions (20), resp. (22), resp. (23), and (21) also the behavior at infinity,

$$\varrho(t,x)\to\varrho_\infty\geq 0, \quad \mathbf{u}(t,x)\to\mathbf{u}_\infty\in\mathbb{R}^3, \quad \vartheta(t,x)\to\vartheta_\infty>0 \tag{24}$$

in some sense, as $|x|\to\infty$.

2.2.3 Global Conservation Properties

Suppose now that the domain Ω is bounded (and sufficiently smooth). Integrating equation for the conservation of global energy under conditions (23), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)\right)\,\mathrm{d}x + \Lambda\int_{\partial\Omega}|\mathbf{u}|^2\mathrm{dS}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x, \tag{25}$$

provided the trio $(\varrho,\vartheta,\mathbf{u})$ is sufficiently smooth in $\overline{Q_T}$; in particular, in the case of boundary conditions (20) and (22), the *total energy* of the system in the volume Ω is conserved, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)\right)\,\mathrm{d}x = \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x. \tag{26}$$

Under the same smoothness requirement, multiplying equation (9) by a positive constant $\overline{\Theta}$, integrating over Ω, and subtracting the result from equations (25) and (26), we get the *dissipation identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2+H_{\overline{\Theta}}(\varrho,\vartheta)\right)\,\mathrm{d}x+\overline{\Theta}\int_\Omega\sigma\mathrm{d}x+\Lambda\int_{\partial\Omega}|\mathbf{u}|^2\mathrm{dS}x=\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x,\tag{27}$$

respectively,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2+H_{\overline{\Theta}}(\varrho,\vartheta)\right)\,\mathrm{d}x+\overline{\Theta}\int_\Omega\sigma\mathrm{d}x=\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x,\tag{28}$$

where the quantity

$$H_\Theta(\varrho,\vartheta)=\varrho\Big(e(\varrho,\vartheta)-\Theta e(\varrho,\vartheta)\Big)\tag{29}$$

is called *Helmholtz function* or *ballistic free energy*. It plays an essential role in the stability analysis of weak solutions.

2.3 Thermodynamic Stability Conditions

The fluid characterized by the pressure $p(\varrho,\vartheta)$ and internal energy $e(\varrho,\vartheta)$ verifies the *thermodynamic stability conditions* if

$$\frac{\partial p(\varrho,\vartheta)}{\partial\varrho}>0,\ \frac{\partial e(\varrho,\vartheta)}{\partial\vartheta}>0\text{ for all }\varrho,\vartheta>0.\tag{30}$$

We easily verify by using Gibbs' relation (8) that

$$\frac{\partial H_{\overline{\vartheta}}}{\partial\vartheta}(\varrho,\vartheta)=\varrho\frac{\vartheta-\overline{\vartheta}}{\vartheta}\frac{\partial e}{\partial\vartheta}(\varrho,\vartheta)\text{ and }\frac{\partial^2H_{\overline{\vartheta}}}{\partial\varrho^2}(\varrho,\overline{\vartheta})=\frac{1}{\varrho}\frac{\partial p}{\partial\varrho}(\varrho,\overline{\vartheta}).\tag{31}$$

Thus, the thermodynamic stability in terms of the function $H_{\overline{\vartheta}}$ can be reformulated as follows:

$$\varrho\mapsto H_{\overline{\vartheta}}(\varrho,\overline{\vartheta})\text{ is strictly convex,}\tag{32}$$

while

$$\vartheta\mapsto H_{\overline{\vartheta}}(\varrho,\vartheta)\text{ attains its global minimum at }\vartheta=\overline{\vartheta}.\tag{33}$$

We notice that if the second thermodynamical condition is satisfied, then the map $\vartheta\mapsto s(\varrho,\vartheta)$ is for any ϱ a (strictly) increasing function of temperature; therefore it admits a limit as $\vartheta\to 0+$ that is 0 or $-\infty$ (after choosing adequately the constant of integration).

2.4 Constitutive Relations

We shall primarily assume a certain minimal regularity of constitutive laws for pressure and internal energy,

$$p \in C^1([0,\infty)\times[0,\infty)), \quad e \in C^1((0,\infty)\times[0,\infty)) \tag{34}$$

We shall always assume that the gas obeys the *second law of thermodynamics* expressed through the Gibbs relation (8) postulating existence of the specific entropy; in particular, it must obey relation

$$\partial_\varrho e(\varrho,\vartheta) = \frac{1}{\varrho^2}\Big(p(\varrho,\vartheta) - \vartheta\partial_\vartheta p(\varrho,\vartheta)\Big), \tag{35}$$

called *Maxwell's relation*.

There are several families of constitutive laws enjoying physical justification and allowing for the satisfactory theory of weak solutions. They can be written down in the following framework

$$p(\varrho,\vartheta) = p_{\rm ra}(\vartheta) + p_{\rm mo}(\varrho,\vartheta) + p_{el}(\varrho), \tag{36}$$

where the indexes "ra," "mo," and "el" refer to "radiative," "molecular," and "elastic" (pressure), respectively. Correspondingly, the internal energy reads

$$e(\varrho,\vartheta) = \frac{1}{\varrho}e_{\rm ra}(\vartheta) + e_{\rm mo}(\varrho,\vartheta) + e_{\rm el}(\varrho), \tag{37}$$

where we have to take

$$e_{\rm ra}(\vartheta) = \vartheta p'_{\rm ra}(\vartheta) - p_{\rm ra}(\vartheta), \quad e_{\rm el}(\varrho) = \int_1^\varrho \frac{p_{\rm el}(z)}{z^2}{\rm d}z$$

in order to comply with Maxwell's relation (35). Under these assumptions, the specific entropy reads

$$s(\varrho,\vartheta) = \frac{1}{\varrho}s_{\rm ra}(\vartheta) + s_{\rm mo}(\varrho,\vartheta) + s_{\rm el}(\varrho), \tag{38}$$

and the Helmholtz function is

$$H_\Theta = H_{{\rm ra},\Theta} + H_{{\rm mo},\Theta} + H_{\rm el}, \tag{39}$$

According to (35), the radiative entropy reads

$$s_{\rm ra}(\vartheta) - s_{\rm ra}(1) = \int_1^\vartheta \frac{e'_{\rm ra}(z)}{z}{\rm d}z. \tag{40}$$

Consequently the radiative Helmholtz function is given by

$$H_{\text{ra},\Theta}(\varrho,\vartheta) = H_{\text{ra},\Theta}(\vartheta) = \int_1^\vartheta \frac{e'_{\text{ra}}(z)}{z}(z-\Theta)\mathrm{d}z + H_{\text{ra},\Theta}(1). \tag{41}$$

The contribution of the elastic components of pressure and internal energy to the specific entropy and to the Helmoholtz function is

$$s_{\text{el}}(\varrho,\vartheta) = 0, \quad H_{\text{el}}(\varrho,\vartheta) = H_{\text{el}}(\varrho) = \varrho \int_1^\varrho \frac{p_{\text{el}}(z)}{z^2}\mathrm{d}z = \varrho e_{\text{el}}(\varrho), \tag{42}$$

respectively, again by virtue of relation (35). In particular,

$$\varrho H'_{\text{el}}(\varrho) - H_{\text{el}}(\varrho) = p_{\text{el}}(\varrho), \tag{43}$$

and in view of (11) function $(t,x) \mapsto H_{\text{el}}(\varrho(t,x))$ verifies

$$\partial_t H_{\text{el}}(\varrho) + \mathrm{div}_x(H_{\text{el}}(\varrho)\mathbf{u}) + p_{\text{el}}(\varrho)\mathrm{div}_x\mathbf{u} = 0. \tag{44}$$

We shall consider two families of molecular pressure constitutive laws:

1. *Real gas phenomenological constitutive laws*
   The molecular pressure and internal energy in many real gases enter into the following general framework

   $$p_{\text{mo}}(\varrho,\vartheta) = \vartheta p_{\text{th}}(\varrho), \quad e_{\text{mo}}(\varrho,\vartheta) = e_{\text{th}}(\vartheta). \tag{45}$$

   In this situation, the specific entropy reads

   $$s_{\text{mo}}(\varrho,\vartheta) = s_{\text{mo},\vartheta}(\vartheta) + s_{\text{mo},\varrho}(\varrho), \quad s_{\text{mo},\vartheta}(\vartheta) = \int_1^\vartheta \frac{e'_{\text{th}}(z)}{z}\mathrm{d}z,$$
   $$s_{\text{mo},\varrho}(\varrho) = -\int_1^\varrho \frac{p_{\text{th}}(z)}{z^2}\mathrm{d}z \tag{46}$$

   and the Helmholtz function is

   $$H_{\text{mo},\Theta} = \varrho\Big(e_{\text{mo}}(\vartheta) - \Theta\int_1^\vartheta \frac{e'_{\text{mo}}(z)}{z}\mathrm{d}z\Big) + \varrho\int_1^\varrho \frac{p_{\text{th}}(z)}{z^2}\mathrm{d}z.$$

2. *Constitutive laws derived in the statistical mechanics*
   They take the general form

   $$p_{\text{mo}}(\varrho,\vartheta) = \vartheta^{\gamma/(\gamma-1)} P\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right), \ \gamma > 1, \tag{47}$$

where

$$P \in C^1[0,\infty). \tag{48}$$

In agreement with Gibbs' relation (8), the (specific) internal energy must be taken as

$$e_{\rm mo}(\varrho,\vartheta) = \frac{1}{\gamma-1}\frac{\vartheta^{\gamma/(\gamma-1)}}{\varrho} P\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right). \tag{49}$$

In this case, the specific entropy reads

$$s_{\rm mo}(\varrho,\vartheta) = S\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right), \quad \text{where } S'(Z) = -\frac{1}{\gamma-1}\frac{\gamma P(Z) - P'(Z)Z}{Z^2}. \tag{50}$$

The reader may consult Eliezer, Ghatak, and Hora [23] and [33, Chapter 3] for the physical background and further discussion concerning the structural hypotheses (47), (48), and (49).

We shall proceed to several concrete examples.

Radiative pressure

The radiative pressure and energy are given by the Stefan-Boltzmann law:

$$p_{\rm ra}(\varrho,\vartheta) = p_{\rm ra}(\vartheta) = \frac{a}{3}\vartheta^4,$$

$$e_{\rm ra}(\varrho,\vartheta) = \frac{a}{\varrho}\vartheta^4 \text{ where } a > 0 \text{ is the Stefan-Boltzmann constant;} \tag{51}$$

consequently we deduce from (40) and (41),

$$s_{\rm ra}(\varrho,\vartheta) = \frac{4}{3}\frac{a}{\varrho}\vartheta^3, \;\; H_{{\rm ra},\Theta}(\vartheta) = a(\vartheta^4 - \frac{4}{3}\Theta\vartheta^3). \tag{52}$$

Examples of real gas phenomenological molecular pressure constitutive laws

Perfect gas – Boyle's law. For the perfect gas,

$$p_{\rm mo}(\varrho,\vartheta) = R\varrho\vartheta, \quad e_{\rm mo}(\varrho,\vartheta) = \underline{c_v}\vartheta^p, \quad p \geq 1 \tag{53}$$

where $R > 0$ is universal gas constant and $\underline{c_v} > 0$; we have for the specific entropy

$$s_{\rm mo}(\varrho,\vartheta) = \left\{ \begin{array}{cc} \underline{c_v}\ln\vartheta - R\ln\varrho, & \text{if } p = 1, \\ \underline{c_v}\frac{p}{p-1}(\vartheta^{p-1} - 1) - R\ln\varrho & \text{if } p > 1 \end{array} \right\}, \tag{54}$$

and

$$H_{\mathrm{mo},\Theta}(\varrho,\vartheta) = \left\{ \begin{array}{cc} \underline{c_v}\varrho\Big((\vartheta - \Theta\ln\vartheta) + \gamma\ln\varrho\Big) & \text{if } p = 1, \\ \underline{c_v}\varrho\Big(\vartheta^p + \frac{p}{p-1}(\vartheta^{p-1} - 1)\Theta + \gamma\ln\varrho\Big) & \text{if } p > 1 \end{array} \right\}. \tag{55}$$

Real gases – virial series. According to Becker [2, Chapter 10], the pressure in the real gas can be expressed through the so-called virial series that takes the form

$$p(\varrho,\vartheta) = R\vartheta\varrho + \sum_{i=1}^{n} B_i(\vartheta)\varrho^i, \quad n \in N.$$

One of the best approximations of this form is the so-called Beattie-Bridgeman state equation (see [106, Sections 3.4, 10.10, Chapter 10] for more details).

Mie-Gruneisen equations of state are of the form

$$p(\varrho,\vartheta) = p_c(\varrho) + \varrho\vartheta G(\varrho,\vartheta),$$

where $p_c(\varrho)$ refers to the "cold" pressure (see [13, 99] for more details).

Examples of molecular pressure constitutive laws from statistical mechanics
In formulas (47), (48), and (49), at least two values of γ are considered to be physically reasonable.

Monoatomic gas. For monoatomic gases, $\gamma = 5/3$.
Relativistic gas. For the so-called relativistic gas, $\gamma = 4/3$.

See [23, Chapter 3] for more details.

Examples of elastic pressure

Nuclear fluids. In a simplified model of nuclear fluids, the molecular pressure is given by the Boyle's law while there is an elastic pressure being composed of two terms:

$$p_{\mathrm{el}}(\varrho) = c_1\varrho^{5/3} + \Big(c_2\varrho^3 - c_3\varrho^2\Big), \quad c_1, c_2, c_3 > 0,$$

where the first term is the so-called Thomas-Fermi-Weizsacker approximation while the second term comes from the so-called Skyrme interaction (see [19]).

Perfect gas in isentropic regime. Supposing that the gas evolves in the regime with the constant entropy $\overline{s}$, we may deduce from $(54)_{p=1}$ and (53)

$$p_{\rm mo}(\varrho,\vartheta) \equiv p_{\rm el}(\varrho) = b\varrho^{\gamma},\ b = Re^{\overline{s}} > 0,\ \gamma = \frac{R+c_v}{c_v}.$$

This is the *pressure law for isentropic gas*. The values of γ (that is called adiabatic constant) ranges in the interval $(1, \frac{5}{3})$. The value $\gamma = 5/3$ corresponds to the *isentropic flows of monoatomic gas*.

2.5 Constraints Imposed by Thermodynamic Stability Conditions

The elastic pressure satisfies thermodynamic stability conditions if and only if

$$p'_{\rm el}(\varrho) > 0 \text{ for all } \varrho > 0. \tag{56}$$

The molecular pressure and internal energy given by formula (45) satisfy thermodynamic stability conditions if and only if

$$p'_{\rm th}(\varrho)>0 \text{ for all } \varrho>0, \quad e'_{\rm th}(\vartheta)>0 \text{ for all } \vartheta>0. \tag{57}$$

Likewise, the pressure and internal energy given by formulas (47), (48), and (49) satisfy thermodynamic stability conditions if and only if

$$P'(Z) > 0, \quad \frac{\gamma P(Z) - P'(Z)Z}{Z} > 0 \quad \text{for all } Z > 0. \tag{58}$$

First point to be noticed at this moment is that by virtue of (58), the function $Z \mapsto P(Z)/Z^{\gamma}$ must be decreasing on $(0,\infty)$ and therefore

$$\lim_{Z\to\infty} P(Z)/Z^{\gamma} = p_{\infty} \in [0,\infty). \tag{59}$$

Second point is that under the thermodynamic stability conditions, function $Z \mapsto S(Z)$ is decreasing on interval $(0,\infty)$ in view of (50); it may be chosen by means of a convenient additive constant in such a way that

$$\lim_{Z\to\infty} S(Z) = S_{\infty},\ \text{where } S_{\infty} = 0 \text{ or } S_{\infty} = -\infty. \tag{60}$$

2.6 Third Law of Thermodynamics

The *third thermodynamical law* postulates that

$$\lim_{\vartheta\to 0+} s(\varrho,\vartheta) = 0 \text{ for all } \varrho > 0. \tag{61}$$

We notice that the perfect gas whose state equation is given by the Boyle's law does not obey the third law (see formula (54)). The gases of mechanical statistics whose pressure and internal energy are given by formulas (47)–(49) obey the third law provided S can be taken (by choosing the integration constant in (50)) in such a way that

$$\lim_{Z\to\infty} S(Z) = 0. \tag{62}$$

The third law imposes further constraints on the constitutive laws in extreme regimes close to values $\vartheta = 0$. It is usually not necessary for building up the existence theory (at least on bounded domains). It may however play an important role when one investigates the stability issues.

2.7 Barotropic Flows

A fluid flow is said to be in *barotropic* regime or the fluid is said to be barotropic if the pressure p depends solely on the density. This can be achieved if we take in (36), $p_{\rm ra} = 0$, and molecular pressure/internal energy given by (45) with $p_{\rm th}(\varrho) = 0$. We thus get

$$p(\varrho) = p_{\rm el}(\varrho), \quad e(\varrho, \vartheta) = e_{\rm el}(\varrho) + e_{\rm th}(\vartheta).$$

Supposing moreover that the viscous stress $\mathbb{S}$ is independent on the absolute temperature, system (3)–(9) in this situation reads

$$\partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = 0, \tag{63}$$

$$\partial_t(\varrho \mathbf{u}) + {\rm div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = {\rm div}_x \mathbb{S}(\varrho, \nabla_x \mathbf{u}) = \varrho \mathbf{f}, \tag{64}$$

$$\partial_t(\varrho e_{\rm th}(\vartheta)) + {\rm div}_x(\varrho e_{\rm th}(\vartheta)\mathbf{u}) + {\rm div}_x \mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta) = \mathbb{S}(\varrho, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}, \tag{65}$$

where we have used identity (44) in order to transform (5) to (65). We observe that equation (65) and system (63)–(64) are decoupled in the sense that once the couple $(\varrho, \mathbf{u})$ is determined from equations (63)–(64), temperature ϑ can be obtained by solving (65) with boundary conditions (21).

Moreover, taking a scalar product of equation (64) with $\mathbf{u}$ and integrating over Ω (under the assumption of enough smoothness of $\varrho, \mathbf{u}$ and positivity of ϱ) yields

$$\partial_t \int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\Big)\, {\rm d}x + \int_\Omega \mathbb{S}(\varrho, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\, {\rm d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\, {\rm d}x, \tag{66}$$

where

$$H(\varrho) = H_{\rm el}(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} {\rm d}z, \tag{67}$$

provided the boundary conditions for velocity are conservative as those exposed in (20) or (22). This equation replaces for the barotropic flows the global dissipation identity (28) valid for the (regular) heat-conducting flows.

System of partial differential equations (63) and (64) is called *compressible Navier-Stokes equations* in *barotropic regime*. It does not describe fully satisfactorily physically realistic situations. However, it is *consistent* with thermodynamics, and it already contains pretty much of the mathematical difficulties encountered when dealing with the full Navier-Stokes-Fourier system. Its investigation is not only of independent interest, but it can be used as a preliminary toy problem before attacking the full system.

The most usual examples of barotropic flows are *isothermal flows* where

$$p(\varrho) = R\overline{\vartheta}\varrho$$

describing the flows of the perfect gas with the constant temperature $\overline{\vartheta} > 0$ and the *isentropic flows*

$$p(\varrho) = Re^{\overline{s}}\varrho^{\gamma}, \quad \gamma = \frac{R + c_v}{c_v}$$

describing the flows of the perfect gas with the constant entropy $\overline{s} \in \mathbb{R}$. Notice, however, that the requirements of constant temperature or constant entropy violate conservation of energy (65) unless specific external heat sources are not added to (65).

3 Specific Mathematical Tools for Compressible Fluids

We shall gather in this section most of mathematical tools needed to investigate weak solutions to the compressible Navier-Stokes equations or to the Navier-Stokes-Fourier system. As far as the notations are concerned, we employ standard notation commonly used in the mathematical analysis and in the theory of partial differential equations, as in the books [30, 33, 59, 88, 102].

3.1 Instantaneous Values of Functions in $L^\infty(0,T;L^1(\Omega))$

Theorem on Lebesgue points (see, e.g., [10, Appendix]) says that for any $v \in L^1(0,T;X)$, X a Banach space, there exists $\tilde{v}^\pm \in L^1(0,T;X)$ such that:

(i)

$$\text{For a. a. } \tau \in (0,T), \lim_{h\to 0+} \frac{1}{h}\int_{B^\pm(\tau;h)} \|v(t) - \tilde{v}^\pm(\tau)\|_X \mathrm{d}t = 0,$$

where $B^+(\tau;h) = (\tau,\tau+h)$, $B^-(\tau,h) = (\tau-h;\tau)$.

(ii)

$$\text{For a. a. } \tau \in (0,T), \tilde{v}^+(\tau) = \tilde{v}^-(\tau).$$

(iii) If $v \in C_{\text{weak}}([0,T]; L^1(\Omega))$, then $\tilde{v}^+(\tau) = \tilde{v}^-(\tau) = v(\tau)$ for *all* $\tau \in [0,T]$.

After this reminder, we are ready to define the *instantaneous values* of functions in $L^\infty(0,T;L^1(\Omega))$. We define *right instantaneous value* of $v \in L^\infty(0,T;L^1(\Omega))$ at $\tau \in [0,T)$ as a continuous linear functional (a measure) $v(\tau+) \in (C(\overline{\Omega}))^*$

$$< v(\tau+), \eta >_{C(\overline{\Omega})} = \liminf_{h\to 0+} \frac{1}{h} \int_{B^+(\tau,h)} v(t,x)\eta(x)\mathrm{d}x \text{ for all } \eta \in C(\overline{\Omega}), \tag{68}$$

and *left instantaneous value* of $v \in L^\infty(0,T;L^1(\Omega))$ at $\tau \in (0,T]$ as a continuous linear functional (a measure) $v(\tau-) \in (C(\overline{\Omega}))^*$

$$< v(\tau-), \eta >_{C(\overline{\Omega})} = \limsup_{h\to 0+} \frac{1}{h} \int_{B^-(\tau,h)} v(t,x)\eta(x)\mathrm{d}x \text{ for all } \eta \in C(\overline{\Omega})). \tag{69}$$

The *instantaneous values* of function v will be defined as follows:

$$\text{inst}[v](0) = v(0^+), \text{inst}[v](\tau) = \frac{1}{2}(v(\tau^+) + v(\tau^-)), \text{ if } \tau \in (0,T), \text{ inst}[v](T) = v(T^-). \tag{70}$$

If v belongs only to $L^\infty(0,T;L^1(\Omega))$, then $v(\tau) = \text{inst}[v](\tau)$ for a.a. $\tau \in (0,T)$. If $v \in L^\infty(0,T;L^p(\Omega))$, $1 < p < \infty$, then $\text{inst}[v](\tau) \in L^p(\Omega)$. Theorem on Lebesgue points described above implies that for any $v \in C_{\text{weak}}([0,T];L^1(\Omega))$,

$$\text{inst}[v](\tau) = v(\tau) \text{ for all } \tau \in [0,T]. \tag{71}$$

Here and in the sequel, $C_{\text{weak}}([0,T];L^p(\Omega))$ is the space of functions in $L^\infty(0,T;L^p(\Omega))$ which are continuous for the weak topology of the space $L^p(\Omega)$, $1 \le p < \infty$.

3.2 Instantaneous Values of Solutions of Conservation Laws

3.2.1 The Case of Variational Identity

Suppose that $d \in L^\infty(0,T;L^1(\Omega))$ verifies identity

$$-\int_0^T \int_\Omega d(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \tag{72}$$

$$= \int_0^T \int_\Omega G(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t + \int_\Omega d_0(x)\varphi(0,x)\,\mathrm{d}x, \text{ with any } \varphi \in C^1_c([0,T)\times\overline{\Omega}),$$

where $(\mathbf{F}, G) \in L^1(Q_T; \mathbb{R}^4)$ and $d_0 \in L^1(\Omega)$. We may take in (72) the test functions $\psi^{\pm}_{\tau,h}(t)\varphi(t,x)$, $\varphi \in C^1_c([0,T)\times\overline{\Omega})$, where $\tau \in (0,T)$ and $h > 0$ is sufficiently small, and

$$\psi^-_{\tau,h} = \left\{ \begin{array}{c} 1 \text{ if } t \in (-\infty, \tau - h] \\ 1 - \frac{1}{h}(t - \tau + h) \text{ if } t \in (\tau - h, \tau) \\ 0 \text{ if } t \in [\tau, \infty) \end{array} \right\},$$

$$\psi^+_{\tau,h} = \left\{ \begin{array}{c} 1 \text{ if } t \in (-\infty, \tau] \\ 1 - \frac{1}{h}(t - \tau) \text{ if } t \in (\tau, \tau + h) \\ 0 \text{ if } t \in [\tau + h, \infty) \end{array} \right\}.$$

(We easily verify by density argument that $\psi^{\pm}_{\tau,h}(t)\varphi(t,x)$ are convenient test functions.) Letting moreover $h \to 0$, we obtain by virtue of (69) and (68) and the theorem on Lebesgue points

$$\begin{aligned} &\int_\Omega d(\tau,x)\varphi(\tau,x)\,\mathrm{d}x - \int_\Omega d_0(x)\varphi(0,x)\,\mathrm{d}x \\ &\quad = \int_0^\tau \int_\Omega d(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_0^\tau \int_\Omega G(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \text{ with any } \varphi \in C^1_c([0,T)\times\overline{\Omega}) \end{aligned} \tag{73}$$

for a.a. $\tau \in (0,T)$, or for all $\tau \in [0,T]$ if $\int_\Omega d(\tau,x)\varphi(\tau,x)\,\mathrm{d}x$ is replaced $< \mathrm{inst}[d](\tau); \varphi(\tau,\cdot) >_{C(\overline{\Omega})}$ according to formula (70). In particular, if $d \in L^\infty(0,T;L^p(\Omega))$ with some $1 < p < \infty$, then $d \in C_{\mathrm{weak}}([0,T];L^p(\Omega))$. We also observe that weak formulations (72) and (73) are equivalent.

3.2.2 The Case of Variational Inequality

We shall now suppose that function $d \in L^\infty(0,T;L^1(\Omega))$ verifies solely the variational inequality

$$\begin{aligned} &-\int_0^T\int_\Omega d(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega Z(t,x)\phi(t,x)\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \\ &\le \int_0^T\int_\Omega G(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t + \int_\Omega d_0(x)\varphi(0,x)\,\mathrm{d}x, \text{ with any } \varphi \in C^1_c([0,T);M), \varphi \ge 0, \end{aligned} \tag{74}$$

where M is a vector subspace of $C^1(\overline{\Omega})$, $(\mathbf{F}, G, Z) \in L^1(Q_T;\mathbb{R}^5)$, $Z \ge 0$, and $d_0 \in L^1(\Omega)$. Replacing in (74) test function φ by $\psi^{\pm}_{\tau,h}\varphi$ and letting $h \to 0$, we get

$$\int_\Omega d(\tau,x)\varphi(\tau,x)\,\mathrm{d}x - \int_\Omega d_0(x)\varphi(0,x)\,\mathrm{d}x + \int_0^\tau \int_\Omega Z(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \tag{75}$$

$$\leq \int_0^\tau \int_\Omega d(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega G(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t$$

for a.a. $\tau \in (0,T)$ with any $\varphi \in C_c^1([0,T);M)$, $\varphi \geq 0$. Formulations (74) and (75) are equivalent.

On the other hand, inequality (74) implies

$$-\int_0^T \psi'(t)\Big[\int_\Omega d(t,x)\eta(x)\,\mathrm{d}x\Big]\mathrm{d}t + \int_0^T \psi(t)\Big[\int_\Omega Z(t,x)\eta(x)\,\mathrm{d}x\Big]\mathrm{d}t$$
$$-\int_0^T \psi(t)\Big[\int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\eta(x)\,\mathrm{d}x\Big]\mathrm{d}t \qquad (76)$$

$$\leq \int_0^T \psi(t)\Big[\int_\Omega G(t,x)\eta(x)\,\mathrm{d}x\Big]\mathrm{d}t + \psi(0)\int_\Omega d_0(x)\eta(x)\,\mathrm{d}x, \text{ with any } \eta \in M, \eta \geq 0$$

for all $\psi \in C_c^1[0,T)$, $\psi \geq 0$.

We deduce from (76) that for any $\eta \in M$, $\eta \geq 0$, there exists a nonnegative linear functional Σ_η on the vector space $C_c^1[0,T)$ defined by

$$-\int_0^T \psi'(t)\Big[\int_\Omega d(t,x)\eta(x)\,\mathrm{d}x\Big]\mathrm{d}t + <\Sigma_\eta,\psi> - \int_0^T \psi(t)\Big[\int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\eta(x)\,\mathrm{d}x\Big]\mathrm{d}t \qquad (77)$$
$$= \int_0^T \psi(t)\Big[\int_\Omega G(t,x)\eta(x)\,\mathrm{d}x\Big]\mathrm{d}t + \psi(0)\int_\Omega d_0(x)\eta(x)\,\mathrm{d}x \text{ for any } \psi \in C_c^1[0,T),$$

verifying

$$<\Sigma_\eta,\psi> \geq \int_0^T \psi(t)\Big[\int_\Omega Z(t,x)\eta(x)\,\mathrm{d}x\Big]\mathrm{d}t \text{ for any } \psi \in C_c^1[0,T), \psi \geq 0.$$

As a nonnegative linear functional on $C_c^1[0,T)$, Σ_η can be extended to a nonnegative linear functional on $C_c[0,T)$ by the standard Schwarz procedure. Indeed, if $\psi \in C_c[0,T)$, we take sequences

$$C_c^1[0,T) \ni \psi_n^- \nearrow [\psi]^-, \quad C_c^1[0,T) \ni \psi_n^+ \nearrow [\psi]^+,$$
$$\text{where } [\psi]^+ = \max\{0,\psi\}, [\psi]^- = -\min\{0,\psi\}, \qquad (78)$$

define $<\Sigma_\eta,[\psi]^->$ and $<\Sigma_\eta,[\psi]^+>$ as the limits of nondecreasing sequences $<\Sigma_\eta,\psi_n^->$ and $<\Sigma_\eta,\psi_n^+>$, respectively, (we notice that these limits do not depend on the choice of the sequences ψ_n^-, resp., ψ_n^+ provided they satisfy (78)) and set

$$<\Sigma_\eta,\psi> = <\Sigma_\eta,[\psi]^+> - <\Sigma_\eta,[\psi]^-> \text{ for all } \psi \in C_c[0,T). \qquad (79)$$

Employing Hölder's inequality in each integral term in (76), we find out that

$$< \Sigma_\eta, \psi^-_{T,h} > \leq c, \tag{80}$$

where $c = \Big(\|d\|_{L^\infty(0,T;L^1(\Omega))} + \|(\mathbf{F}, G)\|_{L^1(Q_T;\mathbb{R}^4)} + \|d_0\|_{L^1(\Omega)}\Big)\|\eta\|_{C^1(\overline{\Omega})}$ is independent of $0 < h < T$. This fact in combination with (79) makes possible to extend Σ_η to a *continuous linear functional* on $C[0, T]$. We will denote this functional by the same symbol Σ_η and indicate the duality pairing subscript $C[0, T]$. In particular,

$$| < \Sigma_\eta, \psi >_{C[0,T]} | \leq \|\Sigma_\eta\|_{(C[0,T])^*} \|\psi\|_{C[0,T]}, \tag{81}$$

where

$$\|\Sigma_\eta\|_{(C[0,T])^*} = \lim_{h\to 0+} < \Sigma_\eta, \psi^-_{T,h} >$$
$$\leq \Big(\|d\|_{L^\infty(0,T;L^1(\Omega))} + \|(\mathbf{F}, G)\|_{L^1(Q_T;\mathbb{R}^4)} + \|d_0\|_{L^1(\Omega)}\Big)\|\eta\|_{C^1(\overline{\Omega})}.$$

According to the Riesz representation theorem, there exists $\sigma-$algebra of measurable sets containing all Borel sets on [0,T] and a unique nonnegative measure μ_{Σ_η} on this $\sigma-$algebra such that

$$< \Sigma_\eta, \psi >_{C[0,T]} = \int_{[0,T]} \psi \, \mathrm{d}\mu_{\Sigma_\eta}, \tag{82}$$

and, moreover, thanks to (81)

$$\|\Sigma_\eta\|_{(C[0,T])^*} = \int_{[0,T]} \mathrm{d}\mu_{\Sigma_\eta}. \tag{83}$$

We may define a restriction of Σ_η on $C[0, \tau)$, resp., $C[0, \tau]$ by setting

$$< \Sigma_\eta, \psi >_{C[0,\tau]} = \int_{[0,\tau]} \psi \, \mathrm{d}\mu_{\Sigma_\eta}, \text{ resp. } < \Sigma_\eta, \psi >_{C[0,\tau)} = \int_{[0,\tau)} \psi \, \mathrm{d}\mu_{\Sigma_\eta}. \tag{84}$$

In particular, functions $\tau \mapsto < \Sigma_\eta, 1 >_{C[0,\tau]}$, and $\tau \mapsto < \Sigma_\eta, 1 >_{C[0,\tau)}$ are *nondecreasing functions*, and, moreover, $< \Sigma_\eta, 1 >_{C[0,\tau)} \leq < \Sigma_\eta, 1 >_{C[0,\tau]}$ for all $\tau \in (0, T]$.

Coming back to identity (77) with test functions $\psi^\pm_{\tau,h}$ and letting $h \to 0+$, we obtain

$$\int_\Omega d(\tau, x)\eta(x) \, \mathrm{d}x - \int_\Omega d_0(x)\eta(x) \, \mathrm{d}x + \frac{1}{2}\Big(\int_{[0,\tau)} d\mu_{\Sigma_\eta} + \int_{[0,\tau]} d\mu_{\Sigma_\eta}\Big) \tag{85}$$

$$= \int_0^\tau \int_\Omega \mathbf{F}(t, x) \cdot \nabla_x \eta(x) \, \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega G(t, x)\eta(x) \, \mathrm{d}x\mathrm{d}t \text{ for any } \eta \in M, \eta \geq 0,$$

for a.a. $\tau \in (0,T)$ (or equivalently for all $\tau \in (0,T]$ if we replace $\int_\Omega d(\tau,x)\eta(x)\,\mathrm{d}x$ by the duality pairing $<\mathrm{inst}[d](\tau);\eta>_{C(\overline{\Omega})}$, where $\mathrm{inst}[d]$ is the instantaneous value of d defined in (70). We can therefore conclude that function $<d(\tau);\eta>_{C[0,T]}$ (with η as in (85)) is a sum of *a nonincreasing function with countable number of jumps* (because $d \in L^\infty(0,T;L^1(\Omega))$) and of an *absolutely continuous function*.

3.2.3 The Particular Case of Variational Inequality on (0; *T*)

By the same token, variational inequalities

$$-\int_0^T d(t)\psi'(t)\mathrm{d}t + \int_0^T \psi(t)Z(t)\mathrm{d}t \leq \int_0^T \psi(t)G(t)\mathrm{d}t + d_0\psi(0) \text{ for all } \psi \in C_c^1[0,T),\, \psi \geq 0, \tag{86}$$

with $Z, G \in L^1(0,T)$, $Z \geq 0$, $d \in L^\infty(0,T)$, and

$$d(\tau)\psi(\tau) - d_0\psi(0) + \int_0^\tau \psi(t)Z(t)\mathrm{d}t \leq \int_0^\tau d(t)\psi'(t)\mathrm{d}t + \int_0^\tau \psi(t)G(t)\mathrm{d}t \tag{87}$$

for all $\psi \in C_c^1[0,T)$, $\psi \geq 0$ and a.a. $\tau \in (0,T)$, are equivalent.

On the other hand, if $d \in L^\infty(0,T)$ verifies variational inequality (86), then there exists a nonnegative finite measure μ on the σ-algebra of Borel sets on interval $[0,T]$ such that

$$d(\tau) - d_0 + \frac{1}{2}\Big(\int_{[0,\tau)} \mathrm{d}\mu + \int_{[0,\tau]} \mathrm{d}\mu\Big) = \int_0^\tau G(t)\mathrm{d}t \text{ for a.a. } \tau \in (0,T), \tag{88}$$

or for all $\tau \in (0,T]$ if we replace $d(\tau)$ by $\mathrm{inst}[d](\tau)$ defined by (70) with $d(\tau-) = \limsup_{h\to 0}\frac{1}{h}\int_{B-(\tau;h)} d(t)\mathrm{d}t$, $d(\tau+) = \liminf_{h\to 0}\frac{1}{h}\int_{B+(\tau;h)} d(t)\mathrm{d}t$. Now we can read from (88) that the map $[0,T] \ni \tau \mapsto \mathrm{inst}[d](\tau)$ is a *sum of nonincreasing function with at most countable number of jumps and an absolutely continuous function.*

If (86) is an identity (with sign "=" instead of "≤" and with $Z = 0$), then it is equivalent to the "integrated form"

$$d(\tau) - d_0 = \int_0^\tau G(t)\mathrm{d}t \text{ for a.a. } \tau \in (0,T). \tag{89}$$

In particular, $\mathrm{inst}[v]$ is an *absolutely continuous function* on $[0,T]$.

3.2.4 The Case 2 with $M = C^1(\overline{\Omega})$

We suppose that $d \in L^\infty(0,T;L^1(\Omega))$ verifies the variational inequality

$$-\int_0^T\int_\Omega d(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega Z(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\mathrm{d}t$$
$$\le \int_0^T\int_\Omega G(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t+\int_\Omega d_0(x)\varphi(0,x)\,\mathrm{d}x,\ \text{ with any } \varphi\in C^1_c([0,T)\times\overline{\Omega}), \varphi\ge 0, \tag{90}$$

where again $(\mathbf{F},G,Z)\in L^1(Q_T;\mathbb{R}^5)$, $Z\ge 0$, and $d_0\in L^1(\Omega)$.

Repeating the reasoning of (76)–(85) with the set $[0,T)\times\overline{\Omega}$ in place of $[0,T)$, we will find that there exists a nonnegative continuous linear functional $\Sigma\in(C([0,T]\times\overline{\Omega}))^*$ such that

$$-\int_0^T\int_\Omega d(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\mathrm{d}t+<\Sigma,\varphi>_{C([0,T]\times\overline{\Omega})}-\int_0^T\int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\int_\Omega G(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t+\int_\Omega d_0(x)\varphi(0,x)\,\mathrm{d}x,\ \text{ with any } \varphi\in C^1_c([0,T)\times\overline{\Omega}), \varphi\ge 0, \tag{91}$$

where

$$<\Sigma,\varphi>_{C([0,T]\times\overline{\Omega}))}\ge\int_0^T\int_\Omega Z(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \text{ with any } \varphi\in C([0,T]\times\overline{\Omega}), \varphi\ge 0.$$

Due to the nonnegativity of Σ,

$$\|\Sigma\|_{C([0,T]\times\overline{\Omega})^*}=\lim_{h\to 0+}<\Sigma,\psi^-_{T,h}>\le\|d\|_{L^\infty(0,T;L^1(\Omega))}. \tag{92}$$

Moreover there exists a unique nonnegative measure μ_Σ on the σ-algebra of Borel sets of $[0,T]\times\overline{\Omega}$ such that

$$<\Sigma,\varphi>_{C([0,T]\times\overline{\Omega})}=\int_{[0,T]\times\overline{\Omega}}\varphi\mathrm{d}\mu_\Sigma. \tag{93}$$

Choosing in (91) test functions $\varphi(t,x)\psi^\pm_{\tau,h}$, we get

$$\int_\Omega d(\tau,x)\varphi(\tau,x)\,\mathrm{d}x-\int_\Omega d_0(x)\varphi(0,x)\,\mathrm{d}x+\frac{1}{2}\Big(\int_{[0,\tau)\times\overline{\Omega}}\varphi\mathrm{d}\mu_\Sigma+\int_{[0,\tau]\times\overline{\Omega}}\varphi\mathrm{d}\mu_\Sigma\Big)$$
$$=\int_0^\tau\int_\Omega d(t,x)\partial_t\varphi(t,x)\,\mathrm{d}xt+\int_0^\tau\int_\Omega \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \tag{94}$$
$$+\int_0^\tau\int_\Omega G(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t,\ \text{ with any } \varphi\in C^1_c([0,T)\times\overline{\Omega}), \varphi\ge 0.$$

Identity (94) holds for a.a. $\tau\in(0,T)$ and it is equivalent to (90).

In particular, one deduces from the choice $\psi(t,x) = \psi^-_{\tau,h}(t)1(x)$, resp., $\psi(t,x) = \psi^-_{z,h}(t)1(x)$ in (90)

$$\int_\Omega d(\tau,x)\,\mathrm{d}x - \int_\Omega d_0(x)\,\mathrm{d}x + \frac{1}{2}\Big(\int_{[0,\tau)\times\overline{\Omega}} \mathrm{d}\mu_\Sigma + \int_{[0,\tau]\times\overline{\Omega}} \mathrm{d}\mu_\Sigma\Big)$$
$$= \int_0^\tau \int_\Omega G(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \tag{95}$$

for a.a. $\tau \in (0,T)$, and

$$\int_\Omega d(\tau,x)\,\mathrm{d}x - \int_\Omega d(z,x)\,\mathrm{d}x + \frac{1}{2}\Big(\int_{[z,\tau)\times\overline{\Omega}} \mathrm{d}\mu_\Sigma + \int_{(z,\tau]\times\overline{\Omega}} \mathrm{d}\mu_\Sigma\Big)$$
$$= \int_z^\tau \int_\Omega G(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t \tag{96}$$

for a.a. $0 < z < \tau < T$ or for all values of τ and z in $[0,T]$ if we replace $\int_\Omega d(\cdot,x)\,\mathrm{d}x$ by $\mathrm{inst}\Big[\int_\Omega d(\cdot,x)\,\mathrm{d}x\Big](\cdot)$.

3.3 Weakly Convergent Sequences in L^1

Theorem 1. *Let $O \subset \mathbb{R}^N$ be a bounded open set and $v_n : O \mapsto \mathbb{R}$ be a sequence of measurable functions such that*

$$\sup_{n\geq 1} \|\Phi(v_n)\|_{L^1(O)} < \infty, \textit{ for a certain application } \Phi \in C[0,\infty).$$

Suppose that

$$\lim_{|z|\to\infty} \frac{|z|}{\Phi(|z|)} = 0.$$

Then there is a subsequence of v_n (not relabeled) such that

$$v_n \rightharpoonup v \textit{ in } L^1(O).$$

3.4 Convexity, Monotonicity, and Weak Convergence

It is well known that convex lower semicontinuous functions give rise to L^1 – sequentially weakly lower semicontinuous functionals – and give rise to a useful criterion of the a.e. convergence. We present here a convenient formulation of these results taken over from [30, Theorem 2.11 and Corollary 2.2] or [33, Theorem 10.20]. (More general formulation can be found in Brezis [10] or in Ekeland, Temam [22].) The corresponding theorems read:

Theorem 2. *Let $O \subset \mathbb{R}^N$ be a measurable set and $\{\mathbf{v}_n\}_{n=1}^\infty$ a sequence of functions in $L^1(O;\mathbb{R}^M)$ such that*

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(O;\mathbb{R}^M).$$

Let $\Phi : \mathbb{R}^M \to (-\infty, \infty]$ be a lower semicontinuous convex function.

Then $\Phi(\mathbf{v}) : O \mapsto \mathbb{R}$ is integrable and

$$\int_O \Phi(\mathbf{v})\mathrm{d}x \leq \liminf_{n\to\infty} \int_O \Phi(\mathbf{v}_n)\mathrm{d}x.$$

Strictly convex lower semicontinuous functions are involved in a useful criterion of the a.e. convergence.

Theorem 3. *Let $O \subset \mathbb{R}^N$ be a measurable set and $\{\mathbf{v}_n\}_{n=1}^\infty$ a sequence of functions in $L^1(O;\mathbb{R}^M)$ such that*

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(O;\mathbb{R}^M).$$

Let $\Phi : \mathbb{R}^M \to (-\infty, \infty]$ be a lower semicontinuous convex function such that $\Phi(\mathbf{v}_n) \in L^1(O)$ for any n, and

$$\Phi(\mathbf{v}_n) \to \overline{\Phi(\mathbf{v})} \text{ weakly in } L^1(O).$$

Then

$$\Phi(\mathbf{v}) \leq \overline{\Phi(\mathbf{v})} \text{ a.e. on } O. \tag{97}$$

If, moreover, Φ is strictly convex on an open convex set $U \subset \mathbb{R}^M$ and

$$\Phi(\mathbf{v}) = \overline{\Phi(\mathbf{v})} \text{ a.e. on } O,$$

then

$$\mathbf{v}_n(\mathbf{y}) \to \mathbf{v}(\mathbf{y}) \text{ for a.a. } \mathbf{y} \in \{\mathbf{y} \in O \mid \mathbf{v}(\mathbf{y}) \in U\} \tag{98}$$

extracting a subsequence as the case may be.

Similar properties are true also for monotone functions as a consequence of the so-called Minti's trick. The following result is taken from [33, Theorem 10.19]:

Theorem 4. *Let $I \subset \mathbb{R}$ be an interval, $Q \subset \mathbb{R}^N$ a domain, and*

$$(P, G) \in C(I) \times C(I) \quad \textit{a couple of nondecreasing functions.} \tag{99}$$

Assume that $\varrho_n \in L^1(Q; I)$ is a sequence such that

$$\left\{ \begin{array}{c} P(\varrho_n) \to \overline{P(\varrho)}, \\ \\ G(\varrho_n) \to \overline{G(\varrho)}, \\ \\ P(\varrho_n)G(\varrho_n) \to \overline{P(\varrho)G(\varrho)} \end{array} \right\} \quad \textit{weakly in } L^1(Q). \tag{100}$$

(i) Then

$$\overline{P(\varrho)}\ \overline{G(\varrho)} \le \overline{P(\varrho)G(\varrho)}. \tag{101}$$

(ii) If, in addition,

$$\begin{array}{c} G \in C(\mathbb{R}), \quad G(\mathbb{R}) = \mathbb{R}, \quad G \textit{ is strictly increasing}, \\ \\ P \in C(\mathbb{R}), \quad P \textit{ is nondecreasing}, \end{array} \tag{102}$$

and

$$\overline{P(\varrho)G(\varrho)} = \overline{P(\varrho)}\ \overline{G\varrho)}, \tag{103}$$

then

$$\overline{P(\varrho)} = P \circ G^{-1}(\overline{G(\varrho)}). \tag{104}$$

(iii) In particular, if $G(z) = z$, then

$$\overline{P(\varrho)} = P(\varrho). \tag{105}$$

3.5 The Inverse of the Div Operator (Bogovskii's Formula)

Theorem 5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

(i) Then there exists a linear mapping $\mathcal{B}$,

$$\mathcal{B} : \{f \mid f \in C_c^\infty(\Omega), \int_\Omega f \, \mathrm{d}x = 0\} \to C_c^\infty(\Omega; \mathbb{R}^N),$$

such that

$$\mathrm{div}_x(\mathcal{B}[f]) = f$$

with the following properties:

(ii) *We have*

$$\|\mathcal{B}[f]\|_{W^{k+1,p}(\Omega;\mathbb{R}^N)} \leq c\|f\|_{W^{k,p}(\Omega)} \text{ for any } 1 < p < \infty,\ k = 0, 1, \ldots, \tag{106}$$

In particular, $\mathcal{B}$ *can be extended in a unique way to a bounded linear operator*

$$\mathcal{B} : \{f \mid f \in L^p(\Omega),\ \int_\Omega f \, \mathrm{d}x = 0\} \to W_0^{1,p}(\Omega; \mathbb{R}^N).$$

(iii) *If* $f \in L^p(\Omega)$, $\int_\Omega f \, \mathrm{d}x = 0$, *and, in addition,* $f = \mathrm{div}_x \mathbf{g}$, *where* $\mathbf{g} \in (L^q(\Omega))^N$, $1 < q < \infty$, *and* $\mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$ *(in the weak sense of normal traces), then*

$$\|\mathcal{B}[f]\|_{L^q(\Omega;\mathbb{R}^N)} \leq c\|\mathbf{g}\|_{L^q(\Omega;\mathbb{R}^N)}. \tag{107}$$

Operator $\mathcal{B}$ has been constructed for the first time by Bogovskii. The reader can consult Galdi [59] or [88, Section 3.3] for more details about this problem.

3.6 Poincaré- and Korn-Type Inequalities

Applications in compressible thermodynamics often require refined versions of Poincaré and Korn inequalities that are not directly covered by the standard theory. We shall list some of them and refer the reader to [33, Appendix, Sections 10.8, 10.9] for more systematic treatment.

Theorem 6. *Let* $1 \leq p \leq \infty$, $0 < \Gamma < \infty$, *and let* $\Omega \subset \mathbb{R}^N$ *be a bounded Lipschitz domain. Let* $V \subset \Omega$ *be a measurable set such that*

$$|V| \geq V_0 > 0.$$

Then there exists a positive constant $c = c(p, \Gamma, V_0)$ *such that*

$$\| v \|_{W^{1,p}(\Omega)} \leq c\Big[\|\nabla_x v\|_{W^{1,p}(\Omega,\mathbb{R}^N)} + \Big(\int_V |v|^\Gamma \mathrm{d}x\Big)^{\frac{1}{\Gamma}}\Big]$$

for any $v \in W^{1,p}(\Omega)$.

Theorem 7. *Let $\Omega \subset \mathbb{R}^N$, $N > 2$ be a bounded Lipschitz domain, and let $1 < p < \infty$, $M_0 > 0$, $K > 0$, $\gamma > 1$. Then there exists a positive constant $c = c(p, M_0, K, \gamma)$ such that inequality*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \leq c\Big(\Big\|\nabla_x \mathbf{v}\Big\|_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega r|\mathbf{v}|\,\mathrm{d}x\Big)$$

holds for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N)$ and any nonnegative function r such that

$$0 < M_0 \leq \int_\Omega r\,\mathrm{d}x, \int_\Omega r^\gamma\,\mathrm{d}x \leq K. \tag{108}$$

The following lemma is often useful in combination with Theorem 6 to investigate positivity of the temperature (see [33, Lemma 2.1]).

Lemma 1. *Let Ω be a bounded Lipschitz domain and $p, \gamma > 1$. Let $S \in C(0,\infty)$ be a strictly decreasing function such that $\lim_{Z\to\infty} S(Z) = S_\infty \in \{-\infty, 0\}$ and*

$$\limsup_{n\to\infty} \int_{\{\varrho_n \leq \vartheta_n^{1/(\gamma-1)}\}} \varrho_n S\Big(\frac{\varrho_n}{\vartheta_n^{1/(\gamma-1)}}\Big)\mathrm{d}x \leq 0$$

whenever $\varrho_n \geq 0$ is bounded in $L^\gamma(\Omega)$ and $0 < \vartheta_n \to 0$ in $L^p(\Omega)$.

Then for any $M_0 > 0$, $\Gamma_0 > 0$, and $\underline{S} \in \mathbb{R}$, there exist $\alpha = \alpha(M_0, \Gamma_0, \underline{S}) > 0$, $\underline{\vartheta} = \underline{\vartheta}(M_0, \Gamma_0, \underline{S}) > 0$ such that for any nonnegative functions ϱ, ϑ satisfying

$$\int_\Omega \varrho \mathrm{d}x \geq M_0, \int_\Omega (\varrho^\gamma + \vartheta^p)\mathrm{d}x \leq \Gamma_0,$$

and

$$\int_\Omega \varrho S\Big(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\Big)\mathrm{d}x \geq \underline{S} > M_0 S_\infty,$$

we have

$$\Big|\{\vartheta \geq \underline{\vartheta}\}\Big| \geq \alpha. \tag{109}$$

The classical Korn's inequality deals with the symmetrized gradients of the vector fields. It reads:

Theorem 8. *Assume that $1 < p < \infty$.*

(i) There exists a positive constant $c = c(p, N)$ such that

$$\|\nabla \mathbf{v}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})} \leq c\|\nabla \mathbf{v} + \nabla^T \mathbf{v}\|_{L^p(\mathbb{R}^N;\mathbb{R}^{N\times N})}$$

for any $\mathbf{v} \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$.

(ii) Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then there exists a positive constant $c = c(p, N, \Omega) > 0$ such that

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \le c\Big(\|\nabla\mathbf{v} + \nabla^T\mathbf{v}\|_{L^p(\Omega,\mathbb{R}^{N\times N})} + \int_\Omega |\mathbf{v}|\,\mathrm{d}x\Big)$$

for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N)$.

In the fluid dynamics of compressible fluids, we often need a version of Korn's inequality involving the *symmetrized and traceless* gradient. It reads:

Theorem 9. *Let $1 < p < \infty$ and $N > 2$.*

(i) There exists a positive constant $c = c(p, N)$ such that

$$\|\nabla\mathbf{v}\|_{L^p(\Omega;\mathbb{R}^{N\times N})} \le c\|\nabla\mathbf{v} + \nabla^T\mathbf{v} - \frac{2}{N}\mathrm{div}\mathbf{v}\,\mathbb{I}\|_{L^p(\Omega;\mathbb{R}^{N\times N})}$$

for any $\mathbf{v} \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$, where $\mathbb{I} = (\delta_{i,j})_{i,j=1}^N$ is the identity matrix.

(ii) Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then there exists a positive constant $c = c(p, N, \Omega) > 0$ such that

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \le c\Big(\|\nabla\mathbf{v} + \nabla^T\mathbf{v} - \frac{2}{N}\mathrm{div}\mathbf{v}\,\mathbb{I}\|_{L^p(\Omega,\mathbb{R}^{N\times N})} + \int_\Omega |\mathbf{v}|\,\mathrm{d}x\Big)$$

for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N)$.

Finally the generalized version of the above inequality reads:

Theorem 10. *Let $\Omega \subset \mathbb{R}^N$, $N > 2$ be a bounded Lipschitz domain, and let $1 < p < \infty$, $M_0 > 0$, $K > 0$, $\gamma > 1$.*

Then there exists $c = c(p, K, M_0, \gamma) > 0$ such that

$$\|\mathbf{v}\|_{W^{1,p}(\Omega;\mathbb{R}^N)} \le c\Big(\Big\|\nabla_x\mathbf{v} + \nabla_x^T\mathbf{v} - \frac{2}{N}\mathrm{div}\mathbf{v}\,\mathbb{I}\Big\|_{L^p(\Omega;\mathbb{R}^N)} + \int_\Omega r|\mathbf{v}|\,\mathrm{d}x\Big)$$

for any $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^N)$ and for any nonnegative function r such that

$$0 < M_0 \le \int_\Omega r\,\mathrm{d}x,\ \int_\Omega r^\gamma\,\mathrm{d}x \le K. \tag{110}$$

3.7 Time Compactness

We report the classical theorem known as Aubin-Lions-Simon lemma [6, Theorem II.5.16].

Theorem 11. *Let $X \subset\subset B \subset Y$ be Banach spaces, where the symbols $\subset\subset$ denotes compact and $\subset$ continuous imbeddings, respectively, and let $1 \le p, q \le \infty$. Let v^n be a sequence of functions such that*

$$v^n \text{ is bounded in } L^p(0,T;X), \quad \partial_t v^n \text{ bounded in } L^q(0,T;Y).$$

Then there exists a subsequence (denoted again by v^n) such that

$$\text{if } p < \infty, \ v^n \to v \text{ (strongly) in } L^p(0,T;B);$$
$$\text{if } p = \infty \text{ and } q > 1, \ v^n \to v \text{ (strongly) in } C([0,T];B).$$

The classical Aubin-Lions lemma is convenient for applications involving time evolution of the quantity v expressed through an equation. It usually cannot be applied to investigate time compactness of quantities evaluating according to differential inequalities. In the latter situation, one may use a weaker variant of the above theorem (see [30, Lemma 6.3]).

Theorem 12. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $1 < p < \infty$. Let v^n be a sequence of functions such that*

$$v^n \text{ is bounded in } L^p(0,T;L^q(\Omega)) \cap L^\infty(0,T;L^1(\Omega)), q > \frac{2N}{2+N},$$
$$\partial_t v^n = g_n + \Sigma_n,$$

where

$$\Sigma_n \text{ is a nonnegative distribution and } g_n \text{ is bounded in } L^1(0,T;W^{-m,r}(\Omega))$$

with some $m \ge 1$, $r > 1$. Then v^n contains a subsequence such that

$$v_n \to v \text{ (strongly) in } L^p(0,T;W^{-1,2}(\Omega)).$$

3.8 Operator $\nabla\Delta^{-1}$ and Riesz-Type Operators

We introduce operators $\mathcal{A} = \nabla_x \Delta^{-1}$ and $\mathcal{R} = \nabla_x \otimes \nabla_x \Delta^{-1}$,

$$(\nabla\Delta^{-1})_j(v) = -\mathcal{F}^{-1}\Big[\frac{i\xi_j}{|\xi|^2}\mathcal{F}(v)(\xi)\Big], \quad (\nabla \otimes \nabla\Delta^{-1})_{ij}(v) = \mathcal{F}^{-1}\Big[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}(v)(\xi)\Big], \tag{111}$$

where $\mathcal{F}$ denotes the Fourier transform

$$[\mathcal{F}(v)](\xi) = \frac{1}{2\pi^3}\int_{\mathbb{R}^3} v(x)\exp(-\mathrm{i}\xi\cdot x)\mathrm{d}x.$$

We recall the basic properties of these operators (see e.g. Feireisl [30], [33, Sections 0.5 and 10.16] or [88] for more details).

Theorem 13. *(i)* *$\mathcal{A}$ is a continuous linear operator from $L^1\cap L^2(\mathbb{R}^3)$ to $L^2+L^\infty(\mathbb{R}^3;\mathbb{R}^3)$ and from $L^p(\mathbb{R}^3)$ to $L^{3p/(3-p)}(\mathbb{R}^3;\mathbb{R}^3)$ for any $1<p<3$.*

(ii) $\mathcal{R}$ is a continuous linear operator from from $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3;\mathbb{R}^{3\times 3})$ for any $1<p<\infty$.

(iii) The following formulas hold

$$\mathcal{R}(v)=\mathcal{R}^T(v),\ \sum_{j=1}^3\mathcal{R}_{jj}(v)=v,\ \ v\in L^p(\mathbb{R}^3),$$

$$\partial_k\mathcal{R}_{ij}(v)=\mathcal{R}_{ij}(\partial_k v),\ \mathcal{R}_{ij}(\partial_k v)=\mathcal{R}_{ik}(\partial_j v),\ v\in W^{1,p}(\mathbb{R}^3),$$

where $1<p<\infty$;

$$\nabla_x A(v)=\mathcal{R}(v),\quad \mathrm{div}\mathcal{A}(v)=v,\ v\in L^p(\mathbb{R}^3),$$

where $1<p<3$;

$$\int_{\mathbb{R}^3}\mathcal{A}(v)w\mathrm{d}x=-\int_{\mathbb{R}^3}v\mathcal{A}(w)\mathrm{d}x,$$

with

$$v\in L^p(\mathbb{R}^3),\ w\in L^q(\mathbb{R}^3),\ \mathcal{A}(w)\in L^{p'}(\mathbb{R}^3),\ \mathcal{A}(v)\in L^{q'}(\mathbb{R}^3),$$

where $1<q,p<3$;

$$\int_{\mathbb{R}^3}\mathcal{R}(v)w\mathrm{d}x=\int_{\mathbb{R}^3}v\mathcal{R}(w)\mathrm{d}x,\ v\in L^p(\mathbb{R}^3),\ w\in L^{p'}(\mathbb{R}^3),$$

where $1<p<\infty$.

3.9 Some Results of Compensated Compactness

We shall start by the celebrated Div-Curl lemma of Murat and Tartar [84] formulated in the form [33, Lemma 10.1].

Theorem 14. *Let $Q \subset \mathbb{R}^N$ be an open set. Assume*

$$\begin{aligned} \mathbf{U}_n &\to \mathbf{U} \text{ weakly in } L^p(Q;\mathbb{R}^N),\\ \mathbf{V}_n &\to \mathbf{V} \text{ weakly in } L^q(Q,\mathbb{R}^N), \end{aligned} \tag{112}$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1.$$

In addition, let

$$\left.\begin{aligned} \operatorname{div} \mathbf{U}_n &\equiv \nabla \cdot \mathbf{U}_n,\\ \operatorname{curl} \mathbf{V}_n &\equiv (\nabla \mathbf{V}_n - \nabla^T \mathbf{V}_n) \end{aligned}\right\} \text{ be precompact in } \begin{cases} W^{-1,s}(Q),\\ W^{-1,s}(Q,\mathbb{R}^{N\times N}), \end{cases} \tag{113}$$

for a certain $s > 1$. Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \text{ weakly in } L^r(Q).$$

The next theorem involving commutator of Riesz operators may be seen as a consequence of the Div-Curl lemma stated above (see Feireisl [30, Section 6] or [33, Theorem 10.27]).

Theorem 15. *Let*

$$\mathbf{V}_\varepsilon \to \mathbf{V} \text{ weakly in } L^p(\mathbb{R}^N;\mathbb{R}^N),$$
$$\mathbf{U}_\varepsilon \to \mathbf{U} \text{ weakly in } L^q(\mathbb{R}^N;\mathbb{R}^N),$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} < 1$. Then

$$\mathbf{U}_\varepsilon \cdot \mathcal{R}[\mathbf{V}_\varepsilon] - \mathcal{R}[\mathbf{U}_\varepsilon] \cdot \mathbf{V}_\varepsilon \to \mathbf{U} \cdot \mathcal{R}[\mathbf{V}] - \mathcal{R}[\mathbf{U}] \cdot \mathbf{V} \text{ weakly in } L^s(\mathbb{R}^N).$$

The next theorem is a compensated compactness result in the spirit of Coifman and Meyer [15] (see [33, Theorem 10.28]).

Theorem 16. *Let $w \in W^{1,r}(\mathbb{R}^N)$ and $\mathbf{V} \in L^p(\mathbb{R}^N;\mathbb{R}^N)$ be given, where*

$$1 < r < N, \ \frac{1}{r} - \frac{1}{N} + \frac{1}{p} < 1.$$

Then there exists $\alpha > 0$ and $q = q(r,p) > 1$ such that

$$\left\| \mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}] \right\|_{W^{\alpha,q}(\mathbb{R}^N;\mathbb{R}^N)} \leq c(r,p)\|w\|_{W^{1,r}(\mathbb{R}^N)}\|\mathbf{V}\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}.$$

Here $W^{\alpha,q}(\mathbb{R}^N)$ denotes the Sobolev-Slobodeckii space.

3.10 Parametrized (Young) Measures

Let $Q \subset \mathbb{R}^N$ be a domain; we say that $\psi : Q \times \mathbb{R}^M$ is a *Carathéodory function* on $Q \times \mathbb{R}^M$ if

$$\left\{ \begin{array}{c} \text{for a. a. } x \in Q, \text{ the function } \lambda \mapsto \psi(x,\lambda) \text{ is continuous on } \mathbb{R}^M; \\ \\ \text{for all } \lambda \in \mathbb{R}^M, \text{ the function } x \mapsto \psi(x,\lambda) \text{ is measurable on } Q. \end{array} \right\} \tag{114}$$

We recall that ν is called a *probability measure* on $\mathbb{R}^M$ if it is a *nonnegative Borel measure*, such that $\nu(\mathbb{R}^M) = 1$. In the sequel, we shall deal with families $\{\nu_x\}_{x\in Q}$ of probability measures ν_x. We say that the family of measures $\{\nu_x\}_{x\in Q}$ is a *family of parametrized measures depending measurably on* x if for almost all $x \in Q$, ν_x is a probability measure and if

$$\left\{ \begin{array}{c} \forall \phi : R^M \to \mathbb{R},\ \phi \in C(\mathbb{R}^M) \cap L^\infty(\mathbb{R}^M), \\ \\ \text{the function } x \to \int_{\mathbb{R}^M} \phi(\lambda)\, \mathrm{d}\nu_x(\lambda) :=< \nu_x, \phi > \text{ is measurable on } Q. \end{array} \right\} \tag{115}$$

Families of parametrized measures are connected to the weak convergence as described in the following theorem (see Pedregal [91, Chapter 6, Theorem 6.2]):

Theorem 17. *Let* $\{\mathbf{v}_n\}_{n=1}^\infty$, $\mathbf{v}_n : Q \subset \mathbb{R}^N \to \mathbb{R}^M$ *be a weakly convergent sequence of functions in* $L^1(Q;\mathbb{R}^M)$*, where* Q *is a domain in* $\mathbb{R}^N$*.*

Then there exist a subsequence (not relabeled) $\{\mathbf{v}_n\}_{n=1}^\infty$ *and a parameterized family* $\{\nu_y\}_{y\in Q}$ *of probability measures on* $\mathbb{R}^M$ *depending measurably on* $y \in Q$ *with the following property:*

For any Carathéodory function $\Phi = \Phi(y,z)$*,* $y \in Q$*,* $z \in \mathbb{R}^M$ *such that*

$$\Phi(\cdot,\mathbf{v}_n) \to \overline{\Phi} \text{ weakly in } L^1(Q),$$

we have

$$\overline{\Phi}(y) = \int_{\mathbb{R}^M} \psi(y,z)\, \mathrm{d}\nu_y(z) \text{ for a.a. } y \in Q.$$

The family of measures $\{\nu_y\}_{y\in Q}$ associated to a sequence $\{\mathbf{v}_n\}_{n=1}^\infty$, $\mathbf{v}_n \rightharpoonup \mathbf{v}$ in $L^1(Q;\mathbb{R}^M)$, is termed *Young measure*. Suppose that $\mathbf{v}_n$ is only a bounded sequence in $L^1(Q)$. Then there still exists an associated parametrized family $\{\nu_y\}_{y\in Q}$ of nonnegative Borel measures with the properties stated in Theorem 17, which, however, do not need to be necessarily probability measures.

3.11 Some Elements of the DiPerna-Lions Transport Theory

In the following theorems, we present some consequences of the DiPerna-Lions transport theory applied to the continuity equation (see [33, Section 10.16].)

Theorem 18. *Let* $N \geq 2$, $\beta, q \in (1, \infty)$, $\frac{1}{q} + \frac{1}{\beta} \in (0, 1]$. *Suppose that the functions* $(\varrho, \mathbf{u}) \in L^{\beta}_{\rm loc}((0, T) \times \mathbb{R}^N) \times L^q_{\rm loc}(0, T; W^{1,q}_{\rm loc}(\mathbb{R}^N; \mathbb{R}^N))$, *where* $\varrho \geq 0$ *a. e. in* $(0, T) \times \mathbb{R}^N$, *satisfy the transport equation*

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = f \tag{116}$$

in $\mathcal{D}'((0, T) \times \mathbb{R}^N)$, *where* $f \in L^1_{\rm loc}((0, T) \times \mathbb{R}^N)$.

Then

$$\partial_t b(\varrho) + \mathrm{div}_x\Big((b(\varrho)\mathbf{u}\Big) + \Big(\varrho b'(\varrho) - b(\varrho)\Big)\mathrm{div}_x \mathbf{u} = f b'(\varrho) \tag{117}$$

in $\mathcal{D}'((0, T) \times \mathbb{R}^N)$ *for any*

$$b \in C^1([0, \infty)), \quad b' \in C_c([0, \infty)). \tag{118}$$

Theorem 19. *Let* $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ *be a bounded Lipschitz domain.*

(i) Suppose that (β, q) *satisfy assumptions of Theorem 18 and that* $(\varrho, \mathbf{u}) \in L^{\beta}((0, T) \times \Omega) \times L^q(0, T; W^{1,q}_0(\Omega))$. *Then there holds: If the couple* $(\varrho, \mathbf{u})$ *satisfies equation (116) in* $\mathcal{D}'((0, T) \times \Omega)$, *then it verifies the same equation also in* $\mathcal{D}'((0, T) \times \mathbb{R}^N)$ *provided* $(\varrho, \mathbf{u})$ *is extended to* $(0, T) \times \mathbb{R}^N$ *as follows:*

$$(\varrho, \mathbf{u})(t, x) = \begin{cases} (\varrho, \mathbf{u})(t, x) \text{ for } (t, x) \in (0, T) \times \Omega, \\ (\varrho_\infty \geq 0, 0) \text{ for } (t, x) \in (0, T) \times (\mathbb{R}^N \setminus \Omega). \end{cases} \tag{119}$$

(ii) Suppose that $(\varrho, \mathbf{u}) \in L^1((0, T) \times \Omega) \times L^q(0, T; W^{1,q}_0(\Omega))$, $q > 1$ *verifies renormalized continuity equation (117) in* $\mathcal{D}'((0, T) \times \Omega)$ *with any* b *belonging to class (118). Then the extension (119) verifies the same equation in* $\mathcal{D}'((0, T) \times \mathbb{R}^N)$ *for the same functions* b.

Theorem 20. *Let* $\Omega \subset \mathbb{R}^N$, $N \geq 2$ *be a bounded domain and let* $(\varrho, \mathbf{u})$, $\varrho \in L^{\infty}(0, T; L^{\beta}(\Omega))$, $\mathbf{u} \in L^q(0, T; W^{1,q}(\Omega))$, $f \in L^q((0, T) \times \Omega)$, $\varrho \mathbf{u} \in L^1((0, T) \times \Omega)$ *satisfies continuity equation (116) in* $\mathcal{D}'((0, T) \times \Omega)$ *and renormalized continuity equation (117) with any* b *in class (118). Then*

$$\varrho \in C([0, T]; L^1(\Omega)).$$

Lemma 2. *Let $N \geq 2$, $\beta, q \in [1, \infty)$, $\frac{1}{q} + \frac{1}{\beta} \in (0, 1]$. Suppose that the functions $(\varrho, \mathbf{u}) \in L^{\beta}_{\rm loc}((0,T)\times\mathbb{R}^N)\times L^q_{\rm loc}(0,T; W^{1,q}_{\rm loc}(\mathbb{R}^N;\mathbb{R}^N))$, where $\varrho \geq 0$ a.e. in $(0,T)\times \mathbb{R}^N$, satisfy the renormalized continuity equation (117) for any b belonging to the class (118).*

Then we have:

(i) If $f \in L^p_{\rm loc}((0,T)\times\mathbb{R}^N)$ for some $p > 1$, $p'(\frac{\beta}{q'} - 1) \leq \beta$, then equation (117) holds for any

$$b \in C^1([0,\infty)),\ \ |b'(s)| \leq cs^{\beta/q'-1},\ \textit{for}\ s > 1. \tag{120}$$

(ii) If $f = 0$, then equation (117) holds for any

$$b \in C([0,\infty)) \cap C^1((0,\infty)),\ sb' - b \in C[0,\infty),\ |b'(s)| \leq cs^{\beta/q'-1}\ \textit{if}\ s \in (1,\infty). \tag{121}$$

3.12 The Gronwall Lemma

We recall a variant of the Gronwall-Bellman lemma. The reader can consult the monograph [89] for the details on this variant and other differential and integral inequalities.

Theorem 21. *Let $\alpha \in L^1(0,T)$, $\beta \in L^1(0,T)$, $\beta \geq 0$ be given functions. Suppose that a function $u \in L^\infty(0,T)$ satisfies inequality*

$$u(\tau) \leq \alpha(\tau) + \int_0^\tau \beta(t)u(t)\mathrm{d}t\ \textit{for a.a.}\ \tau \in (0,T).$$

Then

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s)e^{\int_s^t \beta(z)\mathrm{d}z}\mathrm{d}s\ \textit{for a.a.}\ t \in (0,T).$$

4 Existence of Weak Solutions to the Compressible Navier-Stokes Equations for Barotropic Flows

In this section we shall define and investigate weak solutions to the system (63)–(64) in a time cylinder $Q_T = (0,T) \times \Omega$, where Ω is a bounded domain, with pressure

$$p = p(\varrho), \quad p \in C[0,\infty) \cap C^1(0,\infty),\ p(0) = 0. \tag{122}$$

and stress tensor (12), where

$$\mu = \text{const.} > 0, \quad \eta = \text{const.} \geq 0. \tag{123}$$

The system is completed with initial conditions

$$\varrho(0,\cdot) = \varrho_0(\cdot), \quad \varrho\mathbf{u}(0,\cdot) = \varrho_0\mathbf{u}_0, \tag{124}$$

and no-slip boundary conditions (20), i.e.,

$$\mathbf{u}(t,\cdot)|_{\partial\Omega} = 0. \tag{125}$$

4.1 Weak Formulation and Weak Solutions

We begin with the definition of the Leray-type weak solutions to problem (63)–(64), (122)–(125). It consists of the standard weak formulation of equations (63)–(64). Dissipation identity (66) will be replaced by the dissipation *inequality* "$\le$" in the integral form. In fact identity (66) integrated over time contains the functional $\mathbb{Z} \mapsto \int_0^\tau \int_\Omega \mathbb{S}(\mathbb{Z}) : \mathbb{Z} \, \mathrm{d}x\mathrm{d}t$, $\mathbb{Z} = \nabla\mathbf{u}$ that is not continuous but only sequentially lower weakly semicontinuous with respect to the weak topology of $L^2(Q_\tau;\mathbb{R}^9)$. Consequently, when passing from approximations to a solution, the limit processes will conserve solely the inequality "$\le$."

Definition 1. Let Ω be a bounded domain, and let

$$\varrho_0 : \Omega \to [0,+\infty), \mathbf{u}_0 : \Omega \to \mathbb{R}^3,$$

$$\varrho_0\mathbf{u}_0 = 0, \quad \varrho_0\mathbf{u}_0^2 = 0 \text{ a.e. in the set } \{x \in \Omega | \varrho_0(x) = 0\} \tag{126}$$

with finite energy $E_0 = \int_\Omega (\frac{1}{2}\varrho_0\mathbf{u}_0^2 + H(\varrho_0))\mathrm{d}x$ and finite mass $0 < M_0 = \int_\Omega \varrho_0 \mathrm{d}x$.

We shall say that a pair $(\varrho, \mathbf{u})$ is a *finite energy weak solution* to the problem (63)–(64), (122)–(125) emanating from the initial data $(\varrho_0, \mathbf{u}_0)$ if:

(a)

$$\varrho \in L^\infty(0,T;L^1(\Omega)),\ \varrho \ge 0 \ \text{ a.e. in } (0,T)\times\Omega,\ \ p(\varrho) \in L^1(Q_T), \tag{127}$$

$$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega)),\ \varrho\mathbf{u}, \frac{1}{2}\varrho\mathbf{u}^2,\ H(\varrho) \in L^\infty(0,T;L^1(\Omega)).$$

(b) $\varrho \in C_{\mathrm{weak}}([0,T];L^1(\Omega))$, and the continuity equation (63) is satisfied in the following weak sense

$$\int_\Omega \varrho\varphi \mathrm{d}x\Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\mathrm{d}x\mathrm{d}t, \tag{128}$$

for all $\tau \in [0,T]$ and for all $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$.

(c) $\varrho\mathbf{u} \in C_{\rm weak}([0,T];L^1(\Omega))$, and the momentum equation (64) is satisfied in the weak sense,

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi {\rm d}x\Big|_0^\tau = \int_0^\tau\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u}:\nabla\varphi + p(\varrho){\rm div}\varphi - \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\varphi + \varrho\mathbf{f}\cdot\varphi\Big)\,{\rm d}x{\rm d}t \tag{129}$$

for all $\tau \in [0,T]$ and for all $\varphi \in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3)$.

(d) The dissipation identity (66) is satisfied as inequality in the weak sense:

$$\begin{aligned} &-\int_0^T \psi'(t)\int_\Omega\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\Big){\rm d}x + \int_0^T\psi(t)\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\,{\rm d}x{\rm d}t \\ &\leq \int_0^T\psi(t)\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,{\rm d}x{\rm d}t + E_0\psi(0) \text{ for all } \psi\in C^1[0,T),\ \psi\geq 0. \end{aligned} \tag{130}$$

Here and hereafter the symbol $\int_\Omega g{\rm d}x\,|_0^\tau$ is meant for $\int_\Omega g(\tau,x){\rm d}x - \int_\Omega g_0(x){\rm d}x$. We recall that the Helmholtz function H is defined in (67). Space $C_{\rm weak}([0,T];L^1(\Omega))$ is defined in (71).

Definition 2. A couple $(\varrho,\mathbf{u})$ satisfying all requirements of Definition 1 with exception of the energy inequality (130) which is replaced by

$$\int_\Omega\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\Big){\rm d}x\Big|_0^\tau + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\,{\rm d}x{\rm d}t \leq \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,{\rm d}x{\rm d}t, \tag{131}$$

for almost all $\tau \in (0,T)$ will be called *bounded energy weak solution* of problem (63)–(64), (122)–(125).

Definition 3. We say that the couple

$$(\varrho,\mathbf{u}) \in L^\infty(0,T;L^1(\Omega))\times L^2(0,T;W^{1,p}(\Omega)),\ \varrho\geq 0,\ \varrho\mathbf{u}\in L^1(Q_T),\ p>1 \tag{132}$$

satisfies continuity equation in the renormalized sense iff it satisfies continuity equation (116) in $\mathcal{D}'((0,T)\times\Omega)$ and renormalized continuity equation (117) in $\mathcal{D}'((0,T)\times\Omega)$ with any test function b belonging to the class (118) and with $f=0$.

Weak solution to problem (63)–(64), (122)–(125) satisfying the continuity equation in the renormalized sense will be called *renormalized weak solution.*

Remark 1. 1. Suppose that $(\varrho,\mathbf{u})$ is a renormalized weak solution of the continuity equation such that

$$\varrho \in L^{\infty}(0,T;L^{\gamma}(\Omega)),\ \mathbf{u} \in L^2(0,T;W^{1,2}(\Omega)),\ \gamma > 1,$$

where Ω is a bounded domain. Then

$$\varrho \in C([0,T];L^1(\Omega)).$$

If, moreover, $\mathbf{u}|_{(0,T)\times\partial\Omega} = 0$ and Ω is a Lipschitz domain, then the renormalized continuity equation is satisfied up to the boundary, namely,

$$\int_{\Omega} b(\varrho(\tau,x))\varphi(\tau,x)\,\mathrm{d}x - \int_{\Omega} b(\varrho(0,x))\varphi(0,x)\,\mathrm{d}x \tag{133}$$

$$= \int_0^{\tau}\int_{\Omega}\Big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi - B(\varrho)\mathrm{div}_x\mathbf{u}\varphi\Big)\,\mathrm{d}x\mathrm{d}t = 0$$

for all $\tau \in [0,T]$, for all $\varphi \in C_c^1(\overline{Q}_T)$, and for all b, B belonging to

$$b \in C[0,\infty)\cap C^1(0,\infty),\ |b(z)| \le c(1+z^{\frac{5}{6}\gamma}),\ B \in C[0,\infty),\ |B(\varrho)| \le c(1+\varrho^{\frac{\gamma}{2}}), \tag{134}$$

where b, B are related by the formula $B(z) = zb'(z) - b(z)$. Moreover, $b(\varrho) \in C([0,T];L^1(\Omega))$.

If, in addition $\gamma \ge 2$, then the continuity equation is satisfied up to the boundary, namely,

$$\int_{\Omega}\varrho(\tau,x)\varphi(\tau,x)\,\mathrm{d}x - \int_{\Omega}\varrho(0,x)\varphi(0,x)\,\mathrm{d}x = \int_0^{\tau}\int_{\Omega}\Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\mathrm{d}t$$

for all $\tau \in [0,T]$ and for all $\varphi \in C_c^1(\overline{Q}_T)$.

The above statements follow from the DiPerna-Lions transport theory [18] evoked through Theorems 18, 19, 20, and Lemma 2. The reader can consult [30, Chapter 4, Section 4.1.5], [88, Chapter 6, Section 6.2], [33, Appendix, Section 10.18] for more details and proofs.

2. For any $t \in [0,T]$, the momentum $\varrho\mathbf{u}(t,\cdot)$ vanishes almost everywhere on the vacuum set of function $\varrho(t,\cdot)$. More precisely, properties $\varrho \in C([0,T];L^1(\Omega))$, $\varrho \ge 0$, $\varrho\mathbf{u} \in C_{\mathrm{weak}}([0,T];L^1(\Omega))$, and $\varrho\mathbf{u}^2 \in L^{\infty}(0,T;L^1(\Omega))$, where $\mathbf{u} \in L^1(Q_T)$, are enough to conclude that

$$\varrho\mathbf{u}(t,\cdot) = 0 \text{ a.e. on the set } \{x \in \Omega | \varrho(t,x) = 0\}. \tag{135}$$

Similarly, if in addition to previous hypotheses, $\varrho \in L^{\infty}(0,T;L^{3/2}(\Omega))$, then

$$\varrho\mathbf{u}^2 = 0 \text{ a.e. on the set } \cup_{t\in[0,T]}\Big(\{t\}\times\{x\in\Omega|\varrho(t,x)=0\}\Big). \tag{136}$$

3. We introduce *global kinetic energy* $E_{\rm kin} : [0,T] \mapsto [0,\infty)$ and *global elastic energy* $E_{\rm el} : [0,T] \mapsto [0,\infty)$

$$E_{\rm kin} = \text{inst}\Big[\int_\Omega \frac{1}{2}\varrho\mathbf{u}^2(t,x)\,{\rm d}x\Big],\quad E_{\rm el} = \text{inst}\Big[\int_\Omega H(\varrho)(t,x)\,{\rm d}x\Big], \tag{137}$$

where the instantaneous values were introduced in (70). We define global mechanical energy as

$$E_{\rm mech} \equiv E = E_{\rm kin} + E_{\rm el}. \tag{138}$$

With this notation, in agreement with (86)–(88), inequality (130) can be rewritten as identity

$$E(\tau)\psi(\tau) - \int_0^\tau \psi'(t)E(t){\rm d}t + \frac{1}{2}\Big(\int_{[0,\tau)}\psi(t){\rm d}\mu + \int_{[0,\tau]}\psi(t){\rm d}\mu\Big) \tag{139}$$

$$= \int_0^\tau \psi(t)\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,{\rm d}x{\rm d}t + E_0\psi(0) \text{ for all } \psi\in C^1[0,T], \psi\geq 0,$$

for all $\tau \in [0,T]$, where μ is a nonnegative measure on the $\sigma-$algebra of Borel sets of interval $[0,T]$ satisfying, in particular,

$$\frac{1}{2}\Big(\int_{[0,\tau)}\psi(t){\rm d}\mu + \int_{[0,\tau]}\psi(t){\rm d}\mu\Big) \geq \int_0^\tau \psi(t)\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\,{\rm d}x{\rm d}t, \text{ for all } \psi\in C[0,T], \psi\geq 0.$$

With this definition at hand, we may deduce from inequality (139) in agreement with (86)–(88) that

$$E(\tau) - E_0 + \frac{1}{2}\Big(\int_{[0,\tau)}{\rm d}\mu + \int_{[0,\tau]}{\rm d}\mu\Big) = \int_0^\tau\int_\Omega \varrho\mathbf{u}\cdot\mathbf{f}\,{\rm d}x{\rm d}t \text{ for all } \tau\in[0,T], \tag{140}$$

and

$$E(\tau) - E(z) + \frac{1}{2}\Big(\int_{[z,\tau)}{\rm d}\mu + \int_{(z,\tau]}{\rm d}\mu\Big) = \int_z^\tau\int_\Omega \varrho\mathbf{u}\cdot\mathbf{f}\,{\rm d}x{\rm d}t$$

for all $0 < z < \tau < T$.

In particular, function $\tau \mapsto E(\tau)$ is a sum of a nonincreasing function $\tau \mapsto -\frac{1}{2}\Big(\int_{[0,\tau)}{\rm d}\mu + \int_{[0,\tau]}{\rm d}\mu\Big)$ (that must have at most a countable number of jumps)

and an absolutely continuous function $\tau \mapsto \int_0^\tau \int_\Omega \varrho \mathbf{u}\cdot\mathbf{f}\,\mathrm{d}x\mathrm{d}t$. This representation of E is convenient to use for studying of the *longtime behavior of weak solutions.*
4. Relation (139) implies that any finite energy weak solution is a bounded energy weak solution.

Existence of weak solutions to problems (63)–(64),and (122)–(125) is known provided the pressure verifies in addition to (122) conditions

$$p'(\varrho) \geq a_1\varrho^{\gamma-1} - b, \quad \varrho > 0, \tag{141}$$

$$p(0) = 0, \quad p(\varrho) \leq a_2\varrho^\gamma + b, \quad \varrho \geq 0,$$

with some $\gamma > 3/2$, $a_1 > 0$, $a_2, b \in \mathbb{R}$. The exact statement of the existence result is announced in the following theorem:

Theorem 22 (See [77] for $p(\varrho) \approx \varrho^\gamma$, $\gamma \geq 9/5$, [47, Theorem 1.1] with $p(\varrho) \approx \varrho^\gamma$, $\gamma > 3/2$, [28, Theorem 1.1] for nonmonotone pressure (141) and $\gamma > 3/2$.). *Let Ω be a bounded domain of class $C^{2,\nu}$, $T > 0$ and $\mathbf{f} \in L^\infty(Q_T)$, where $Q_T = (0,T)\times\Omega$. Suppose that the initial data satisfy (126) and that the pressure p belongs to the regularity class (122) and satisfies condition (141) with $\gamma > 3/2$. Then the problem (63)–(64), (124), (125) admits a renormalized finite energy weak solution with the following additional properties*

$$\varrho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^\gamma(\Omega)) \cap L^{p_0}(Q_T),\ p_0 = \min\left\{\frac{5\gamma-3}{3}, \frac{3}{2}\gamma\right\}, \tag{142}$$

$$p(\varrho) \in L^{p_1}(Q_T),\ p_1 = p_0/\gamma > 1, \tag{143}$$

$$\varrho\mathbf{u} \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap C_{\mathrm{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega)). \tag{144}$$

The main ideas of the proof of Theorem 22 will be explained in the more general context of heat-conducting gases in Sect. 7. The detailed existence proof can be found in paper [47] for the monotone pressure and $\gamma > 3/2$ and in [28] for nonmonotone pressure. More details about this problem are available in monographs [30, 77, 88].

On unbounded domains, Definitions 1, 2, and 3 of finite (bounded/renormalized) weak solutions for the problem (63)–(64), (123) (124), (125) must be slightly modified in order to be able to accommodate conditions at infinity (24). We shall first consider the case

$$\varrho_\infty = 0,\ \mathbf{u}_\infty = 0,\ \text{cf. (24)}. \tag{145}$$

Definition 4. Let Ω be an unbounded domain. We say that couple $(\varrho, \mathbf{u})$ is (i) finite energy weak solution, (ii) bounded energy weak solution, (iii) renormalized weak solution of problem (63)–(64), (122)–(125) with zero conditions at infinity (145) iff: it belongs to class (127) with $\mathbf{u}$ belonging to $L^2(0,T;D_0^{1,2}(\Omega;\mathbb{R}^3))$ (in place of $\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$), $\varrho \in C_{\text{weak}}([0,T];L^1(K))$, $\varrho\mathbf{u} \in C_{\text{weak}}([0,T];L^1(K;\mathbb{R}^3))$ with any compact $K \subset \overline{\Omega}$ and

(i) it satisfies all requirements of Definition 1 (for finite energy weak solution);
(ii) it satisfies all requirements of Definition 2 (for bounded energy weak solution);
(iii) it satisfies all requirements of Definition 3 (for renormalized weak solution).

In the above, we have denoted by $D_0^{1,2}(\Omega)$ the homogenous Sobolev space given by

$$\text{closure}_{\|\nabla_x \cdot\|_{L^2(\Omega;R^3)}}\Big(C_c^\infty(\Omega)\Big).$$

The weak solutions designed in Definitions 1, 2, 3, and 4 enjoy the following stability condition with respect to the variations of the domain:

Theorem 23 (See [48, Theorem 1.1]). *Let Ω_n be a sequence of domains in $\mathbb{R}^3$ and Ω be a domain, such that:*

(i) For any compact set $K \subset \Omega$, there is a natural number n_0 such that for all $n \geq n_0$, $K \subset \Omega_n$;
(ii) Sets $\Omega_n \setminus \Omega$ enjoy the property $\text{cap}_2(\overline{\Omega_n \setminus \Omega}) \to 0$, where

$$\text{cap}_2(M) = \inf\{\int_{R^3} |\nabla_x \phi| \mathrm{d}x \mid \phi \in C_c^\infty(\mathbb{R}^3),\ \phi \geq 1 \ on\ M\}.$$

Let $(\varrho_n, \mathbf{u}_n)$ be a sequence of bounded energy renormalized weak solutions to problem (63)–(64), (123), (124), (125) (and (24), (145) if Ω is unbounded) with pressure p satisfying (122), (141) with $\gamma > 3/2$ with initial conditions $(\varrho_{n,0} \geq 0, \mathbf{u}_{n,0})$ and external forces $\mathbf{f}_n$ such that

$$\Big(\varrho_{n,0},\ \varrho_{n,0}\mathbf{u}_{n,0}\Big) \to \Big(\varrho_0, \varrho_0\mathbf{u}_0\Big) \ in\ L^1(\mathbb{R}^3;\mathbb{R}^4)\ (when\ extended\ by\ (0,\mathbf{0})\ to\ \mathbb{R}^3),$$

$$E_{n,0} = \int_{\Omega_n}\Big(\varrho_{n,0}|\mathbf{u}_{n,0}|^2 + H(\varrho_{n,0})\Big)\mathrm{d}x \to E_0,\ \mathbf{f}_n \to \mathbf{f}\ in\ L^\infty \cap L^1((0,T)\times\mathbb{R}^3;\mathbb{R}^3).$$

Then, extending $(\varrho_n, \mathbf{u}_n)$ by $(0,\mathbf{0})$ in $(0,T)\times(\mathbb{R}^3\setminus\Omega)$ and passing to a subsequence as the case may be, we have

$$\varrho_n \to \varrho\ in\ C([0,T];L^1(\mathbb{R}^3)),\ \mathbf{u}_n \rightharpoonup \mathbf{u}\ in\ L^2(0,T;W^{1,2}(\mathbb{R}^3)),$$

where $(\varrho, \mathbf{u})$ is a bounded energy renormalized weak solution of the same problem on $(0,T)\times\Omega$ for initial conditions $(\varrho_0, \mathbf{u}_0)$.

Remark 2. 1. A sufficient condition guaranteeing (ii) is, for example, $\Omega_n \setminus \Omega$ bounded, $|\partial\Omega_n| = |\partial\Omega| = 0$, and $|\Omega_n \setminus \Omega| \to 0$.

2. Existence of weak solutions on nonsmooth domains. Theorem 23 asserts existence of *bounded energy* weak solutions on a large class of nonsmooth bounded domains. These weak solutions are however not finite energy weak solutions. *Finite energy weak solutions* do not exist in general on nonsmooth domains, but they are known to exist on domains that are Lipschitz (or even slightly less regular than Lipschitz; see [73]).

   More exactly, the conclusion of Theorem 22 is valid under the same assumptions for bounded Lipschitz domains. To see this fact, one may approximate domain Ω by "larger" smooth domains Ω_n and construct the finite energy weak solutions $(\varrho_n, \mathbf{u}_n)$ on domains Ω_n according to Theorem 22. Since Ω is Lipschitz, we obtain the crucial estimate $H(\varrho_n)$ up to the boundary (in $L^p((0,T) \times \Omega)$, $p > 1$), thanks to the properties of the Bogovskii operator on Lipschitz domains (see Theorem 5 and its application exposed in *item* 6 of Sect. 7.1). This estimate suffices to pass to the limit in the differential form of the dissipation inequality (130).

   This is in sharp contrast with the case of a nonsmooth bounded domain when the Bogovskii operator provides only local estimates out of the boundary for the sequence ϱ_n. Under this circumstance one does not have almost everywhere convergence of sequence $H(\varrho_n)$ up to the boundary, and one must use the lower weak semi-continuity and the weaker integral form (131) of the dissipation inequality for the limit passage. The reader can consult Kukucka [73], Poul [48, 94], and comments in [88, Section 7.12] for related material.

3. Existence of weak solutions on unbounded domains (case $\varrho_\infty, \mathbf{u}_\infty) = (0, \mathbf{0})$).

   Large class of unbounded domains (in particular, exterior domains, but many others) can be approximated by $C^{2,\nu}$ domains in the sense of convergence postulated in Theorem 23.

   Theorem 23 in combination with the existence Theorem 22 thus guarantees existence of *bounded energy weak solutions* to problem (63)–(64), (123) (124), (125) endowed *with conditions at infinity* (145) on an unbounded domain Ω in the class described in the above alinea, provided hypotheses of Theorem 22 are satisfied on Ω, and $\mathbf{f}$ belongs additionally to $L^1((0,T) \times \Omega)$. Existence of finite energy weak solutions in this situation is not known.

4. Existence of (bounded) energy weak solutions on unbounded domains (case $\varrho_\infty > 0, \mathbf{u}_\infty \in \mathbb{R}^3$).

   If $\mathbf{u}_\infty = \mathbf{0}$, the definition of the bounded energy weak solutions has to be changed as follows: (1) as far as the functional spaces, we must take $\varrho \in L^\infty(0,T; L^1_{\rm loc}(\overline{\Omega})) \cap C_{\rm weak}([0,T]; L^1(K))$ (K any compact subset of $\overline{\Omega}$), $\mathbf{u} \in L^2(0,T; D_0^{1,2}(\Omega; \mathbb{R}^3))$, $\varrho\mathbf{u} \in L^\infty(0,T; L^1_{\rm loc}(\overline{\Omega}; \mathbb{R}^3)) \cap C_{\rm weak}([0,T]; L^1(K; \mathbb{R}^3))$, and $p(\varrho) \in L^\infty(0,T; L^1_{\rm loc}(\overline{\Omega}))$; (2) weak formulations to the continuity and momentum equations remain without changes (see (128), (129)); and (3) the dissipation inequality (131) must be replaced by

$$\int_\Omega \Big(\varrho \mathbf{u}^2 + H(\varrho) - H'(\varrho_\infty)(\varrho - \varrho_\infty) - H(\varrho_\infty)\Big)\mathrm{d}x\Big|_0^\tau \tag{146}$$
$$+\int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\, \mathrm{d}x\mathrm{d}t \le \int_0^\tau \int_\Omega \varrho \mathbf{f}\cdot\mathbf{u}\, \mathrm{d}x\mathrm{d}t.$$

Bounded energy weak solutions are known to exist on a large class of uniformly bounded Lipschitz domains, provided $\mathbf{f} \in L^1 \cap L^\infty((0,T)\times\Omega;\mathbb{R}^3)$ for the initial data with finite energy $\int_\Omega \Big(\varrho_0\mathbf{u}_0^2 + H(\varrho_0) - H'(\varrho_\infty)(\varrho_0 - \varrho_\infty) - H(\varrho_\infty)\Big)\mathrm{d}x$. Existence of finite energy weak solutions (where the dissipation inequality (146) is replaced by its differential counterpart) is not known in this situation. The reader can consult [88] and [70] for more details and related material on unbounded domains in this situation.

The treatment when $\mathbf{u}_\infty \neq 0$ is slightly more involved. It is investigated in [88, Definition 7.78, Theorem 7.79] in the case of an exterior domain.

5. One can consider the same problem (63)–(64), (124) with the *complete slip* (22) or with the *Navier slip* (23) boundary conditions for the velocity (instead of $\mathbf{u}|_{\partial\Omega} = 0$) on a bounded domain provided one modifies appropriately the definition of weak solutions. For example, in the case of Navier's boundary conditions, the necessary modifications in the definition of finite energy weak solutions are the following: (1) In functional spaces (see formula (127)), one has to require $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$ and $\mathbf{u}\cdot\mathbf{n}|_{(0,T)\times\partial\Omega} = 0$ in the sense of traces instead of $\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$. (2) In the weak formulation of the momentum equation (129), one has to add to the right-hand side term $-\Lambda\int_0^T\int_{\partial\Omega}\mathbf{u}\cdot\varphi \mathrm{d}S_x\mathrm{d}t$ and to consider test function $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$, $\varphi\cdot\mathbf{n}|_{[0,T]\times\partial\Omega} = 0$. 3) One has to add term $\Lambda\int_0^T\psi(t)\int_{\partial\Omega}|\mathbf{u}|^2\mathrm{d}S_x\mathrm{d}t$ to the left-hand side of the dissipation inequality (130).

   Once these modifications are done, one can prove existence of *finite energy* weak solutions under the same assumptions on the regularity of the domain, initial data, external force, constitutive relations, and transport coefficients as in Theorem 22. The solutions constructed in this way enjoy all additional properties mentioned in Theorem 22. Also in this situation, any finite energy weak solution is also a bounded energy weak solution. The reader can consult [77], [88, Section 7.12.2], [33, Chapter 3] for related considerations.
6. Likewise one can consider finite (and bounded) energy weak solutions to the problem (63)–(64), (124) with periodic boundary conditions (i.e., Ω is replaced by the *periodic cell* $([0,1]|_{\{0,1\}})^3$ (1− periodic torus)- with period 1 for simplicity. In this case, all function spaces entering into the definition of weak solutions are replaced by the functional spaces of (periodic) functions on the torus with the same regularity and integrability properties. Theorem 22 holds also in this situation.
7. The case of non-homogenous boundary conditions. The reasonable (and natural) definition of weak solutions of problem (63)–(64), (124) with nonzero inflow-outflow boundary conditions

$$\left\{ \begin{array}{c} \varrho(t,x) = \varrho_\infty(t,x) \text{ on } \cup_{t\in(0,T)} \Big(\{t\} \times \Gamma_{\rm in}(t)\}\Big), \\ \mathbf{u}(t,x) = \mathbf{u}_\infty(t,x) \text{ on } (0,T) \times \partial\Omega \end{array} \right\}, \tag{147}$$

where $\Gamma_{\rm in}$ is the inflow part of the boundary,

$$\Gamma_{\rm in}(t) = \{x \in \partial\Omega \,|\, \mathbf{u}_\infty(t,x)\cdot\mathbf{n}(x) < 0\},$$

has been suggested in [88, Section 7.12.5]. Existence of this weak solution has been proved in Novo [85] (for Ω a ball and ϱ_∞, $\mathbf{u}_\infty$ constant) and in Girinon [62] (where the domain and boundary data can be more general, but the inflow boundary must be convex and contained in the cone and the inflow velocity must verify the so-called no-reflux condition). The general result without these limitations has been obtained recently in [12].

8. Theorem 22 is true also for bounded two-dimensional domains provided $\gamma > 1$. In the borderline cases ($\gamma = 3/2$ for the three-dimensional domains and $\gamma = 1$ for the two-dimensional domains), the main difficulty in proving the existence of weak solutions comes from the limit passage in the convective term (at least in two dimensions). The two-dimensional case has been solved only recently (see [92]); the three-dimensional case still resists. These problems are subject of a separate chapter of the handbook.
9. The progress within the framework of the Lions' theory (with limitation $\gamma \geq 9/5$) has been made also in another directions. It concerns the relaxation of certain hypotheses on the pressure (allowing more general nonmonotonicity than stipulated in (141)) and the relaxation in the conditions in the form of the viscous stress tensor (allowing small anisotropic perturbations of the stress tensor (12) in the case of constant viscosities (123)) (see D. Bresch and P.E. Jabin [9]).
10. Existence of time periodic solutions is subject of papers [46, 47].

Remark 3. 1. Sometimes, it may be convenient to use another representation of mechanical energy than the representation (137). To this end we introduce lower continuous convex function

$$\mathfrak{e} : \mathbb{R}\times\mathbb{R}^3 \mapsto (-\infty,\infty],\ \mathfrak{e}(r,\mathbf{q}) = \left\{ \begin{array}{c} \frac{1}{2}\frac{\mathbf{q}^2}{r} \text{ if } r > 0, \\ 0 \text{ if } (r,\mathbf{q}) = (0,\mathbf{0}), \\ +\infty \text{ if } r \leq 0, (r,\mathbf{q}) \neq (0,\mathbf{0}). \end{array} \right. \tag{148}$$

We realize that under hypothesis (141),

$$H(\varrho) = A(\varrho) + B(\varrho),\ A(\varrho) = \varrho\int_1^\varrho \frac{p(z)-p(1)+bz}{z^2}{\rm d}z,\ B(\varrho) = \varrho\int_1^\varrho \frac{p(1)-bz}{z^2}{\rm d}z,$$

where A is convex continuous function on $[0,\infty)$, $|A(z)| \leq c(1+z^\gamma)$ and B is continuous on $[0,\infty)$, $B(z) \leq c(1+z|\ln z|)$ with some $c > 0$, for all $z \in (0,\infty)$. We introduce mechanical energy

$$[0,T] \ni t \mapsto \mathfrak{E}_{\rm mech}(t) = \mathfrak{E}(t) = \int_\Omega \mathfrak{e}\Big(\varrho(t,x), \varrho\mathbf{u}(t,x)\Big)\,{\rm d}x + \int_\Omega H(\varrho(t,x))\,{\rm d}x. \tag{149}$$

Since $\int_\Omega \frac{1}{2}\varrho(t,x)\mathbf{u}^2(t,x)\,{\rm d}x = \int_\Omega \mathfrak{e}\Big(\varrho(t,x), \varrho\mathbf{u}(t,x)\Big)\,{\rm d}x$ for a.a. $t \in (0,T)$, we have

$$\mathfrak{E}(t) = E(t) \text{ for almost all } t \in (0,T). \tag{150}$$

Moreover, according to theorem of lower weak semi-continuity of convex functionals in form formulated in Theorem 2, function

$$[0,T] \ni t \mapsto \mathfrak{E}(t) \text{ is lower semicontinuous function,} \tag{151}$$

in particular

$$\mathfrak{E}(0) = E_0 \le \liminf_{t\to 0+} \mathfrak{E}(t).$$

5 Dissipative Solutions, Relative Energy Inequality, and Weak-Strong Uniqueness Principle

5.1 Relative Energy and Relative Energy Functional

Let us now introduce the notion of the *relative energy*. We first introduce the *relative energy function*

$$\begin{aligned} E &: [0,\infty) \times (0,\infty) \to \mathbb{R}, \\ &(\varrho, r) \mapsto E(\varrho|r) = H(\varrho) - H'(r)(\varrho - r) - H(r), \end{aligned} \tag{152}$$

where H is defined by (67). If the pressure verifies the monotonicity hypothesis

$$p'(\varrho) > 0 \text{ for all } \varrho > 0, \tag{153}$$

the Helmoholtz function H is strictly convex on $[0,\infty)$, and therefore

$$E(\varrho|r) \ge 0 \quad \text{and} \quad E(\varrho|r) = 0 \ \Leftrightarrow \ \varrho = r.$$

In fact function $E(\cdot|\cdot)$ possesses better coercivity properties than stated above. This is subject of the following lemma whose proof is an easy application of the real analysis of functions of two variables.

Lemma 3. *Let* $0 < a < b < \infty$ *and let*

$$p \in C[0,\infty) \cap C^1(0,\infty), \ \ p(0) \ge 0, \quad p'(\varrho) > 0.$$

Then there exists a number $c = c(a,b) > 0$ *such that for all* $\varrho \in [0,\infty)$ *and* $r \in [a,b]$,

$$E(\varrho|r) \geq c(a,b)\Big(1_{\mathcal{O}_{\rm res}}(\varrho) + \varrho 1_{\mathcal{O}_{\rm res}}(\varrho) + (\varrho - r)^2 1_{\mathcal{O}_{\rm ess}}(\varrho)\Big), \tag{154}$$

where E *is defined in* (152) *and*

$$\mathcal{O}_{\rm ess} = [a/2, 2b],\ \mathcal{O}_{res} = [0,\infty) \setminus \mathcal{O}_{\rm ess}. \tag{155}$$

In order to measure the "distance" between a weak solution $(\varrho, \mathbf{u})$ of the compressible Navier-Stokes system and any other state $(r, \mathbf{U})$ of the fluid, we introduce the relative energy functional, defined by

$$\mathcal{E}(\varrho, \mathbf{u}\Big| r, \mathbf{U}) = \int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + E(\varrho\,|\,r)\Big)\mathrm{d}x. \tag{156}$$

It appears that any (bounded energy) weak solution satisfies an inequality involving the relative energy functional called *relative energy inequality* regardless whether the pressure satisfies the thermodynamic stability condition. It is however to be noticed that the relative energy functional measures "a distance" between weak solution and any other state of the fluid only provided thermodynamic stability condition (153) is satisfied.

This fact is formulated in the following theorem:

Theorem 24. *If* $(\varrho, \mathbf{u})$ *is a weak solution to problem* (63)–(64), (122)–(125) *emanating from the finite energy initial data* $(\varrho_0, \mathbf{u}_0)$ *specified in* (126) *and external force* $\mathbf{f} \in L^\infty(Q_T; \mathbb{R}^3)$, *then*

$$\mathcal{E}\Big(\varrho, \mathbf{u}\Big| r, \mathbf{U}\Big)(\tau)\Big|_0^\tau + \int_0^\tau\int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u} - \mathbf{U})\Big) : \nabla_x(\mathbf{u} - \mathbf{U})\,\mathrm{d}x\mathrm{d}t \leq \tag{157}$$

$$\int_0^\tau\int_\Omega \mathbb{S}\Big(\nabla_x \mathbf{U}\Big) : \nabla_x(\mathbf{U} - \mathbf{u})\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \varrho\partial_t\mathbf{U}\cdot(\mathbf{U} - \mathbf{u})\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \varrho\mathbf{u}\cdot\nabla\mathbf{U}\cdot(\mathbf{U} - \mathbf{u})\,\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega p(\varrho)\mathrm{div}\mathbf{U}\,\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \frac{r - \varrho}{r}\partial_t p(r)\,\mathrm{d}x\mathrm{d}t$$
$$-\int_0^\tau\int_\Omega \frac{\varrho}{r}\nabla_x p(r)\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t - \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot(\mathbf{U} - \mathbf{u})\,\mathrm{d}x\mathrm{d}t$$

for a.a. $\tau \in (0,T)$, *and for any pair of test functions*

$$r \in C^1([0,T]\times\overline{\Omega}),\ r > 0,\ \mathbf{U} \in C^1_c([0,T]\times\overline{\Omega};\mathbb{R}^3),\ \mathbf{U}|_{(0,T)\times\partial\Omega} = 0. \tag{158}$$

Remark 4. 1. Theorem 24 remains true if one replaces the Dirchlet boundary conditions (125) with the slip (22) or Navier's conditions (23). In the latter case, we have to add to the left-hand side of the relative energy inequality term $\Lambda \int_0^\tau \int_{\partial\Omega} |\mathbf{u} - \mathbf{U}|^2 \mathrm{d}S_x \mathrm{d}t$, and the test functions $(r, \mathbf{U})$ must be taken in the class (158), where however condition $\mathbf{U}|_{(0,T)\times\partial\Omega} = 0$ must be replaced by $\mathbf{U}\cdot\mathbf{n}|_{(0,T)\times\partial\Omega} = 0$ (see [50, Section 3.2.1]).
2. Theorem 24 remains valid if one replaces bounded domain with an unbounded domain and considers in addition conditions $(\varrho_\infty \geq 0, \mathbf{u}_\infty = 0)$ at infinity (cf. (24) and *items* 3, 4 in Remark 2). In this case the test functions $(r, \mathbf{U})$ must be taken in class (158), where $r - \varrho_\infty \in C_c^1(([0,T]\times\overline{\Omega})$ (see [50, Theorem 2.4]).

Theorem 24 has been formulated in [50] (see also [49]) under assumptions that p additionally complies with the assumptions (141) of the existence theory and satisfies the thermodynamic stability conditions. The proof from [50] can be repeated line by line without those additional assumptions. The reader can consult similar and more involved proof of Theorem 39 (dealing with the full Navier-Stokes-Fourier system). Under thermodynamic stability conditions, relative energy inequality becomes a powerful tool with many applications, in singular limit investigation [35, 36, 38, 52, 80, 100] and in numerical analysis [60], to name only a few. In what follows, we shall concentrate to the applications closely related to the problem of well posedness of weak solutions: weak-strong uniqueness principle and longtime behavior of weak solutions.

5.2 Dissipative Solutions

Inspired by Theorem 24, and following the philosophy of P.L. Lions [76] for the Euler equations (that can be traced back to Prodi [95] and Serin [98] in the case of incompressible Navier-Stokes equations), we define for the compressible Navier-Stokes equation the notion of *dissipative solutions* that is weaker than weak solutions.

Definition 5. The couple $(\varrho, \mathbf{u})$ is a *dissipative solution* of problem (63)–(64), (122)–(125) iff:

(a) It belongs to class (127).
(b) It satisfies relative energy inequality (157).

Remark 5. 1. According to Theorem 24, under assumptions of the existence Theorem 22, problem (63)–(64), (122)–(125), admits at least one dissipative solution.
2. Any bounded energy weak solution $(\varrho, \mathbf{u})$ to problem (63)–(64), (122)–(125) is a dissipative solution (regardless the thermodynamic stability condition and the asymptotic behavior of $\varrho \mapsto p(\varrho)$ for large values of ϱ). The validity of the

opposite statement is an open problem; it is not known whether any dissipative solution is a weak solution (even if condition (153) holds).
3. Under the hypotheses (141) of the existence theory (invoked in Theorem 22) and under the thermodynamic stability conditions (153), finite energy weak solutions satisfying relative energy inequality to system (63)–(64), (122)–(125) have been for the first time constructed in [49].

5.3 Relative Energy Inequality with a Strong Solution as a Test Function

If the test functions $(r, \mathbf{U})$ in the relative energy inequality (157) obey equations (63)–(64) almost everywhere in Q_T, the right-hand side of the relative energy becomes quadratic in differences $(\varrho - r, \mathbf{u} - \mathbf{U})$. This observation is subject of the following lemma:

Lemma 4. *Let Ω be a bounded Lipschitz domain and $\mathbf{f} \in L^\infty(Q_T)$. Let $(\varrho, \mathbf{u})$ be a weak solution to the Navier-Stokes equations with initial and boundary conditions (124)–(125). Let $(r, \mathbf{U})$ that belongs to the class*

$$0 < \underline{r} \le r \le \overline{r} < \infty; \quad \mathbf{U} \in L^\infty(0, T; L^\infty(\Omega)), \tag{159}$$
$$\partial_t r,\ \partial_t \mathbf{U},\ \nabla_x r,\ \nabla_x \mathbf{U} \in L^2(0, T; L^\infty(\Omega)),$$

be another (weak) solution of the same equations with initial data $(r(0), \mathbf{U}(0)) = (r_0, \mathbf{U}_0)$. Then, under assumptions of Theorem 24,

$$\mathcal{E}\Big(\varrho, \mathbf{u}\Big| r, \mathbf{U}\Big)\Big|_0^\tau + \int_0^\tau \int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u}_\varepsilon - \mathbf{U})\Big) : \nabla_x(\mathbf{u} - \mathbf{U})\, \mathrm{d}x\mathrm{d}t \tag{160}$$

$$\le \int_0^\tau \int_\Omega (\rho - r)(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x \mathbf{U})\cdot(\mathbf{U} - \mathbf{u})\, \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \rho(\mathbf{u} - \mathbf{U})\cdot\nabla_x \mathbf{U}\cdot(\mathbf{U} - \mathbf{u})\, \mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \frac{\nabla_x p(r)}{r}(r - \rho)\cdot(\mathbf{u} - \mathbf{U})\, \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega \Big(p(\rho) - p'(r)(\rho - r) - p(r)\Big)\mathrm{div}\mathbf{U}\, \mathrm{d}x\mathrm{d}t$$

for a.a. $\tau \in (0, T)$.

Sketch of the proof. We deduce from regularity (159) and weak formulation of the momentum equation (129) that $\nabla_x^2 \mathbf{U} \in L^2(0, T; L^\infty(\Omega; \mathbb{R}^{27}))$. The couple $(r, \mathbf{U})$ is in fact a strong solution and satisfies momentum and continuity equations a.e. in Q_T:

$$\partial_t r + \mathrm{div}(r\mathbf{U}) = 0 \text{ a.e. in } (0, T) \times \Omega, \tag{161}$$

$$r\partial_t \mathbf{U} + r\mathbf{U}\cdot\nabla\mathbf{U} + \nabla p(r) = \mathrm{div}\mathbb{S}(\nabla\mathbf{U}) + r\mathbf{f} \text{ a.e. in } (0, T) \times \Omega. \tag{162}$$

The scalar product of (162) and $\mathbf{u}-\mathbf{U}$ integrated over Ω yields

$$\int_{\Omega}\Big(r\partial_t\mathbf{U}+r\mathbf{U}\cdot\nabla\mathbf{U}-r\mathbf{f}+\nabla p(r)\Big)\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x+\int_{\Omega}\mathbb{S}(\nabla\mathbf{U}):\nabla(\mathbf{u}-\mathbf{U})\,\mathrm{d}x=0, \tag{163}$$

where we have used the integration by parts in the last integral.

Now we put together identity (163) and relative energy inequality (157). Formula (160) appears after a straightforward calculation. This finishes proof of Lemma 4.

5.4 Stability and Weak-Strong Uniqueness

We shall show here three versions of theorems on stability of strong solutions in the class of weak solutions and of weak-strong uniqueness theorems.

In the first theorem, we shall require for the pressure solely the thermodynamic stability condition, while we shall suppose that the weak solution has density bounded from below and from above by positive constants.

Theorem 25. *Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the pressure p is twice continuously differentiable on $(0,\infty)$ and verifies thermodynamic stability condition (153).*

Let $(\varrho,\mathbf{u})$ be a weak solution to the Navier-Stokes equations (63)–(64), (124)–(125) emanating from initial data $(\varrho_0,\mathbf{u}_0)$ specified in (126) in the time interval $[0,T)$, $T>0$ such that

$$0<\underline{\varrho}<\varrho(t,x)<\overline{\varrho}<\infty. \tag{164}$$

Let $(r,\mathbf{U})$ be a strong solution of the same equations in the regularity class (159), with initial data $(r_0,\mathbf{U}_0)$ satisfying (126).

Then

$$\int_{\Omega}\Big(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+|\varrho-r|^2\Big)(\tau)\,\mathrm{d}x\leq c\int_{\Omega}\Big(\frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{U}_0|^2+|\varrho_0-r_0|^2\Big)\,\mathrm{d}x.$$

Theorem 25 has a drawback: it is a conditional result in the sense that it is not known whether one can construct global in time weak solutions satisfying the additional condition (164). In the second and third theorems, we require for pressure slightly more than the thermodynamic stability conditions. As a counterpart we can deal with bounded energy weak solutions without any additional assumptions. This allows us to get unconditional results.

Theorem 26. *Let Ω be a bounded Lipschitz domain. Suppose that pressure satisfies in addition to the thermodynamic stability condition (153)*

$$c_1+c_2\varrho+H(\varrho)\geq p(\varrho)\ \textit{for all}\ \varrho\geq\overline{R}, \tag{165}$$

where $\overline{R}$*,* c_1*,* c_2 *are some positive constants. Assume further that pressure belongs to the regularity class* (122) *and is twice continuously differentiable on* $(0, \infty)$ *and that viscosities* μ*,* η *verify* (123)*. Assume that the external force* $\mathbf{f} \in L^\infty(Q_T, \mathbb{R}^3)$*.*

Let $(\varrho, \mathbf{u})$ *be a weak solution to the Navier-Stokes equations* (63)–(64)*,* (124)–(125) *emanating from initial data* $(\varrho_0, \mathbf{u}_0)$ *specified in* (126)*. Let* $(r, \mathbf{U})$ *be a strong solution of the same equations with initial data* $(r_0, \mathbf{U}_0)$ *as in* (126) *that belongs to the class* (159)*.*

Then there exists a positive number c (dependent on $\mu, T, |\Omega|, \mathrm{diam}\Omega, \underline{r}, \overline{r}$*,* $\|p\|_{C^2([\underline{r}/2, 2\overline{r}])}$*,* $\|\mathbf{f}, \mathbf{U}\|_{L^\infty(Q_T;\mathbb{R}^6)}$*,* $\|\partial_t \mathbf{U}, \nabla \mathbf{U}, \nabla r\|_{L^2(0,T;L^\infty(\Omega;\mathbb{R}^{15}))}$ *but independent of the weak solution itself) such that*

$$\mathcal{E}(\varrho, \mathbf{u}|r, \mathbf{U})(\tau) \leq c\mathcal{E}(\varrho_0, \mathbf{u}_0|r_0, \mathbf{U}_0) \tag{166}$$

for a.a. $\tau \in (0, T)$*.*

In particular, if $(\varrho_0, \mathbf{u}_0) = (r_0, \mathbf{U}_0)$*, then*

$$\varrho = r, \ \mathbf{u} = \mathbf{U} \text{ in } [0, T] \times \Omega. \tag{167}$$

The third variant of the weak-strong uniqueness theorem is the following:

Theorem 27. *Conclusions* (166)–(167) *of Theorem 26 remain true if we replace the class of strong solutions* (159) *with the larger class*

$$0 < \underline{r} \leq r \leq \overline{r} < \infty, \quad \mathbf{U} \in L^\infty((0, T) \times \Omega), \tag{168}$$

$$\nabla_x r \in L^2(0, T; L^q(\Omega; \mathbb{R}^3)), \ \nabla_x^2 \mathbf{U} \in L^2(0, T; L^q(\Omega)), \ q > \max\{3, \frac{6\gamma}{5\gamma - 6}\}.$$

and the hypothesis (165) *by the stronger hypothesis* (141) *with* $\gamma > 6/5$*.*

Remark 6. 1. One may verify by using the definition of Helmholtz function H that if pressure satisfies assumptions of Lemma 3 and condition

$$0 < \frac{1}{p_\infty} \leq \liminf_{\varrho \to \infty} \frac{p(\varrho)}{\varrho^\gamma} \leq \limsup_{\varrho \to \infty} \frac{p(\varrho)}{\varrho^\gamma} \leq p_\infty < \infty, \text{ where } \gamma > 0,$$

then it satisfies condition (165). In particular, any pressure satisfying the thermodynamic stability condition (153) and assumption $(141)_{\gamma>1}$ verifies condition (165). Consequently, weak solutions constructed in Theorem 22 verify the weak-strong uniqueness principle, provided the pressure is, in addition to the hypotheses in Theorem 22, twice continuously differentiable on $(0, \infty)$ and verifies thermodynamic stability condition (153).

2. Under assumptions that Ω is a bounded domain of class C^4, $p \in C^3(0, \infty)$, $\mathbf{f} \in L^2_{\mathrm{loc}}([0, \infty); W^{2,2}(\Omega; \mathbb{R}^3))$, $\partial_t \mathbf{f} \in L^2_{\mathrm{loc}}([0, \infty); L^2(\Omega; \mathbb{R}^3))$

and $\varrho_0 \in W^{3,2}(\Omega)$, $\inf_\Omega \varrho_0 > 0$, $\mathbf{u}_0 \in W^{3,2}(\Omega;\mathbb{R}^2)$ satisfying the *compatibility condition at the boundary* $\frac{1}{\varrho_0}\Big(-\nabla_x p(\varrho_0) + \mathrm{div}\mathbb{S}(\nabla_x \mathbf{u}_0) + \varrho_0 \mathbf{f} - \varrho_0 \mathbf{u}_0 \cdot \nabla_x \mathbf{u}_0\Big)\Big|_{\partial\Omega} = 0$, Valli [104, Theorem A] constructed a unique strong solution to problem (63)–(64), (123)–(125) in the regularity class $\varrho \in C([0,T_M);W^{3,2}(\Omega))$, $\mathbf{u} \in L^2(0,T_M;W^{4,2}(\Omega;\mathbb{R}^3))$, $\partial_t \varrho \in L^2(0,T_M;W^{2,2}(\Omega))$, $\partial_t \mathbf{u} \in L^2([0,T_M);W^{2,2}(\Omega;\mathbb{R}^3))$, $0 < \underline{r} \equiv \inf_{(t,x)\in(0,T_M)\times\Omega} \varrho(t,x)$ on a short time interval $[0,T_M)$ (dependent on the size of the initial data). This class is contained in class (159).

This means that any weak solution emanating from Valli initial data coincides with the Valli strong solution at least on a (short) time interval $[0,T_M)$ provided pressure satisfies assumptions of Theorem 26 (and Ω, $\mathbf{f}$ satisfy the Valli regularity hypotheses).

3. Under assumption $p \in C^1[0,\infty)$, Ω bounded C^3 domain, $\varrho_0 \in W^{1,q}(\Omega)$, $\inf_{x\in\Omega} \varrho_0 > 0$, $\mathbf{u}_0 \in W_0^{1,2} \cap W^{2,2}(\Omega)$, $\mathbf{f} \in C([0,\infty),L^2(\Omega;\mathbb{R}^3)) \cap L^2_{\mathrm{loc}}([0,\infty);L^q(\Omega;\mathbb{R}^3))$, $\partial_t \mathbf{f} \in L^2_{\mathrm{loc}}([0,\infty);W^{-1,2}(\Omega;\mathbb{R}^3))$, $q \in (3,6]$, Cho, Choe, Kim [14, Proposition 5] constructed a unique strong solution to problem (63)–(64), (123)–(125) in the regularity class $\varrho \in C([0,T_M);W^{1,q}(\Omega))$, $\mathbf{u} \in C([0,T_M);W^{2,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T_M;W^{2,q}(\Omega;\mathbb{R}^3))$, $\partial_t\varrho \in L^2(0,T_M;L^q(\Omega))$, $\partial_t \mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$, $\sqrt{\varrho}\partial_t \mathbf{u} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ on a (short) maximal existence time interval $[0,T_M)$ (dependent on the size of initial data).

   Theorem 27 implies, in particular, that any weak solution emanating from the Cho, Choe, and Kim initial data coincides with the strong solution at least on the maximal existence time interval $[0,T_M)$ of the Cho, Choe, and Kim strong solution provided pressure satisfies hypotheses of Theorem 27 (and Ω, $\mathbf{f}$ satisfy the Cho, Choe, Kim regularity hypotheses).
4. Under additional assumptions (141) with $\gamma > 1$ and $p'(\varrho) > 0$, and if

$$\eta < \frac{23\mu}{3}, \tag{169}$$

Sun, Wang, and Zhang [101, Theorem 1.3] showed that if in the previous statement on existence of strong solutions the maximal existence time interval $T_M < \infty$, then necessarily

$$\lim_{\tau\to T_M-} \|\varrho\|_{L^\infty(Q_\tau)} = \infty. \tag{170}$$

Criterion (170) is a blow-up criterion for strong solutions. These criterions are widely investigated in the mathematical literature (see [101] and references quoted there). Loosely speaking, weak-strong uniqueness principle turns most of *blow-up criterions for strong solutions* to the *regularity criterions for weak solutions*.

In particular, any weak solution on the (arbitrary large) time interval $(0,T)$ and on C^3-bounded domain emanating from Cho-Choe-Kim's initial data and external forces is in fact a strong solution (in the Cho-Choe-Kim's class) on the whole time interval $[0, T)$, provided assumptions of Theorem 27 with $\gamma > 3/2$ and (169) are satisfied, as long as the density component ϱ of the weak solution remains bounded (see [49, Theorem 4.6]).

Another consequence of the weak-strong uniqueness principle is the fact that the density in the weak solution must exhibit blowup before developing vacuum. More precisely, if all assumptions of the previous alinea are satisfied, and if density of the weak solution (that exists on the large interval $(0, T)$) verifies

$$\operatorname{ess}\inf_{x\in\Omega} \varrho(\tau, x) = 0 \text{ for a certain } \tau \in (0, T),$$

then

$$\limsup_{t\to\tau-} \left[\operatorname{ess}\sup_{x\in\Omega} \varrho(t, x)\right] = \infty$$

(see [49, Corollary 4.7]).

5. Theorems 26 and 27 hold with obvious modifications with slip boundary conditions (22) or with Navier's boundary conditions (23) (see [50, Section 4.1.2]). It can be easily extended to a large class of unbounded domains with boundary conditions at infinity ($\varrho_\infty \geq 0, \mathbf{u}_\infty = 0$) (see [50, Section 4.2.2 and Theorem 4.6]).

Sketch of the proof of Theorems 25, 26, and 27. We shall outline here the main ideas of proof of Theorems 25, 26, and 27. The reader can find all complementary details in [50, Theorem 4.1].

5.4.1 Main Idea: The Gronwall Inequality

The main idea is to use the relative energy inequality (157) with the strong solution $(r, \mathbf{U})$ of system (63)–(64), (124)–(125) in the form derived in Lemma 4. The goal is to find an estimate of the left-hand side of (160) from below by

$$c \int_0^\tau \|\mathbf{u} - \mathbf{U}\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \mathrm{d}t - \overline{c}' \int_0^\tau \mathcal{E}(\varrho, \mathbf{u} \big| r, \mathbf{U}) \mathrm{d}t + \mathcal{E}(\varrho, \mathbf{u} \big| r, \mathbf{U})\Big|_0^\tau, \tag{171}$$

and the right-hand side from above by

$$\delta \int_0^\tau \|\mathbf{u} - \mathbf{U}\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)} \mathrm{d}t + c'(\delta) \int_0^\tau a(t) \mathcal{E}(\varrho, \mathbf{u} \big| r, \mathbf{U}) \mathrm{d}t \tag{172}$$

with any $\delta > 0$, where $c > 0$ is independent of δ, $\overline{c}' \geq 0$, $c' = c'(\delta) > 0$, and $a \in L^1(0, T)$. This process leads to the estimate

$$\mathcal{E}(\varrho, \mathbf{u} \big| r, \mathbf{U})(\tau) \leq \mathcal{E}(\varrho_0, \mathbf{u}_0 \big| r(0), \mathbf{U}(0)) + c \int_0^\tau a(t) \mathcal{E}(\varrho, \mathbf{u} \big| r, \mathbf{U}) \mathrm{d}t, \tag{173}$$

which implies estimate (166) by the Gronwall inequality invoked in Theorem 21. In the rest of this section, we shall perform this program.

5.4.2 Bound from Below of the Dissipation

By virtue of the Korn inequality invoked in Theorem 9 and the standard Poincaré inequality,

$$c\int_0^\tau \|\mathbf{u}-\mathbf{U}\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\mathrm{d}t \leq \int_0^\tau\int_\Omega \mathbb{S}(\nabla(\mathbf{u}-\mathbf{U}) : \nabla(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\mathrm{d}t. \tag{174}$$

5.4.3 Essential and Residual Sets

We introduce essential and residual sets in Ω. To this end we take in (155) $a = \underline{r}$, $b = \overline{r}$ and define for a.e. $t \in (0,T)$ the residual and essential subsets of Ω as follows:

$$N_{\mathrm{ess}}(t) = \{x \in \Omega \Big| \varrho(t,x) \in \mathcal{O}_{\mathrm{ess}}\},\ \ N_{\mathrm{res}}(t) = \Omega \setminus N_{\mathrm{ess}}(t). \tag{175}$$

With this definition at hand and having assumption (165) in mind, we deduce from Lemma 3

$$c\int_\Omega \Big(\Big[1\Big]_{\mathrm{res}} + \Big[\varrho\Big]_{\mathrm{res}} + \Big[p(\varrho)\Big]_{\mathrm{res}} + \Big[\varrho - r\Big]^2_{\mathrm{ess}}\Big)\ \mathrm{d}x \leq \int_\Omega E(\varrho,\mathbf{u}\Big|r,\mathbf{U})\ \mathrm{d}x \tag{176}$$

with some $c = c(\underline{r},\overline{r}) > 0$, where we have set

$$[h]_{\mathrm{ess}} = h1_{N_{\mathrm{ess}}},\ [h]_{\mathrm{res}} = h1_{N_{\mathrm{res}}}.$$

for a function h defined a.e. in $(0,T)\times\Omega$.

5.4.4 Estimates of the Right-Hand Side of Inequality (160) for Theorem 25

We observe that on essential set, N_{ess} expressions $E(r|\varrho)$ and $(\varrho - r)^2$ are uniformly equivalent, meaning that there are $\underline{c} = \underline{c}(\underline{r},\overline{r}) > 0$ and $\overline{c} = \overline{c}(\underline{r},\overline{r}) > 0$,

$$\underline{c}(\varrho - r)^2 \leq E(\varrho|r) \leq \overline{c}(\varrho - r)^2 \ \text{ whenever } \varrho \in N_{\mathrm{ess}}, \underline{r} \leq r \leq \overline{r}, \tag{177}$$

provided $p \in C^2(0,\infty)$, regardless the structural properties of p near zero and infinity.

Now, we split all integrals over Ω at the right-hand side of inequality (160) to the integrals over the essential sets N_{ess} and residual sets N_{res}; more precisely, we write $\int_0^\tau\int_\Omega = \int_0^\tau\int_{N_{\mathrm{ess}}(t)} + \int_0^\tau\int_{N_{\mathrm{res}}(t)}$.

In the case of Theorem 25, all integrals over the residual sets are zero. (Indeed, we may suppose that $\underline{r} \leq \underline{\varrho}$, $\overline{r} \geq \overline{\varrho}$.) By virtue of the Cauchy-Schwarz inequality and Taylor's formula, the upper bound of the integrals over the essential set is

$$c\int_0^\tau \int_{N_{\rm ess}(t)} \Big(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + |\varrho - r|^2\Big){\rm d}x \le c'\int_0^\tau \mathcal{E}(\varrho,\mathbf{u}|r,\mathbf{U}){\rm d}t, \tag{178}$$

where the last estimate holds due to (176). Implementing these observations into (160), we arrive at inequality (173) and conclude the proof of Theorem 25 by the Gronwall lemma (see Theorem 21) applied to (173).

5.4.5 Estimates of the Right-Hand Side of Inequality (160) for Theorems 26 and 27

The essential part of the right-hand side will be treated exactly as in the previous case. The structural assumptions of the pressure will play a role only for the estimates of the residual part of integrals at the right-hand side of inequality (160). Let us show a typical reasoning on the example of the first term of the right-hand side of (160) in the situation of Theorem 26.

Recall that $N_{\rm res} = \{\varrho \le \underline{r}/2\} \cup \{\varrho \ge 2\overline{r}\}$. We shall estimate the integrals over the sets $\{\varrho \le \underline{r}/2\}$ and $\{\varrho \ge 2\overline{r}\}$ separately.

$$\begin{aligned}
&\int_0^\tau \int_\Omega 1_{\{\varrho \le \underline{r}/2\}}(\rho - r)(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})\ {\rm d}x{\rm d}t\\
&\qquad \le 2\overline{r}\int_0^\tau \int_\Omega 1_{\rm res}\Big|\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}\Big|\ \Big|\mathbf{U}-\mathbf{u}\Big|\ {\rm d}x{\rm d}t\\
&\le 2\overline{r}\int_0^\tau \Big\|\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}\Big\|_{L^\infty(\Omega;\mathbb{R}^3)}\Big\|1_{\rm res}\Big\|_{L^2(\Omega)}\Big\|\mathbf{u}-\mathbf{U}\Big\|_{L^2(\Omega;\mathbb{R}^3)}{\rm d}t\\
&\le \delta\int_0^\tau \Big\|\mathbf{u}-\mathbf{U}\Big\|^2_{L^2(\Omega;\mathbb{R}^3)}{\rm d}t + c(\delta,\underline{r},\overline{r})\int_0^\tau a(t)\mathcal{E}\Big(\varrho,\mathbf{u}\Big|r,\mathbf{U}\Big){\rm d}t,
\end{aligned}$$

where $a = \|\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}\|^2_{L^\infty(\Omega;\mathbb{R}^3)} \in L^1(0,T)$, and

$$\begin{aligned}
&\int_0^\tau \int_\Omega 1_{\{\varrho \ge 2\overline{r}\}}(\varrho)(\varrho - r)(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})\ {\rm d}x{\rm d}t\\
&\qquad \le 2\int_0^\tau \int_\Omega [1]_{\rm res}\sqrt{\varrho}\Big|\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}\Big|\sqrt{\varrho}\Big|\mathbf{U}-\mathbf{u}\Big|\ {\rm d}x{\rm d}t\\
&\le \int_0^\tau \Big\|\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}\Big\|_{L^\infty(\Omega;\mathbb{R}^3)}\Big\|\Big[\varrho\Big]_{\rm res}\Big\|^{1/2}_{L^1(\Omega)}\Big\|\varrho\Big(\mathbf{u}-\mathbf{U}\Big)^2\Big\|^{1/2}_{L^1(\Omega)}{\rm d}t\\
&\qquad \le c(\underline{r},\overline{r})\int_0^\tau a(t)\mathcal{E}\Big(\varrho,\mathbf{u}\Big|r,\mathbf{U}\Big){\rm d}t
\end{aligned}$$

with the same a as before. In all the above three formulas, we have employed (176) in the passage to their last lines.

The remaining terms at the right-hand side of the relative energy inequality (160) may be estimated in a similar way. Finally, one gets estimate (173) and applies the

Gronwall lemma invoked in Theorem 21. This finishes the sketch of the proof of Theorems 25, 26, and 27.

6 Longtime Behavior of Barotropic Flows

In this section, results on the longtime behavior of weak solutions to the barotropic system (63)–(64) with homogenous Dirichlet boundary conditions (125) with viscosities (123) and pressure (122) are discussed.

Further restrictions on the pressure, typically

$$p(\varrho) = a\varrho^{\gamma},\ a > 0, \tag{179}$$

where $\gamma \geq 1$ will be required later in most of statements starting from Sect. 6.2, mostly for the sake of simplicity. We do not restrict ourself to bounded domains.

The barotropic model may be viewed as a special case of the Navier-Stokes-Fourier system with constant temperature or with constant entropy, as described in Sect. 2.7. In this model the mechanical motion is completely separated from thermal effects. The simplified system (63), (64), when considered independently of the thermal energy equation (65), may feature rather different properties than the complete system. For instance, in contrast to the full system, it admits bounded absorbing sets for nonconservative forcing term, $\mathbf{f} \not\equiv \nabla F$, or even nontrivial periodic solutions provided the driving force is time periodic, which is impossible in the full system in domains with thermally insulated boundary (see Sect. 12, Corollary 3, Remark 26 and compare with [51]).

In Sect. 6.2, the large-time dynamics of weak solutions to the problem (63)–(64), (125) where the external force is *a gradient of a scalar potential* F, bounded and Lipschitz continuous on Ω will be discussed.

Formally, the problem (63)–(64), (125) represents a gradient flow which admits a Lyapunov function – the *total energy*

$$E_F(t) = \int_{\Omega} \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho) - F\varrho\Big)\,\mathrm{d}x,$$

satisfying the *energy inequality*

$$\frac{\mathrm{d}E_F}{\mathrm{d}t} + \int_{\Omega} \Big((\frac{4}{3}\mu + \eta)|\nabla\mathbf{u}|^2 + \eta|\mathrm{div}\,\mathbf{u}|^2\Big)\,\mathrm{d}x \leq 0 \tag{180}$$

(see *item 3* in Remark 1). Consequently, it is plausible to anticipate that, at least for some sequences $t_n \to \infty$,

$$\varrho(t_n) \to \varrho_s,\ \varrho\mathbf{u}(t_n) \to \mathbf{0},$$

where ϱ_s is a solution to the corresponding stationary problem. Uniqueness of stationary solutions is discussed in Sect. 6.1.

6.1 Uniqueness of Equilibria

In this section, static (equilibrium) solutions to the problem (63)–(64), (125) are examined in the case that the external force **f** is a gradient of a potential F which is assumed to be locally Lipschitz continuous on Ω. The system reads

$$\nabla p(\varrho) = \varrho \nabla F,\ \varrho \geq 0,\ \int_\Omega \varrho \, \mathrm{d}x = m, \tag{181}$$

where the parameter m represents the total mass conserved by the flow. Beir$\tilde{a}$o da Veiga [1] obtained a necessary and sufficient condition for the existence of a strictly positive solutions of (181) expressed in terms of F and structural properties of p. It is easy to show that such a solution is necessarily unique. On the other hand, this restriction excludes an important class of solutions with vacuum states. The following theorem applies to any domain $\Omega \subset \mathbb{R}^n$ and a broad class of nonlinearities p. The uniqueness condition is expressed in terms of the upper level sets of the potential F,

$$[F > k] \equiv \{x \in \Omega;\ F(x) > k\}.$$

Theorem 28 ([39, Theorem 2.1]). *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain. Suppose that pressure p satisfies condition (122) and thermodynamic stability condition (153). Let F be a locally Lipschitz continuous function on Ω, and, in addition, suppose that the the upper level sets*

$$[F > k] \textit{ are connected in } \Omega \textit{ for any } k. \tag{182}$$

Then, given $m > 0$, there is at most one function $\varrho \in L^\infty_{loc}(\Omega)$ satisfying (181).

Moreover, if such a function exists, it is given by the formula

$$\varrho(x) = Q^{-1}(F(x) - k_\Omega) \tag{183}$$

for a certain constant k_Ω, where

$$Q(z) = \begin{cases} \int_0^z \frac{\mathrm{d}p(s)}{s} \textit{ if } P_0 = \int_0^1 \frac{\mathrm{d}p(s)}{s} \textit{ is finite} \\ \int_1^z \frac{\mathrm{d}p(s)}{s} \textit{ if } \int_0^1 \frac{\mathrm{d}p(s)}{s} = +\infty. \end{cases}$$

Theorem 28 provides the following corollary:

Corollary 1. *Let p satisfy assumptions of Theorem 28; let P_0 be finite, $|\Omega| = \infty$, $F \geq 0$, and*

$$\int_{\Omega} Q^{-1}(F(x))\,\mathrm{d}x = m_0 > 0 \text{ finite}.$$

Then there are no solutions of (181) *with the mass* $m > m_0$.

Proof of Corollary 1. If there is such a solution ϱ, then, by virtue of Theorem 28, it would hold

$$\varrho(x) = Q^{-1}(F(x) + c)$$

with $c > 0$ and, consequently,

$$\int_{\Omega} \varrho(x)\,\mathrm{d}x \geq Q^{-1}(c)|\Omega| = \infty.$$

□

Remark 7. Two examples involving pressure (179) and different potential forces are given.

1. In the case that $p(\varrho) = a\varrho^{\gamma}$, the solution formula reads

$$\varrho(x) = \left(\frac{\gamma - 1}{a\gamma}[F(x) + c]^{+}\right)^{\frac{1}{\gamma-1}} \text{ for a certain constant } c \in \mathbb{R}.$$

2. Let F be the gravity potential of a solid ball surrounded by a viscous gas, i.e.,

$$F(x) = -\frac{\omega}{|x|},\ x \in \Omega = \{x \in \mathbb{R}^3 \mid |x| \geq r\}$$

for certain positive constants ω, r. Consider

$$p(z) = z^{\gamma},\ 1 < \gamma < \frac{4}{3}.$$

A straightforward computation gives

$$Q^{-1}(z) = \left(\frac{\gamma - 1}{\gamma}\right)^{\frac{1}{\gamma-1}} z^{\frac{1}{\gamma-1}} \text{ for } z \geq 0,$$

and, consequently,

$$\int_{\Omega} Q^{-1}(F(x))\,\mathrm{d}x = c(\gamma, \omega)\int_{\Omega} |x|^{\frac{-1}{\gamma-1}}\,\mathrm{d}x,$$

where the last integral is finite provided $1 < \gamma < \frac{4}{3}$. Applying Corollary 1, it is possible to deduce the existence of a finite critical mass m_0 for ϱ, such that the problem (181) does not possess any solution for $m > m_0$. In such a situation, one can anticipate that any solution of the evolution problem (63), (64), (125) with the initial mass $m > m_0$ should divide into two parts, one of which will converge to a stationary state and the other tending locally to zero.

The importance of the assumption (182) is illustrated by the following statement:

Theorem 29. *Let p satisfy the hypotheses of Theorem 28. Assume P_0 is finite, $\int_1^\infty \frac{\mathrm{d}p(s)}{s} = \infty$, and there exists k such that the set $[F > k]$ has two disjoint bounded open components.*

Then there is $m > 0$ and a nonempty interval I such that the problem (181) admits a one-parameter family of solutions $\varrho_\nu, \ \nu \in I$ satisfying

$$\int_\Omega \varrho_\nu(x) \, \mathrm{d}x = m \textit{ for all } \nu \in I.$$

Proof of Theorem 29. Consider the two disjoint components $\mathcal{O}_1, \mathcal{O}_2$ from the hypotheses of the theorem. As F is continuous, there exists $\nu_0 > k$ such that the function

$$\varrho_{\nu_0}(x) = 1_{\mathcal{O}_1} Q^{-1}(F(x) - \nu_0) + 1_{\mathcal{O}_2} Q^{-1}(F(x) - \nu_0)$$

is a solution of (181) with

$$\int_\Omega \varrho_{\nu_0}(x) \, \mathrm{d}x = m > 0$$

for a certain finite m. Using continuous dependence of the integral on parameters and monotonicity of Q^{-1}, one can find a small interval I containing ν_0 and a nonincreasing function $q : I \mapsto I$ such that

$$\varrho_\nu(x) = 1_{\mathcal{O}_1} Q^{-1}(F(x) - \nu) + 1_{\mathcal{O}_2} Q^{-1}(F(x) - q(\nu)), \ \nu \in I$$

are solutions of (181) satisfying

$$\int_\Omega \varrho_\nu(x) \, \mathrm{d}x = m.$$

□

The next result applies to the pressure $p(\varrho) = a\varrho^\gamma$. In Theorem 28, the solution is uniquely determined by its mass m. One can expect that, prescribing in addition the potential energy e

$$\int_{\Omega} \frac{a}{\gamma - 1} \varrho^{\gamma} - \varrho F \, \mathrm{d}x = e, \tag{184}$$

the geometrical condition on the upper level set $[F > k]$ could be relaxed. This is really the case as stated in the following:

Theorem 30 ([42, Theorem 1.2]). *Let $\Omega \subset R^N$ be an arbitrary domain. Assume F is locally Lipschitz continuous function on Ω, $p(\varrho) = a\varrho^{\gamma}$, $\gamma > 1$. Moreover, suppose Ω can be decomposed as*

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \ \Omega_1 \cap \Omega_2 = \emptyset, \tag{185}$$

where $\Omega_1, \ \Omega_2 \subset R^N$ are domains (one of them possibly empty) and that

$$[F > k] \cap \Omega_i \text{ is connected in } \Omega_i \text{ for } i = 1, 2 \text{ and for any } k \in \mathbb{R}. \tag{186}$$

Then, given m, e, the problem (181), (184) *admits at most two solutions.*

The proof, where some elements of convex analysis are used, can be found in [42].

Remark 8. 1. Saying that ϱ is a solution of (181), we require, in particular, all the integrals being convergent, i.e., $\varrho \in L^1 \cap L^{\gamma}(\Omega)$, $\varrho F \in L^1(\Omega)$.
2. The previous results were generalized by Erban [24] for F locally Lipschitz continuous and bounded, $p(\varrho) = a\varrho^{\gamma}$, $\gamma > 1$. He showed that there exists critical mass $\tilde{m}$ such that:
   - The system (181) has at most one solution for the mass $m \in [\tilde{m}, \infty)$.
   - There is continuum of solutions of the system (181) for the mass $m \in (0, \tilde{m})$.

     Moreover, he defined a critical mass m_c such that:
   - If $m \in [m_c, \infty)$, then the stationary problem (181), (187) admits at most two solutions for each energy $e \in \mathbb{R}$.
   - If $m \in (0, m_c)$, then there exists an energy $e \in \mathbb{R}$ such that the system (181), (187) has continuum of solutions.

Some consequences of Theorem 28 with $p(\varrho) = a\varrho^{\gamma}$ finish this section. Since the upper level sets $[F > k]$ are connected in Ω, any solution of the stationary problem (181) with finite mass may be written in the form:

$$\varrho_s(x) = \left(\frac{\gamma - 1}{a\gamma} [F(x) - k]^{+} \right)^{\frac{1}{\gamma - 1}}, \tag{187}$$

where k is uniquely determined by the mass

$$m[\varrho_s] = \int_\Omega \varrho_s \, \mathrm{d}x.$$

The mass $m[\varrho_s]$ considered as a function of the parameter k,

$$m[\varrho_s] : \mathbb{R} \mapsto [0, \infty],$$

is continuous nonincreasing. Moreover, clearly,

$$m[\varrho_s](k) = 0 \text{ for all } k \geq \sup_{x \in \Omega} F(x),$$

and $m[\varrho_s]$ is strictly decreasing on any open interval on which it is finite and strictly positive.

We have the following assertion:

Lemma 5. *Let F be as in Theorem 28. Given $m_0 \geq 0$, there exists a stationary solution $\overline{\varrho}_s$ such that*

$$\int_\Omega \overline{\varrho}_s \, \mathrm{d}x \leq m_0,$$

and

$$\varrho_s \leq \overline{\varrho}_s$$

for any stationary solution ϱ_s such that

$$\int_\Omega \varrho_s \, \mathrm{d}x \leq m_0. \tag{188}$$

Proof of Lemma 5. All stationary solutions are given by the formula (187). Take

$$\overline{k} = \inf\{k | \ \varrho_s \text{ given by (187) satisfies (188)}\}$$

and set

$$\overline{\varrho}_s(x) = \left(\frac{\gamma - 1}{a\gamma} [F(x) - \overline{k}]^+ \right)^{\frac{1}{\gamma - 1}}.$$

□

To conclude, consider the energy

$$e[\varrho_s] = \int_\Omega \frac{a}{\gamma - 1} \varrho_s^\gamma - F \varrho_s \, \mathrm{d}x.$$

as a function of the parameter k.

Lemma 6. *Let F satisfy the hypotheses of Theorem 28. Then the energy $e[\varrho_s]$ is a nondecreasing function of k with values in $[-\infty, 0]$. Moreover, e is strictly increasing on any open interval on which $m[\varrho_s]$ is finite and strictly positive.*

Proof of Lemma 6. Expressing ϱ_s by means of the formula (187), one has to observe that

$$k \mapsto \frac{a}{\gamma-1}\Big(\frac{\gamma-1}{a\gamma}[F-k]^+\Big)^{\frac{\gamma}{\gamma-1}} - F\Big(\frac{\gamma-1}{a\gamma}[F-k]^+\Big)^{\frac{1}{\gamma-1}}$$

is a nondecreasing function of k which may be verified by a direct computation. □

Corollary 2. *For F satisfying the hypotheses of Theorem 28 and E_∞ a given number, there is at most one stationary solution ϱ_s with finite mass and such that*

$$e[\varrho_s] = E_\infty.$$

6.2 Convergence to Equilibria

The aim of this section is to show that *any* weak solution converges to a fixed stationary state as time goes to infinity, more precisely,

$$\varrho(t) \to \varrho_s \text{ strongly in } L^\gamma(\Omega),\ \sqrt{\varrho}\mathbf{u}(t) \to \mathbf{0} \text{ strongly in } [L^2(\Omega)]^3 \text{ as } t \to \infty,$$

under the two basic hypotheses:

$$\partial\Omega \text{ is Lipschitz and compact}$$

and the upper level sets satisfy (182):

$$[F > k] = \{x \in \Omega |\ F(x) > k\} \text{ are connected in } \Omega \text{ for all } k.$$

The above assumptions hold in many physically interesting cases, in particular in the situation when Ω is an exterior domain with spherical boundary and F is the gravitational potential, specifically,

$$\Omega = \{x \in \mathbb{R}^3 |\ |x| \geq R\},\ F(x) = -\frac{\omega}{|x|},$$

$\omega > 0$, modeling the motion of a viscous barotropic gas surrounding a star, considered in [82].

For the sake of simplicity, assume $(179)_{\gamma>1}$, i.e., $p(\varrho) = a\varrho^\gamma$. Further restrictions on values of γ will be required later according to the investigated cases.

Given a positive number m, the condition (182) is both necessary and sufficient for the stationary problem (181) to admit at most one weak solution ϱ_s uniquely determined by its mass

$$m[\varrho_s] = \int_\Omega \varrho_s \, \mathrm{d}x, \tag{189}$$

(cf. Theorem 28). On the other hand, the mass $m[\varrho(t)]$ is a conserved quantity even for the weak solutions of the problem (63)–(64), (125) so one is tempted to believe the condition (189) picks up the right candidate to describe the large-time behavior of the density ϱ. This is certainly true for bounded domains, while, if Ω is unbounded, such a conjecture is false, in general, due to possible "loss of mass at infinity" (cf. Remark 7).

It seems interesting that for Ω bounded and a nonconstant potential F, there always exists $m > 0$ large in comparison with F such that the unique solution of (181) with the given mass m contain vacuum zones (cf. formula (187)). Thus for any nonconstant F, global solutions approach rest states with vacuum regions as time goes to infinity. We should remark in this context that there are many formal results on convergence of isentropic flows to a stationary state under various hypotheses including uniform (in time) boundedness away from zero of the density (see, e.g., [90]). As just observed, this could be rigorously verified only for solutions representing perturbations of strictly positive rest states (cf. [66, 81]). In particular, it is never true when the driving force ∇F is large in comparison with the total mass of the data.

The main result of this section reads as follows:

Theorem 31. *Let $\Omega \subset \mathbb{R}^3$ be a domain with compact and Lipschitz boundary. Let the potential F is bounded and Lipschitz continuous on Ω, and let the upper level sets $[F > k]$ be connected in Ω for any $k < \sup_{x\in\Omega} F(x)$. Moreover, if Ω is unbounded, assume*

$$\lim_{R\to\infty} \operatorname{ess\,sup}_{x\in\Omega, |x|\geq R} \big(|F(x)| + |\nabla F(x)|\big) = 0. \tag{190}$$

Finally, let p verify (179) with $\gamma > 3/2$, namely,

$$p(\varrho) = a\varrho^\gamma \ , \ a > 0, \ \gamma > \frac{3}{2}. \tag{191}$$

Then for any finite energy weak solution $\varrho, \mathbf{u}$ of the problem (63), (64), (125) there exists a stationary state ϱ_s such that

$$\varrho(t) \to \varrho_s \text{ strongly in } L^\gamma(\Omega), \ \sqrt{\varrho}|\mathbf{u}|(t) \to 0 \text{ strongly in } L^2(\Omega) \text{ as } t \to \infty. \tag{192}$$

The proof consists of several steps: energy estimates, local and boundary estimates, compactness result, and, in the case of unbounded Ω, also estimates at infinity. See [40] for details.

Remark 9. 1. Observe that the quantities ϱ and $\sqrt{\varrho}|\mathbf{u}|$ are continuous as functions of t in the space $L^\gamma(\Omega)$ and $L^2(\Omega)$, respectively, endowed with the weak topology, and, consequently, (192) makes sense.

2. The condition (191) seems restrictive from the physical point of view but natural for the mathematical treatment of the problem ensuring local integrability of the product terms appearing in the equations. In fact, such a condition is not necessary provided we know that ϱ is bounded in $L^q(\Omega)$ uniformly in t for a certain $\frac{3}{2} < q \leq \infty$, in particular when the density is uniformly bounded as it is the case for radially symmetric data (cf. [82]).
3. As already mentioned, the mass $m[\varrho_s]$ of the limiting solution may be strictly less than $m[\varrho(t)] = m_0$. Probably the simplest example is $F = 0$, Ω unbounded, when, according to Theorem 31, the density $\varrho(t)$ converges to zero in $L^\gamma(\Omega)$.
4. Another example is furnished by *item* 2 in Remark 7, where $1 < \gamma < \frac{4}{3}$. As shown in the previous section, there is a critical mass $\overline{m}$ such that there is no solution of the stationary problem with a finite mass greater than $\overline{m}$. Taking radially symmetric data, it can be shown that the density $\varrho(t)$ remains bounded uniformly in $t \to \infty$ (see [82, Proposition 1]). In accordance with the Remark 8, Theorem 31 applies even though (191) is not satisfied, yielding convergence for any radially symmetric data. It is clear that the limit mass can never exceed $\overline{m}$.

Remark 10. The proof of Theorem 31 can be carried out without essential modifications in the following situations (see [40]):

1. If Ω is a *bounded* regular domain in $\mathbb{R}^2$, the conclusion of Theorem 31 holds with the same condition (191) with $\gamma > 1$. However, the case of an *exterior* domain exhibits some additional difficulties because of the lack of the Sobolev inequality for functions in $W^{1,2}(\mathbb{R}^2)$.
2. p is a general strictly increasing function of the density,

$$p(z) \approx z^\gamma \text{ for large } z$$

and p' bounded in a neighborhood of zero. Moreover, if Ω is unbounded, we need

$$\int_0^1 \frac{p'(z)}{z} \, \mathrm{d}z \text{ finite.}$$

3. The viscosity coefficients $\mu, \ \eta$ may depend on $\varrho, \mathbf{u}$, and a nonpotential and even time-dependent external force $\mathbf{f}$ may be added to ∇F provided it vanishes in a certain sense for large t.

6.3 Bounded Absorbing Sets

In this part, globally defined finite energy weak solutions of the problem (63)–(64), (125) on a *bounded* Lipschitz domain Ω, will be dealt with. More exactly, assume that ϱ, $\mathbf{u}$ belong to the classes

$$\varrho \in L^\infty_{loc}\Big(\mathbb{R}^+; L^\gamma(\Omega)\Big), \ \mathbf{u} \in L^2_{loc}\Big(\mathbb{R}^+; (W_0^{1,2})^3(\Omega)\Big), \tag{193}$$

the equations (63), (64) hold in $\mathcal{D}'(R^+ \times \Omega)$, and the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\varrho, \mathbf{u}](t) + (\frac{4}{3}\mu + \eta) \int_\Omega |\nabla \mathbf{u}(t)|^2 \,\mathrm{d}x + \eta \int_\Omega |\mathrm{div}_x\, \mathbf{u}(t)|^2 \,\mathrm{d}x \le \int_\Omega \varrho(t)\mathbf{f}(t).\mathbf{u}(t)\,\mathrm{d}x \tag{194}$$

is satisfied in $\mathcal{D}'(\mathbb{R}^+)$, where the energy $E[\varrho, \mathbf{u}]$ is given by the formula

$$E[\varrho, \mathbf{u}](t) = \frac{1}{2} \int_\Omega \varrho(t)|\mathbf{u}(t)|^2 \,\mathrm{d}x + \frac{a}{\gamma - 1} \int_\Omega \varrho^\gamma(t)\,\mathrm{d}x.$$

The following result establishes the existence of an absorbing ball for *any* finite energy weak solution.

Theorem 32 ([43, Theorem 1.1]). *Let p satisfies (179) with*

$$\gamma > \frac{5}{3}, \tag{195}$$

and let $\mathbf{f}$ *be a bounded measurable function,*

$$\Big\{\mathrm{ess} \sup_{t \in \mathbb{R}^+,\, x \in \Omega} |\mathbf{f}(t, x)|\Big\} \le K. \tag{196}$$

Then there exists a constant E_∞, depending solely on γ, K and on the total mass m, having the following property:

Given E_0, there exists a time $T = T(E_0)$ such that

$$E[\varrho, \mathbf{u}](t) \le E_\infty \text{ for a.e. } t > T \tag{197}$$

provided

$$\mathrm{ess} \limsup_{t \to 0+} E[\varrho, \mathbf{u}](t) \le E_0, \tag{198}$$

and ϱ, $\mathbf{u}$ *is a (finite energy) weak solution of the problem (63)–(64), (123), (125), satisfying the hypotheses (193)–(194).*

Remark 11. 1. Theorem 32 was proved in [43] under the additional assumption

$$\varrho \in L^2_{loc}\Big(\mathbb{R}^+; L^2(\Omega)\Big), \tag{199}$$

which is satisfied provided $\gamma \geq 9/5$ (see [41]). In fact, the condition (199) is not necessary in the proof of Theorem 32; it is sufficient to have estimates of the form (262), which are valid for the pressure satisfying (195). Note that estimate of pressure in L^p up to the boundary (whose main ideas are presented in *item* 6 of Sect. 8.2; see also "pressure estimates" in [44, Section 4.2]) is one of the prerequisites to obtain energy inequality in the differential form, and due to this reason, it constitutes one of the building blocks of the proof of Theorem 32.

2. In agreement with *item 3* of Remark 1, the instantaneous values $E = \text{inst}[E[\varrho, \mathbf{u}]]$ (defined in Sect. 3.2) satisfy inequality (131) everywhere in $\mathbb{R}^+$, and consequently inequality (197) is valid for any $t > T$, provided one replaces $E[\varrho, \mathbf{u}]$ by its instantaneous value E.

The proof of Theorem 32 is based on the following Lemma and Proposition.

Lemma 7. *Assume* **f** *satisfies* (196)*. Let* ϱ*,* $\mathbf{u}$ *belong to the classes* (193)*,* (199) *and comply with the energy inequality* (194)*.*

Then, being redefined on a set of measure zero if necessary, the (instantaneous value of) energy E *has locally bounded variation on* $\mathbb{R}^+$*, and*

$$E(t+) = \lim_{s\to t+} E(s) \leq \lim_{s\to t-} E(s) = E(t-) \textit{ for any } t \in \mathbb{R}^+. \tag{200}$$

Moreover,

$$E(t_2-) \leq \Big(1 + E(t_1+)\Big)e^{\sqrt{2m}K(t_2-t_1)} - 1 \textit{ for all } 0 < t_1 < t_2. \tag{201}$$

Sketch of the proof of Lemma 7. It follows from the energy inequality (194) – see *item* 3 in Sect. 3.2 and *item* 3 in Remark 1 – that E can be written as a sum of a nonincreasing function and an absolutely continuous one, and, consequently, E is continuous except a countable set of points in which (200) holds.

By virtue of (196), the right-hand side of (194) may be estimated as follows:

$$\int_\Omega \varrho \mathbf{f}.\mathbf{u}\,\mathrm{d}x \leq K\Big(\int_\Omega \varrho\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_\Omega \varrho|\mathbf{u}|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \leq \sqrt{2m}K(1+E),$$

whence (201) is a straightforward consequence of the Gronwall lemma.

The following assertion plays a crucial role in the proof of Theorem 32.

Proposition 1. *Under the hypotheses of Theorem 32, there exists a constant L, depending solely on γ, K, and m, enjoying the following property: If*

$$E((T+1)-) > E(T+) - 1 \text{ for a certain } T \in \mathbb{R}^+, \tag{202}$$

then

$$\sup_{t \in (T,T+1)} E(t+) \leq L.$$

The proof of this proposition is carried over by a series of auxiliary results (see [43, Proposition 3.1]).

Sketch of proof of Theorem 32. With Lemma 7 and Proposition *1* at hand, Theorem 32 can be proved. To begin, observe there exists $T = T(E_0)$ such that

$$E(t_0+) \leq L \text{ for a certain } t_0 < T,$$

where L is the constant from Proposition 1. Indeed, if it was not the case then, by virtue of Proposition 1, the energy would become negative.

Next claim is that for any integer $n \geq 0$

$$E((t_0+n)+) \leq L. \tag{203}$$

By induction, assume

$$E((t_0+n)+) \leq L.$$

By Proposition 1, either

$$\sup_{t \in (t_0+n, t_0+n+1)} E(t+) \leq L,$$

and, consequently,

$$E((t_0+n+1)-) \leq L,$$

or

$$E((t_0+n+1)+) \leq E((t_0+n+1)-) \leq E((t_0+n)+) - 1 \leq L - 1.$$

Finally, by virtue of Lemma 7 and (203), take

$$E_\infty = (1+L)e^{\sqrt{2mK}} - 1.$$

This completes the proof of Theorem 32.

6.4 Existence of Attractors

In this part, results from the publication [29, Sections 3–5] are presented. Throughout this section, assume

$$\left\{\begin{array}{l} p(\varrho) = a\varrho^\gamma, \ \ \gamma > \frac{5}{3}, \ \ \Omega \text{ is a bounded Lipschitz domain,} \\ \\ \mathbf{f} \in \mathcal{F}, \ \text{where } \mathcal{F} \text{ denotes a bounded subset of } L^\infty(\mathbb{R} \times \Omega). \end{array}\right\} \tag{204}$$

First, observe that the finite energy weak solution satisfies

$$\varrho \in C_{\text{week}}([0,T]; L^\gamma(\Omega)), \ \ \mathbf{q} \equiv \varrho\mathbf{u} \in C_{\text{week}}([0,T]; L^p(\Omega)) \ \text{ with } p = \frac{2\gamma}{\gamma+1},$$

and, moreover, the fact that the continuity equation holds in $\mathcal{D}'([0,T] \times \mathbb{R}^3)$ makes it possible to employ the regularizing machinery in the spirit of DiPerna and Lions [18] to deduce

$$\varrho \in C([0,T); L^\alpha(\Omega)) \text{ for any } \ 1 \le \alpha < \gamma,$$

cf. Theorem 20 in Sect. 3.11. These relations enable to justify the observation that

$$(\varrho\mathbf{u})(t,x) = 0 \text{ for a.e. } \ x \in V(t) = \{x; \ \varrho(t,x) = 0\} \ \text{ for any } t \in [0,T],$$

(cf. *item* 2 in Remark 1).

Now, redefining the total energy on a set of measure zero if necessary, set

$$\mathfrak{E}[\varrho, \varrho\mathbf{u}](t) \equiv \mathfrak{E}(t) = \frac{1}{2}\int_{\varrho(t)>0} \frac{|(\varrho\mathbf{u})|^2}{\varrho}(t)\,\mathrm{d}x + \frac{a}{\gamma-1}\int_\Omega \varrho^\gamma\,\mathrm{d}x, \tag{205}$$

where $t \mapsto \mathfrak{E}(t)$ is lower semicontinuous function on $\mathbb{R}^+$ (cf. Remark 3).

The first result deals with complete bounded trajectories, i.e., the finite energy weak solutions defined on the whole line $\mathbb{R}$ whose energy is uniformly bounded on $\mathbb{R}$. Their importance is shown in Proposition 2. Denote

$$\mathcal{F}^+ = \left\{\mathbf{f}; \mathbf{f} = \lim_{\tau_n \to \infty} \mathbf{h}_n(. + \tau_n) \text{ weak star in } L^\infty(\mathbb{R} \times \Omega) \text{ for a certain } \mathbf{h}_n \in \mathcal{F} \text{ and } \tau_n \to \infty\right\}.$$

We introduce an analogue of the so-called short trajectory in the spirit of [78].

$$U^s[E_0, \mathcal{F}](t_0, t) = \Big\{[\varrho(\tau), \mathbf{q}(\tau)], \ \tau \in [0,1]; \ \varrho(\tau) = \varrho(t+\tau), \ \mathbf{q}(\tau) = (\varrho\mathbf{u})(t+\tau),$$

$$\text{where } \varrho, \mathbf{u} \text{ is a finite energy weak solution of the problem (63)–(64), (125) on an open interval } I,$$

$$(t_0, t+1] \subset I, \ \text{with } \mathbf{f} \in \mathcal{F}, \ \text{and such that } \limsup_{t \to t_0} \mathfrak{E}(t) \le E_0 \equiv \mathfrak{E}(0)\Big\}.$$

Proposition 2. *Assume* $[\varrho_n, \mathbf{q}_n] \in U^s[E_0, \mathcal{F}](t_0, t_n)$ *for a certain sequence* $t_n \to \infty$.

Then there is a subsequence (not relabeled) such that

$$\varrho_n \to \overline{\varrho} \text{ in } L^\gamma((0,1)\times\Omega) \text{ and in } C([0,1]; L^\alpha(\Omega)) \text{ for } 1 \le \alpha < \gamma, \tag{206}$$

$$\mathbf{q}_n \to (\overline{\varrho}\,\overline{\mathbf{u}}) \text{ in } L^p((0,1)\times\Omega) \text{ and in } C_{\text{week}}([0,1]; L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \text{ for any } 1 \le p < \frac{2\gamma}{\gamma+1}, \tag{207}$$

and

$$\mathfrak{E}[\varrho_n, \mathbf{q}_n] \to \mathfrak{E}[\overline{\varrho}, \overline{\varrho}\,\overline{\mathbf{u}}] \text{ in } L^1(0,1), \tag{208}$$

where $\overline{\varrho}, \overline{\mathbf{u}}$ *is a finite energy weak solution of the problem* (63)–(64)*,* (123)*,* (125)*,* (204) *defined on the whole real line* $I = \mathbb{R}$ *such that* $\mathfrak{E}[\overline{\varrho}, \overline{\varrho}\,\overline{\mathbf{u}}] \in L^\infty(\mathbb{R})$ *and* $\mathbf{f} \in \mathcal{F}^+$.

For the proof of Proposition 2, see [29, Proposition 3.1].

A straightforward consequence of Proposition 2 is the next theorem, which says that the set

$$\mathcal{A}^s[\mathcal{F}] = \Big\{[\varrho(\tau), \mathbf{q}(\tau)],\ \tau \in [0,1];\ \varrho, \mathbf{q} = (\varrho\mathbf{u}) \text{ is a finite energy weak solution of the problem (63)–(64), (123), (125), (179) } on\ I = \mathbb{R} \text{ with } \mathbf{f} \in \mathcal{F}^+ \text{ and } \mathfrak{E} \in L^\infty(\mathbb{R})\Big\}$$

is a global attractor on the "space" of short trajectories.

Theorem 33. *Let the assumption* (204) *be satisfied. Then the set* $\mathcal{A}^s[\mathcal{F}]$ *is compact in* $L^\gamma((0,1)\times\Omega) \times [L^p((0,1)\times\Omega)]^3$ *and*

$$\sup_{[\varrho,\mathbf{q}]\in U^s[E_0,\mathcal{F}](t_0,t)} \left[\inf_{[\overline{\varrho},\overline{\mathbf{q}}]\in\mathcal{A}^s[\mathcal{F}]} (\|\varrho - \overline{\varrho}\|_{L^\gamma((0,1)\times\Omega)} + \|\mathbf{q} - \overline{\mathbf{q}}\|_{L^p((0,1)\times\Omega)})\right] \to 0$$

as $t \to \infty$ *for any* $1 \le p < \frac{2\gamma}{\gamma+1}$.

The following assertion is an easy consequence of Theorem 28:

Proposition 3. *Let* $\mathcal{F} = \{\mathbf{f}\}$, $\mathbf{f}(x) = \nabla F(x)$ *such that* $[F > k]$ *are connected for all* $k \in \mathbb{R}$. *Then the set* $\mathcal{A}^s[\{\mathbf{f}\}]$ *of bounded trajectories is a singleton and consists of the quantity* $[\varrho_s, 0]$ *where* ϱ_s *is the unique solution of the stationary problem* (181).

Next, define

$$\mathcal{A}[\mathcal{F}] = \Big\{[\varrho, \mathbf{q}]; \varrho = \varrho(0), \ \mathbf{q} = (\varrho\mathbf{u})(0) \text{ where } \varrho, \mathbf{u} \text{ is a finite energy weak solution of the problem (63)–(64), (123), (125), (179) } on\ I = \mathbb{R} \text{ with } \mathbf{f} \in \mathcal{F}^+ \text{ and } \mathfrak{E} \in L^\infty(\mathbb{R})\Big\}.$$

A direct consequence of Proposition 2 and Theorem 33 is the following:

Theorem 34. *Let the assumption* (204) *be satisfied. Then* $\mathcal{A}[\mathcal{F}]$ *is compact in* $L^\alpha \times L^{\frac{2\gamma}{\gamma+1}}_{\text{week}}(\Omega)$ *and*

$$\sup_{[\varrho,\mathbf{q}]\in U[E_0,\mathcal{F}](t_0,t)} \left[\inf_{[\overline{\varrho},\overline{\mathbf{q}}]\in\mathcal{A}[\mathcal{F}]} (\|\varrho - \overline{\varrho}\|_{L^\alpha(\Omega)} + \int_\Omega (|\mathbf{q} - \overline{\mathbf{q}})\phi \,\mathrm{d}x|)\right] \to 0$$

as $t \to \infty$ *for any* $1 \leq \alpha < \gamma$ *and any* $\phi \in [L^{\frac{2\gamma}{\gamma-1}}(\Omega)]^3$.

Assume, in addition to the hypotheses of Theorem 34, that the energy is sequentially continuous on $\mathcal{A}[\mathcal{F}]$. Then the densities converge strongly in L^γ and the momenta in L^1:

Theorem 35. *Let* (204) *hold, and, moreover, let*

$$\mathfrak{E}[\overline{\varrho_n}, \overline{\mathbf{q}_n}] \to \mathfrak{E}[\overline{\varrho}, \overline{\mathbf{q}}] \tag{209}$$

for any sequence

$$\{[\overline{\varrho_n}, \overline{\mathbf{q}_n}]\} \subset \mathcal{A}[\mathcal{F}] \text{ such that } \overline{\varrho_n} \to \overline{\varrho} \text{ in } L^1(\Omega), \ \overline{\mathbf{q}}_n \to \overline{\mathbf{q}} \text{ weakly in } L^1(\Omega).$$

Then

$$\sup_{[\varrho,\mathbf{q}]\in U[E_0,\mathcal{F}](t_0,t)} \left[\inf_{[\overline{\varrho},\overline{\mathbf{q}}]\in\mathcal{A}[\mathcal{F}]} (\|\varrho - \overline{\varrho}\|_{L^\gamma(\Omega)} + \|\mathbf{q} - \overline{\mathbf{q}})\|_{L^1(\Omega)}\right] \to 0 \text{ as } t \to \infty.$$

See [29, Theorem 4.2] for the proof.

There is an important particular case, when the assumption (209) is satisfied, namely, if $\mathcal{A}[\mathcal{F}]$ consists of a single stationary solution. In particular, making use of Proposition 3, the following generalization of the convergence result [87] and Theorem 31 holds:

Theorem 36. *Assume* (204)*, and let* $\mathbf{f}$ *be measurable function bounded uniformly on* $\mathbb{R}^+$*. Let* F *be globally Lipschitz continuous on* $\overline{\Omega}$ *such that the upper level sets*

$[F > k]$ *are connected for any* $k \in \mathbb{R}$. *Moreover, let*

$$(\mathbf{f}(. + \tau) - \nabla F) \to 0 \text{ weakly in } L^1((0,1) \times \Omega) \text{ as } \tau \to \infty. \tag{210}$$

Then any finite energy weak solution $\varrho, \mathbf{u}$ *of the problem* (63)–(64), (123), (125), (179) *on* $I = \mathbb{R}^+$ *satisfies*

$$\varrho(t) \to \varrho_s \text{ in } L^\gamma(\Omega), \text{ and the kinetic energy } \frac{1}{2}\int_{\varrho>0} \frac{|\mathbf{q}|^2}{\varrho}\,\mathrm{d}x \to 0 \text{ as } t \to \infty,$$

where ϱ_s *is the unique solution of the stationary problem* (181).

The convergence in (210) is very weak. It requires only that integral means taken with respect to space and time approach a potential driving force. In other words, both the density and the momenta are robust with respect to possible random fluctuations of the driving force both in space and time. Finally, we discuss the dependence of the attractor on the driving force $\mathbf{f}$. The result, in the case of a perturbation of a potential force ∇F satisfying (182), may be formulated as follows:

Theorem 37. *Let the assumptions of Theorem 36 be satisfied. Fix* $\alpha \in [1, \gamma)$.
Then given any $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\limsup_{t\to\infty} \|\varrho(t) - \varrho_s\|_{L^\alpha(\Omega)} < \varepsilon$$

whenever

$$\limsup_{t\to\infty} \|\mathbf{f}(t) - \nabla F\|_{L^\infty(\Omega)} < \delta$$

for any density component ϱ *of a finite energy weak solution of the problem* (63)–(64), (123), (125), (179) *with the driving force* $\mathbf{f}$ *measurable and bounded on* $\mathbb{R}^+$. *Here* ϱ_s *is the unique solution of the stationary problem* (181).

The proof, similarly as the proof of Theorem 33, follows from the compactness property stated in Proposition 2.

7 Navier-Stokes-Fourier System in the Internal Energy Formulation

7.1 Definition of Weak Solutions

In this section we shall deal with the Navier-Stokes-Fourier system (3)–(5) with the stress tensor and heat flux given by (12)–(13) and with the pressure and internal energy obeying (34)–(37), where the molecular pressure $p_{\rm mo}$ satisfies (45). The material of this section is mostly taken from [30].

In this situation, one can use identity (44) in order to rewrite the internal energy conservation in the simplified form

$$\partial_t \varrho\Big(e_{\rm th}(\vartheta) + e_{\rm ra}(\varrho,\vartheta)\Big) + {\rm div}_x \varrho\mathbf{u}\Big(e_{\rm th}(\vartheta) + e_{\rm ra}(\varrho,\vartheta)\Big) \tag{211}$$

$$+{\rm div}_x \mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta) + \Big(\vartheta p_{\rm th}(\varrho) + p_{\rm ra}(\vartheta)\Big){\rm div}_x\mathbf{u} = \mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}.$$

The right hand of the above identity contains the positive term $\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}$ which will give rise in the weak formulation to the functional of type $\nabla_x\mathbf{u} \mapsto \int_0^T\int_\Omega \mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\,{\rm d}x{\rm d}t$. This functional cannot certainly be continuous, but can be solely lower weakly semicontinuous with respect to the weak topology of the space $L^2(Q_T;\mathbb{R}^9)$. Therefore, we must replace in the weak formulation of equation (211) the equality sign by the inequality sign "$\geq$." In order to compensate the lack of information caused by this operation we add to the weak formulation of the system the total energy balance (26) with sign "$\leq$."

This motivates the following definition of weak solutions that we shall formulate for the heat flux of a specific form

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x\vartheta = -\nabla_x\mathcal{K}(\vartheta),\ \text{where}\ \mathcal{K}(\vartheta) = \int_0^\vartheta \kappa(z){\rm d}z. \tag{212}$$

Definition 6. Let Ω be a bounded domain, and let the initial conditions $(\varrho_0,\mathbf{u}_0,\vartheta_0)$ satisfy

$$\varrho_0 : \Omega \to [0,+\infty), \mathbf{u}_0 : \Omega \to \mathbb{R}^3,\ \vartheta_0 : \Omega \to (0,\infty), \tag{213}$$

where

$$\varrho_0\mathbf{u}_0 = 0 \text{ and } \varrho_0\mathbf{u}_0^2 = 0 \text{ a.e. in the set } \{x \in \Omega | \varrho_0(x) = 0\}$$

with finite energy $E_0 = \int_\Omega(\frac{1}{2}\varrho_0\mathbf{u}_0^2 + H_{\rm el}(\varrho_0) + \varrho_0 e_{\rm th}(\vartheta_0) + \varrho_0 e_{\rm ra}(\varrho_0,\vartheta_0)){\rm d}x$ and finite mass $0 < M_0 = \int_\Omega \varrho_0{\rm d}x$.

We shall say that a trio $(\varrho,\vartheta,\mathbf{u})$ is a *weak solution* to the Navier-Stokes-Fourier system (3)–(5) with boundary conditions (20)–(21), with viscous stress and heat flux (12)–(15), (212), and with pressure and internal energy (34)–(37), where $p_{\rm mo}$ obeys (45), emanating from the initial data $(\varrho_0,\vartheta_0,\mathbf{u}_0,)$ if:

(a)

$$\varrho \in L^\infty(0,T;L^1(\Omega)),\ \vartheta \in L^1(Q_T),\ \varrho \geq 0, \vartheta > 0 \ \text{ a.e. in } (0,T)\times\Omega, \tag{214}$$

$$\mathbf{u}\in L^2(0,T;W_0^{1,2}(\Omega));\ \ \varrho\mathbf{u},\ \frac{1}{2}\varrho\mathbf{u}^2,\ H_{\rm el}(\varrho),\ \varrho(e_{\rm th}(\vartheta)+e_{\rm ra}(\varrho,\vartheta))\in L^\infty(0,T;L^1(\Omega)),$$

$$\vartheta(p_{\rm th}(\varrho)+p_{\rm ra}(\vartheta)),\ \mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u},\ \mathcal{K}(\vartheta)\in L^1(Q_T).$$

(b) $\varrho \in C_{\rm weak}([0,T];L^1(\Omega))$, and the continuity equation (3) is satisfied in the following weak sense

$$\int_\Omega \varrho\varphi {\rm d}x\Big|_0^\tau = \int_0^\tau\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\Big){\rm d}x{\rm d}t, \tag{215}$$

for all $\tau \in [0,T]$ and for all $\varphi \in C_c^\infty([0,T]\times\overline{\Omega})$.

(c) $\varrho\mathbf{u} \in C_{\rm weak}([0,T];L^1(\Omega))$, and the momentum equation (4) is satisfied in the weak sense,

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi {\rm d}x\Big|_0^\tau = \int_0^\tau\int_\Omega\Big(\varrho\mathbf{u}\cdot\partial_t\varphi+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla\varphi+p(\varrho,\vartheta){\rm div}\varphi-\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\varphi+\varrho\mathbf{f}\cdot\varphi\Big){\rm d}x{\rm d}t = 0 \tag{216}$$

for all $\tau \in [0,T]$ and for all $\varphi \in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3)$.

(d) Balance of thermal energy (211) is satisfied as an inequality

$$\begin{aligned}
&\int_\Omega \Big[\varrho\Big(e_{\rm th}(\vartheta)+e_{\rm ra}(\varrho,\vartheta)\Big)\Big](\tau)\,{\rm d}x - \int_\Omega \varrho_0\Big(e_{\rm th}(\vartheta_0)+e_{\rm ra}(\varrho_0,\vartheta_0)\Big)\,{\rm d}x\\
&\quad\geq \int_0^\tau\int_\Omega \Big(\varrho\Big(e_{\rm th}(\vartheta)+e_{\rm ra}(\varrho,\vartheta)\Big)\partial_t\varphi + \varrho\Big(e_{\rm th}(\vartheta)+e_{\rm ra}(\varrho,\vartheta)\Big)\mathbf{u}\cdot\nabla_x\varphi + \mathcal{K}(\vartheta)\Delta\varphi\\
&\quad - \Big(\vartheta\, p_{\rm th}(\varrho)+p_{\rm ra}(\vartheta)\Big){\rm div}_x\mathbf{u}\varphi + \mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\varphi\Big)\,{\rm d}x{\rm d}t
\end{aligned} \tag{217}$$

for a.a. $\tau \in (0,T)$ and for all $\varphi \in C_c^1([0,T];C^2(\overline{\Omega}))$, $\nabla_x\varphi\cdot\mathbf{n}|_{(0,T)\times\partial\Omega} = 0$, $\varphi \geq 0$.

(e) The balance of the total energy (26) is satisfied in the weak sense as inequality

$$\begin{aligned}
&-\int_0^T \psi'(t)\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\rm el}(\varrho) + \varrho\Big(e_{\rm th}(\vartheta)+e_{\rm ra}(\varrho,\vartheta)\Big)\Big)\,{\rm d}x{\rm d}t \leq \int_0^T \psi(t)\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,{\rm d}x{\rm d}t\\
&\quad + \psi(0)\int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + H_{\rm el}(\varrho_0) + \varrho_0\Big(e_{\rm th}(\vartheta_0)+e_{\rm ra}(\varrho_0,\vartheta_0)\Big)\Big)\,{\rm d}x
\end{aligned} \tag{218}$$

for all $\psi \in C_c^1[0,T)$, $\psi \geq 0$.

We recall that $\int_\Omega g{\rm d}x\,|_0^\tau$ means $\int_\Omega g(\tau,x){\rm d}x - \int_\Omega g_0(x){\rm d}x$. The Helmholtz function $H_{\rm el}$ is defined in (67), and the space $C_{\rm weak}([0,T];L^1(\Omega))$ is defined in (71).

Definition 7. Weak solution whose density-velocity component $(\varrho, \mathbf{u})$ satisfies the continuity equation in the renormalized sense (116)–(117) in $\mathcal{D}'(Q_T)$ with $f = 0$, with any test function b belonging to (118) is called renormalized weak solution.

Remark 12. 1. According to (88) the total energy balance formulation (218) implies

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\rm el}(\varrho) + \varrho\Big(e_{\rm th}(\vartheta) + e_{\rm ra}(\varrho,\vartheta)\Big)\Big)\mathrm{d}x\Big|_0^\tau \le \int_0^\tau \int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\mathrm{d}x\mathrm{d}t, \tag{219}$$

for almost all $\tau \in (0, T)$.

2. According to (85) applied to the thermal energy conservation (217), the right and left instantaneous values $[\varrho e_{\rm th}(\vartheta) + \varrho e_{\rm ra}(\varrho,\vartheta)](\tau+)$ and $[\varrho e_{\rm th}(\vartheta) + \varrho e_{\rm ra}(\varrho,\vartheta)](\tau-)$ defined in (68)–(69) are continuous linear functionals on $C(\overline{\Omega})$ satisfying

$$[\varrho e_{\rm th}(\vartheta) + \varrho e_{\rm ra}(\varrho,\vartheta)](\tau+) \ge [\varrho e_{\rm th}(\vartheta) + \varrho e_{\rm ra}(\varrho,\vartheta)](\tau-). \tag{220}$$

3. We deduce from (85) (with $\eta = 1$) applied to the thermal energy balance (217) that the function of instantaneous values of thermal energy

$$[0,T] \ni \tau \mapsto E_{\rm th}(\tau) \equiv \mathrm{inst}\Big[\int_\Omega \varrho\Big(e_{\rm th}(\vartheta(\cdot,x)) + e_{\rm ra}(\varrho(\cdot,x),\vartheta(\cdot,x))\Big)\,\mathrm{d}x\Big](\tau) \tag{221}$$

is *a sum of an absolutely continuous function and a nondecreasing function (with at most countable number of jumps).*

4. Likewise, according to (86)–(89) applied to (218), the function of the instantaneous values of total energy of the weak solution

$$[0,T] \ni \tau \mapsto E(\tau) \equiv \mathrm{inst}\Big[\int_\Omega \Big(\frac{1}{2}\varrho\mathbf{u}^2(\cdot,x) + \varrho e(\varrho,\vartheta)(\cdot,x) + H_{\rm el}(\varrho(\cdot,x))\Big)\,\mathrm{d}x\Big](\tau) \tag{222}$$

is a sum of an absolutely continuous function and a nonincreasing function (with a countable number of jumps).

It seems that a significant piece of information is lost when replacing the internal energy *equation* (5) by the *variational inequality* (217). However, to compensate this loss, we require that the weak solution obeys the total energy inequality (218). This makes from Definitions 6 and 7 "good" definitions. Indeed, any sufficiently regular weak solution is a classical solution as stated in the following lemma whose proof can be found in Feireisl [30, Section 6].

Lemma 8. *Let the trio* $(\varrho, \vartheta, \mathbf{u})$ *be a weak solution to problem* (3)–(9) *with the same constitutive laws for pressure, internal energy, stress tensor, and heat flux as in Definition 6, with boundary conditions* (20)–(21) *and initial conditions* $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ *verifying* (213) *on a Lipschitz bounded domain* Ω *in the regularity class*

$$(\varrho, \vartheta, \mathbf{u}) \in C^1(\overline{Q}_T) \times C^1(\overline{Q}_T) \cap C([0,T]; C^2(\overline{\Omega})) \times C^1(\overline{Q}_T; \mathbb{R}^3) \cap C([0,T]; C^2(\overline{\Omega}; \mathbb{R}^3)) \quad \varrho > 0,\ \vartheta > 0. \tag{223}$$

Then $(\varrho, \vartheta, \mathbf{u})$ *is a classical solution to the Navier-Stokes-Fourier system. In particular, it satisfies all energy balance laws* (5)*,* (211)*,* (7)*,* (9)–(10) *as identities on* Q_T.

7.2 Existence of Weak Solutions

We start by specifying the assumptions under which the existence theorem on weak solutions will be investigated. We shall consider the flow without radiation (i.e., $p_{\rm ra} = 0$, $e_{\rm ra}$) for which the present weak formulation is more appropriate. The reader is invited to confront these assumptions with the physically motivated requirements (34)–(42), (45)–(46), (56)–(57), (16)–(18):

(1) *Pressure and internal energy.*

$$p(\varrho, \vartheta) = p_{\rm el}(\varrho) + \vartheta\, p_{\rm th}(\varrho),\ \ e(\varrho, \vartheta) = e_{\rm el}(\varrho) + e_{\rm th}(\vartheta), \tag{224}$$

where $p_{\rm el}$ is the same as in the barotropic case, namely,

$$p_{\rm el} \in C[0,\infty) \cap C^1(0,\infty),\ \ p_{\rm el}(0) = 0,\ \left\{ \begin{array}{l} p_{\rm el}(\varrho) \le a_1 \varrho^\gamma + b, \\ p'_{\rm el}(\varrho) \ge a_2 \varrho^{\gamma-1} - b, \end{array} \right\} \tag{225}$$

for some $\gamma \ge 1, a_1, a_2, b > 0$,

$$\begin{array}{c} p_{\rm th} \in C[0,\infty) \cap C^1(0,\infty),\ \ p_{\rm th}(0) = 0,\ \ p'_{\rm th}(\varrho) \ge 0, \\ p_{\rm th}(\varrho) \le c(1+\varrho^\Gamma) \text{ for some } 0 \le \Gamma. \end{array} \tag{226}$$

In agreement with (45), the thermal energy is given by

$$e_{\rm th}(\vartheta) = \int_0^\vartheta c_v(z) {\rm d}z,\ \ c_v \in C^1[0,\infty),\ \ \inf_{z\in[0,\infty)} c_v(z) \equiv \underline{c_v} > 0, \tag{227}$$

$$c_v(\vartheta) \le c(1+\vartheta^{\frac{\alpha}{2}-1}) \text{ where } c > 0, \alpha \ge 0.$$

In agreement with (42), elastic energy is calculated from the elastic pressure $p_{\rm el}$ through the formula

$$\varrho e_{\rm el}(\varrho) \equiv H_{\rm el}(\varrho) = \varrho \int_1^{\varrho} \frac{p_{\rm el}(z)}{z^2} {\rm d}z. \tag{228}$$

(2) *Viscous stress and heat flux*. The fluid is Newtonian with the viscous stress given by (12) with the constant viscosity coefficients

$$\mu > 0, \ \eta \geq 0. \tag{229}$$

Heat flux is given by the Fourier law (212), where

$$\kappa \in C^2[0,\infty), \ c_1(1+\vartheta^\alpha) \leq \kappa(\vartheta) \leq c_2(1+\vartheta^\alpha), \ \text{with constants } c_1, c_2 > 0, \text{ and } \alpha \geq 0. \tag{230}$$

Under the above assumptions, the Navier-Stokes-Fourier systems admits a *weak solution* provided the constants γ, Γ, and α verify some further restrictions. This statement is subject of the following theorem reported from [30, Theorem 7.1].

Theorem 38. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary of class $C^{2,\nu}$, $\nu > 0$. Suppose that pressure, internal energy, viscous stress tensor, and heat flux satisfy assumptions* (224)–(230) *with*

$$\gamma > 3/2, \ 0 \leq \Gamma \leq \gamma/3, \ \alpha \geq 2.$$

Then the Navier-Stokes-Fourier system (3)–(5) *with boundary conditions* (20)–(21) *and initial conditions* (213) *with*

$$\operatorname*{ess\,inf}_{x\in\Omega} \vartheta_0(x) > 0$$

admits a renormalized weak solution with the following additional properties:

$$\varrho \in C([0,T]; L^1(\Omega)) \cap L^\infty(0,T; L^\gamma(\Omega)) \cap L^{p_0}(Q_T), \ p_0 = \min\{\frac{5\gamma-3}{3}, \frac{3}{2}\gamma\}, \tag{231}$$

$$p_{\rm el}(\varrho) \in L^{p_1}(Q_T), \ \vartheta p_{\rm th}(\varrho) \in L^2(Q_T), \ p_1 = p_0/\gamma > 1, \tag{232}$$

$$\varrho \mathbf{u} \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega)), \tag{233}$$

$$\vartheta \in L^{\alpha+1}(Q_T), \tag{234}$$

$$\vartheta^{\underline{\alpha}}, \ [e_{\rm th}(\vartheta)]^{\frac{2}{\alpha}\underline{\alpha}} \in L^2(Q_T) \ \textit{for all} \ \underline{\alpha} \in [0, \frac{\alpha+1}{2}], \ \ \ln\vartheta \in L^2(Q_T), \tag{235}$$

$$\varrho e_{\rm th}(\vartheta) \in L^\infty(0,T; L^1(\Omega)) \cap L^2(0,T; L^{\frac{6\gamma}{\gamma+6}}(\Omega)), \tag{236}$$

$$\text{ess}\lim_{t\to 0+}\int_\Omega \varrho e_{\rm th}(\vartheta)(t,x)\eta(x)\,\mathrm{d}x = \int_\Omega \varrho_0 e_{\rm th}(\vartheta_0)\eta\,\mathrm{d}x,\ \ \eta\in C_c^\infty(\Omega). \tag{237}$$

There exists $\tilde{\vartheta}\in L^2(0,T;W^{1,2}(\Omega))$ *such that* $\vartheta=\tilde{\vartheta}$ *a.e. in* $\{(t,x)|\varrho(t,x)>0\}$. (238)

8 Main Ideas of the Proof of Theorem 38

As in the case of the "simple" barotropic situation, the main issue in the proof of the existence theorem is the understanding of the propagation of the density oscillations. This phenomenon is coupled with the the thermal energy balance and gives rise to further difficulties linked especially to the vanishing density. In fact in the context of weak solutions, we cannot avoid the formation of vacuum regions of nonzero Lebesgue measure.

Rather than existence, we shall prove the weak stability of the set of (sufficiently smooth) weak solutions. We shall formulate this property in the subsequent Lemma 9. The proof of this lemma will contain already all main ingredients of the proof of the existence theorem. The reader should however be aware that even after Lemma 9 is established, the construction of solutions remains a hard and tricky job with great amount of difficulties.

The construction of weak solutions to this problem goes far beyond the scope of the handbook. There are so far two methods available in the mathematical literature: (1) a functional analytic method based on several levels of approximations by partial differential equations involving several (small) parameters similar to the one reported through (402)–(415), whose details can be found in [30, Chapter 7], and (2) numerical method based on the finite volumes/finite element approximations whose details can be found in [55]. This method needs a further restriction on the adiabatic coefficient γ, namely, $\gamma>3$.

Lemma 9. *Let* $\Omega\subset\mathbb{R}^3$ *be a bounded domain with boundary of class* $C^{2,\nu}$, $\nu>0$. *Suppose that pressure, internal energy, viscous stress tensor, and heat flux satisfy assumptions* (224)–(230) *with* $\gamma>3/2$, $0\le\Gamma\le\gamma/3$, *and* $\alpha\ge 2$. *Let* $(\varrho_n,\vartheta_n,\mathbf{u}_n)$ *in the regularity class* (223) *be a sequence of finite energy renormalized weak solutions to problem* (3)–(5) *with boundary conditions* (20)–(21) *and initial conditions* $(\varrho_{n,0},\vartheta_{n,0},\mathbf{u}_{n,0})$ *satisfying*

$$\varrho_{n,0}\rightharpoonup\varrho_0\ \ \text{in } L^1(\Omega),\ \ \varrho_{n,0}\mathbf{u}_{n,0}\rightharpoonup\varrho_0\mathbf{u}_0\ \ \text{in } L^1(\Omega;\mathbb{R}^3), \tag{239}$$

$$\varrho_{n,0}e_{\rm th}(\vartheta_{n,0})\rightharpoonup\varrho_0 e_{\rm th}(\vartheta_0)\ \text{in } L^1(\Omega),$$

$$\int_\Omega\Big(\frac{1}{2}\varrho_{n,0}|\mathbf{u}_{n,0}|^2+\varrho_{n,0}e_{\rm th}(\vartheta_{n,0})+H_{\rm el}(\varrho_{n,0})\Big)\,\mathrm{d}x$$

$$\to\int_\Omega\Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2+\varrho_0 e_{\rm th}(\vartheta_0)+H_{\rm el}(\varrho_0)\Big)\,\mathrm{d}x,$$

with bounded from below entropy

$$\int_{\Omega} \varrho_{n,0} s(\varrho_{n,0}, \vartheta_{n,0})\,\mathrm{d}x \geq \underline{S} \in \mathbb{R},$$

where $(\varrho_{n,0}, \vartheta_{n,0}, \mathbf{u}_{n,0})$ *and* $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ *verify* (213) *with* $M_{n,0} > 0,\ E_{n,0} \in \mathbb{R}$, *and* $M_0 > 0,\ E_0 \in \mathbb{R}$, *respectively. Then there exists a subsequence (denoted again* $(\varrho_n, \vartheta_n, \mathbf{u}_n)$*) such that*

$$\varrho_n \to \varrho \text{ weakly-* in } L^{\infty}(0,T; L^{\gamma}(\Omega)), \tag{240}$$

$$\text{where } \varrho \in C([0,T]; L^1(\Omega)) \cap L^r(Q_T), 0 < r \leq \min\{\frac{3}{2}\gamma, \frac{5}{3}\gamma - 1\},$$

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)),$$

$$\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \text{ weakly-* in } L^{\infty}(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^3)),\ \text{where } \varrho\mathbf{u} \in C_{\text{weak}}(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)),$$

$$\mathcal{K}_{\omega}(\vartheta_n) \rightharpoonup \Theta_{\omega} \text{ as } n \to \infty \text{ (weakly) in } L^1(Q_T),$$

$$\Theta_{\omega} \to \Theta \text{ a.e. in } Q_T \text{ as } \omega \to 0+,$$

with

$$\mathcal{K}_{\omega}(\vartheta) = \int_0^{\vartheta} h_{\omega}(z)\kappa(z)\mathrm{d}z, \quad h_{\omega}(z) = \frac{1}{(1+z)^{\omega}}, \tag{241}$$

where the trio

$$(\varrho, \vartheta = \mathcal{K}^{-1}(\Theta), \mathbf{u})$$

is a renormalized weak solution of (3)–(5) *with boundary conditions* (20)–(21) *and initial conditions* $(\varrho_0, \vartheta_0, \mathbf{u}_0)$.

Remark 13. 1. It should be noticed that the gradient of the temperature component ϑ of the weak solution is not square integrable, as one would expect from the presence of dissipation in the thermal energy balance. One can show that there is $\tilde{\vartheta} \in L^2(0,T; W^{1,2}(\Omega))$ such that

$$\vartheta_n \rightharpoonup \tilde{\vartheta} \text{ in } L^2(0,T; W^{1,2}(\Omega))$$

coinciding with the temperature component ϑ of the weak solution almost everywhere outside the vacuum set. Consequently, $\mathcal{K}(\tilde{\vartheta}) = \mathcal{K}(\vartheta)$ almost everywhere outside vacua; however it may happen that $\mathcal{K}(\tilde{\vartheta}) \neq \mathcal{K}(\vartheta)$ in a subset of the vacuum set with the nonzero measure.

8.1 Equations Verified by the Sequence

By virtue of Lemma 8, the trio $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ satisfies equations (3), (4), (5), $(211)_{e_{\rm ra}=0}$, (7), (9)–(10) together with boundary conditions (20)–(21). In particular,

$$\partial_t \varrho_n + {\rm div}_x(\varrho_n \mathbf{u}_n) = 0 \text{ in } [0,T]\times\mathbb{R}^3 \text{ provided } (\varrho_n, \mathbf{u}_n) \text{ is extended by } (0,0) \text{ outside } \Omega, \tag{242}$$

$$\partial_t(\varrho_n \mathbf{u}_n) + {\rm div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n, \vartheta_n) = {\rm div}_x \mathbb{S}(\nabla_x \mathbf{u}_n) = \varrho_n \mathbf{f} \text{ in } \overline{Q_T}, \tag{243}$$

$$\partial_t(\varrho_n e_{\rm th}(\vartheta_n)) + {\rm div}_x(\varrho_n \mathbf{u}_n e_{\rm th}(\vartheta_n)) \tag{244}$$

$$+ {\rm div}_x \mathbf{q}(\nabla_x \vartheta_n) + \vartheta_n p_{\rm th}(\varrho_n){\rm div}_x \mathbf{u}_n = \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \text{ in } \overline{Q_T},$$

$$\partial_t(\varrho_n s(\varrho_n, \vartheta_n)) + {\rm div}_x(\varrho_n s(\varrho_n, \vartheta_n)\mathbf{u}_n) + {\rm div}_x\left(\frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n)}{\vartheta_n}\right) \tag{245}$$

$$= \frac{1}{\vartheta_n}\left(\mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n - \frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n)\cdot \nabla_x \vartheta_n}{\vartheta_n}\right) \text{ in } \overline{Q_T},$$

$$\frac{{\rm d}}{{\rm d}t}\int_\Omega \left(\frac{1}{2}\varrho_n |\mathbf{u}_n|^2 + \varrho_n e_{\rm th}(\vartheta_n) + H_{\rm el}(\varrho_n)\right) {\rm d}x = \int_\Omega \varrho_n \mathbf{f}\cdot \mathbf{u}_n \,{\rm d}x \text{ for all } t \in [0,T], \tag{246}$$

$$\partial_t b(\varrho_n) + {\rm div}_x\Big(b(\varrho_n)\mathbf{u}_n\Big) + \Big(\varrho_n b'(\varrho_n) - b(\varrho_n)\Big){\rm div}_x \mathbf{u}_n = 0, \ b \text{ as in (134)}, \tag{247}$$

$$\text{in } [0,T]\times\mathbb{R}^3 \text{ provided } (\varrho_n, \mathbf{u}_n) \text{ is extended by } (0,0) \text{ outside } \Omega.$$

This implies

$$\partial_t T_k(\varrho_n) + {\rm div}_x\Big(T_k(\varrho_n)\mathbf{u}_n\Big) + \Big(\varrho_n T_k'(\varrho_n) - T_k(\varrho_n)\Big){\rm div}_x \mathbf{u}_n = 0, \tag{248}$$

and

$$\partial_t \varrho_n L_k(\varrho_n) + {\rm div}_x\Big(\varrho_n L_k(\varrho_n)\mathbf{u}_n\Big) + T_k(\varrho_n){\rm div}_x \mathbf{u}_n = 0 \tag{249}$$

$$\text{in } [0,T]\times\mathbb{R}^3 \text{ provided } (\varrho_n, \mathbf{u}_n) \text{ is extended by } (0,0) \text{ outside } \Omega,$$

where

$$T_k(z) = kT(z/k), \quad L_k(z) = \int_1^z \frac{T_k(w)}{w^2}{\rm d}w, \tag{250}$$

$$T \in C^1[0,\infty), \quad T(z) = \begin{cases} z \text{ if } z \in [0,1], \\ \text{concave on } [0,\infty), \\ 2 \text{ if } z \geq 3. \end{cases}$$

8.2 A Priori Estimates

1. *Bounds due to the mass conservation.*

   Integrating equation (242) yields

$$\int_\Omega \varrho_n(\tau)\,\mathrm{d}x = \int_\Omega \varrho_{n,0}\,\mathrm{d}x,$$

in particular

$$\|\varrho_n\|_{L^\infty(0,T;L^1(\Omega))} \leq c(M_0). \tag{251}$$

2. *Bounds due to the global energy conservation.*

   Balance of total energy in the volume Ω (246) (that is equation (7) integrated over Ω) yields

$$\int_\Omega \Big(\frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + \varrho_n e_{\mathrm{th}}(\vartheta_n) + H_{\mathrm{el}}(\varrho_n)\Big)(\tau)\,\mathrm{d}x$$
$$= \int_\Omega \Big(\frac{1}{2}\varrho_{n,0}|\mathbf{u}_{n,0}|^2 + \varrho_{n,0}e_{\mathrm{th}}(\vartheta_{n,0}) + H_{\mathrm{el}}(\varrho_{n,0})\Big)\,\mathrm{d}x + \int_0^\tau \int_\Omega \varrho_n \mathbf{f}\cdot\mathbf{u}_n\,\mathrm{d}x\mathrm{d}t.$$

Recalling definition of H_{el} (42) and (225), we verify that $c_1\varrho^\gamma \leq H_{\mathrm{el}}(\varrho) + c_2(1+\varrho\ln\varrho)$ with $c_1, c_2 > 0$ (dependent on a_1, a_2, b). Further $|\int_0^\tau\int_\Omega \varrho_n\mathbf{f}\cdot\mathbf{u}_n\,\mathrm{d}x\mathrm{d}t| \leq \|\mathbf{f}\|_{L^\infty(Q_\tau;\mathbb{R}^3)}\int_0^\tau\Big(\sqrt{\int_\Omega \varrho_n\,\mathrm{d}x}\,\sqrt{\int_\Omega \varrho_n(\mathbf{u}_n)^2\,\mathrm{d}x}$ by virtue of the Cauchy-Schwarz inequality. Employing these facts, the Gronwall lemma (see Theorem 21) and assumptions (227) on the form of e_{th}, we derive from the last center-lined identity the bounds

$$\|\varrho_n\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq c(M_0, E_0, F_0, T), \tag{252}$$

$$\|\varrho_n e_{\mathrm{th}}(\vartheta_n)\|_{L^\infty(0,T;L^1(\Omega))},\ \|\varrho_n\vartheta_n\|_{L^\infty(0,T;L^1(\Omega))} \leq c(M_0, E_0, F_0, T), \tag{253}$$

$$\|\varrho_n|\mathbf{u}_n|^2\|_{L^\infty(0,T;L^1(\Omega))} \leq c(M_0, E_0, F_0, T), \tag{254}$$

where here and hereafter, we denote

$$F_0 \equiv \|\mathbf{f}\|_{L^\infty(Q_T;\mathbb{R}^3)}.$$

3. *Bounds due to the entropy balance.*

   Entropy balance (245) integrated over the space time cylinder Q_τ, while taking into account the boundary conditions (20), (21), yields

$$\int_0^\tau \int_\Omega \Big(\frac{1}{\vartheta_n} \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n + \kappa(\vartheta_n) \frac{|\nabla_x \vartheta_n|^2}{(\vartheta_n)^2} \Big) \, \mathrm{d}x\mathrm{d}t$$
$$= \int_\Omega \varrho_n s(\varrho_n, \vartheta_n)(\tau) \, \mathrm{d}x - \int_\Omega \varrho_{n,0} s(\varrho_{n,0}, \vartheta_{n,0}) \, \mathrm{d}x,$$

   where the specific entropy $s(\varrho, \vartheta) = s_{\mathrm{mo}}(\varrho, \vartheta) = s_{\mathrm{mo},\vartheta}(\vartheta) + s_{\mathrm{mo},\varrho}(\varrho)$ is given by formula (46). Employing (46) and assumptions (226), (227) we find pointwise estimates

$$s_{\mathrm{mo},\vartheta}(\vartheta) 1_{\{\vartheta \geq 1\}}(\vartheta) \leq c(1 + e_{\mathrm{th}}(\vartheta)) 1_{\{\vartheta \geq 1\}}(\vartheta),$$
$$-s_{\mathrm{mo},\vartheta}(\vartheta) 1_{\{\vartheta < 1\}}(\vartheta) \geq \underline{c_v} |\ln \vartheta| 1_{\{\vartheta < 1\}}(\vartheta),$$

   and

$$s_{\mathrm{mo},\varrho}(\varrho) \leq c(1 + \varrho^\Gamma).$$

   Consequently, we deduce from the entropy balance the following bounds

$$\|\varrho_n \ln \vartheta_n\|_{L^\infty(0,T;L^1(\Omega))} \leq c(M_0, E_0, \underline{S}, F_0, T), \tag{255}$$

$$\|\nabla \ln \vartheta_n\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\nabla (\vartheta_n)^{\alpha/2}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leq c(M_0, E_0, \underline{S}, F_0, T). \tag{256}$$

   Estimate (256) in combination with (255) and (253) yields

$$\| \ln \vartheta_n\|_{L^2(0,T;W^{1,2}(\Omega))} + \|(\vartheta_n)^{\alpha/2}\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(M_0, E_0, \underline{S}, F_0, T), \tag{257}$$

   by virtue of the Poincaré inequality stated in Theorem 7. Next, we may use the Sobolew imbedding to get

$$\| \ln \vartheta_n\|_{L^2(0,T;L^6(\Omega))} + \|(\vartheta_n)^{\underline{\alpha}}\|_{L^2(0,T;L^6(\Omega))} \leq c(M_0, E_0, \underline{S}, F_0, T), \ 0 \leq \underline{\alpha} \leq \alpha/2. \tag{258}$$

   Finally, estimate (257) in combination with assumption (227) yields, in particular,

$$\|e_{\mathrm{th}}(\vartheta_n)\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(M_0, E_0, \underline{S}, F_0, T). \tag{259}$$

4. *Bounds due to the thermal energy balance I*

   Integrating the thermal energy balance (244) while taking into account boundary conditions (20)–(21), we get

$$\int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \, \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_\Omega \vartheta_n p_{\mathrm{th}}(\varrho_n) \mathrm{div}_x \mathbf{u}_n \, \mathrm{d}x\mathrm{d}t$$
$$+ \int_\Omega \varrho_n e_{\mathrm{th}}(\vartheta_n)(\tau, x) \, \mathrm{d}x - \int_\Omega \varrho_{n,0} e_{\mathrm{th}}(\vartheta_{n,0}) \, \mathrm{d}x.$$

The first term at the right-hand side is bounded from above by virtue of Hölder and Young inequalities by $c(\delta)\|\vartheta_n\|^2_{L^2(0,\tau;L^6(\Omega))}\|p_{\mathrm{th}}(\varrho_n)\|^2_{L^\infty(0,\tau,L^3(\Omega))} + \delta\|\mathrm{div}_x \mathbf{u}_n\|^2_{L^2(0,\tau;L^2(\Omega))}$ with any $\delta > 0$, while the left-hand side is bounded from below by $c\|\nabla_x \mathbf{u}_n\|^2_{L^2(0,\tau;L^2(\Omega;\mathbb{R}^9))}$ in view of Korn's inequality stated in Theorem 9. Next, we use the known upper bounds (258), (252) together with assumption (226) where $0 \le \Gamma \le \gamma/3$, and (253) to obtain

$$\|\nabla_x \mathbf{u}_n\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^9))} \le c(M_0, E_0, \underline{S}, F_0, T).$$

Finally, the classical Poincaré inequality gives

$$\|\mathbf{u}_n\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \le c(M_0, E_0, \underline{S}, F_0, T). \tag{260}$$

5. *Bounds due to the thermal energy balance II*

Multiplying the thermal energy balance (244) by $(\vartheta_n)^{-\beta}$, $0 < \beta < 1$, and integrating over Q_τ, we get

$$\beta \int_0^\tau \int_\Omega \kappa(\vartheta_n) \frac{|\nabla_x \vartheta_n|}{\vartheta_n^{\beta+1}} \, \mathrm{d}x\mathrm{d}t = \int_\Omega \varrho_n \mathcal{H}_\beta(\vartheta_n)(\tau, x) \, \mathrm{d}x - \int_\Omega \varrho_{n,0} \mathcal{H}_\beta(\vartheta_{n,0}) \, \mathrm{d}x$$
$$- \int_0^\tau \int_\Omega \frac{1}{(\vartheta_n)^\beta} \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \, \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega (\vartheta_n)^{1-\beta} p_{\mathrm{th}}(\varrho_n) \mathrm{div}_x \mathbf{u}_n \, \mathrm{d}x\mathrm{d}t,$$

where $\mathcal{H}_\beta(\vartheta) = \int_0^\vartheta z^{-\beta} c_v(z)\mathrm{d}z$. We write $z^{-\beta} c_v(z) \le \frac{1}{z} 1_{\{0<z<1\}}(z) + c_v(z) 1_{\{z \ge 1\}}(z)$; whence the first two terms are bounded by virtue of (253), (255). We already know from the entropy balance (see *item* 3.) and from the thermal energy balance (see *item* 4.) that $\int_0^\tau \int_\Omega \left(1 + \frac{1}{\vartheta_n}\right) \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \, \mathrm{d}x\mathrm{d}t \le c(M_0, E_0, \underline{S}, F_0, T)$; whence the third term at the right-hand side is bounded by $c(M_0, E_0, \underline{S}, F_0, T)$. Finally, the absolute value of the last term is estimated by Hölder's inequality and (252), (258)–(260). As a conclusion, after application of Theorem 7

$$\|(\vartheta_n)^{\frac{\alpha+1-\beta}{2}}\|_{L^2(0,T;W^{1,2}(\Omega))} \le c(M_0, E_0, \underline{S}, F_0, T, \beta) \text{ with any } 0 < \beta < 1. \tag{261}$$

6. *Pressure estimates*

We multiply momentum equation (243) by the test function $\varphi(t, x) = \eta(t)\mathcal{B}\Big[\varrho_n^\omega - \frac{1}{|\Omega|}\int_\Omega \varrho_n^\omega \, \mathrm{d}x\Big]$ and integrate over the space time cylinder Q_T, where $\mathcal{B}$ is the inverse of the divergence operator defined in Theorem 5, ω is a positive number (that will be specified later), and η represents a conveniently chosen family of $C_c^1(0,T)$ cutoff functions. Employing the renormalized continuity equation (247) and the properties of operator $\mathcal{B}$, one gets after several integrations by parts

$$\int_0^T \eta \int_\Omega p(\vartheta_n, \varrho_n)(\varrho_n)^\omega \, \mathrm{d}x\mathrm{d}t = \sum_{i=1}^7 I_i,$$

where

$$I_1 = \frac{1}{|\Omega|}\int_0^T \eta\Big(\int_\Omega p(\vartheta_n, \varrho_n)\,\mathrm{d}x \int_\Omega (\varrho_n)^\omega \,\mathrm{d}x\Big)\mathrm{d}t,$$

$$I_2 = \int_0^T \eta \int_\Omega \varrho_n \mathbf{u}_n \cdot \mathcal{B}\Big[\mathrm{div}_x((\varrho_n)^\omega \mathbf{u}_n)\Big]\,\mathrm{d}x\mathrm{d}t,$$

$$I_3 = (\omega - 1)\int_0^T \eta \int_\Omega \varrho_n \mathbf{u}_n \cdot \mathcal{B}\Big[(\varrho_n)^\omega \mathrm{div}_x \mathbf{u}_n - \frac{1}{|\Omega|}\int_\Omega (\varrho_n)^\omega \mathrm{div}_x \mathbf{u}_n \,\mathrm{d}x\Big]\,\mathrm{d}x\mathrm{d}t,$$

$$I_4 = -\int_0^T \eta' \int_\Omega \varrho_n \mathbf{u}_n \cdot \mathcal{B}\Big[\varrho^\omega - \frac{1}{|\Omega|}\int_\Omega (\varrho_n)^\omega \,\mathrm{d}x\Big]\,\mathrm{d}x\mathrm{d}t,$$

$$I_5 = -\int_0^T \eta \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla_x \mathcal{B}\Big[\varrho^\omega - \frac{1}{|\Omega|}\int_\Omega (\varrho_n)^\omega \,\mathrm{d}x\Big]\,\mathrm{d}x\mathrm{d}t,$$

$$I_6 = \int_0^T \eta \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathcal{B}\Big[\varrho^\omega - \frac{1}{|\Omega|}\int_\Omega (\varrho_n)^\omega \,\mathrm{d}x\Big]\,\mathrm{d}x\mathrm{d}t,$$

$$I_7 = -\int_0^T \eta \int_\Omega \varrho_n \mathbf{f} \cdot \mathcal{B}\Big[\varrho^\omega - \frac{1}{|\Omega|}\int_\Omega (\varrho_n)^\omega \,\mathrm{d}x\Big]\,\mathrm{d}x\mathrm{d}t.$$

Writing moreover the left-hand side as the sum

$$\int_0^T \eta \int_\Omega p_{\mathrm{el}}(\varrho_n)(\varrho_n)^\omega \,\mathrm{d}x\mathrm{d}t + J_1, \quad J_1 = \int_0^T \eta \int_\Omega \vartheta_n p_{\mathrm{th}}(\varrho_n)(\varrho_n)^\omega \,\mathrm{d}x\mathrm{d}t,$$

we shall use the Hölder inequality, assumptions (225)–(226), and Theorem 5 together with already established estimates (namely, (252)–(253), (258)–(261)) in order to get bound

$$\|\varrho_n\|_{L^{\gamma+\omega}(Q_T)} \le c(M_0, E_0, \underline{S}, F_0, T),\; 0 < \omega \le \min\{\frac{2}{3}\gamma - 1, \frac{1}{2}\gamma\}, \qquad (262)$$

in particular,

$$\|p(\varrho_n, \vartheta_n\|_{L^q(Q_T)} \le c(M_0, E_0, \underline{S}, F_0, T) \text{ with some } q > 1. \qquad (263)$$

7. *Temperature estimates*

Temperature estimates (258), (261) are still not sufficient to give sense to the term containing $\mathcal{K}(\vartheta)$ in the weak formulation (217) of the thermal energy balance. We need further estimates of the temperature.

An improved estimate outside of vacuum regions is quite straightforward. Since $\int_G \vartheta^{(\alpha+1)p} \mathrm{d}x \leq (\int_G \vartheta \mathrm{d}x)^{1/s'} (\int_G \vartheta^{((\alpha+1)p-1/s')s} \mathrm{d}x)^{1/s}$, $1 \leq p < \infty$, $1 < s < \infty$, we have by virtue of (261), (253)

$$\|\vartheta_n\|_{L^p(\{\varrho_n \geq \varepsilon\})} \leq c(M_0, E_0, \underline{S}, F_0, T, p, \varepsilon) \text{ with } 0 < \varepsilon < M/|\Omega|. \tag{264}$$

Similar estimate near the vacuum regions is more tricky. To this end, we multiply by test function

$$\varphi(t,x) = \eta(t)(\psi(t,x) - \overline{\psi}), \quad \text{with } \overline{\psi} = \inf_{(t,x)\in Q_T} \psi(t,x),$$

and integrate over Q_T the thermal energy balance equation (244), where ψ is the unique solution with the zero mean of the Neumann problem

$$\Delta\psi = h(\varrho_n(t,x)) - \frac{1}{|\Omega|}\int_\Omega h(\varrho_n)\,\mathrm{d}x \text{ in } \Omega, \quad \nabla_x\psi \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{265}$$

in which

$$C^1(\mathbb{R}) \ni h \text{ nonincreasing}, \; h(z) = 0 \text{ if } z \leq \varepsilon, \; h(z) = -1 \text{ if } z \geq 2\varepsilon,$$

and $\eta \in C^1_c(0,T)$ represents the same family of test functions as used in the step dealing with pressure estimates. This testing yields the integral identity

$$\int_0^T \eta \int_\Omega \mathcal{K}(\vartheta_n)\Big(h(\varrho_n) - \frac{1}{|\Omega|}\int_\Omega h(\varrho_n)\,\mathrm{d}x\Big)\,\mathrm{d}x\mathrm{d}t = \sum_{i=1}^5 I_i,$$

where

$$I_1 = \int_0^T \eta' \int_\Omega (\overline{\psi} - \psi)\varrho_n e_{\mathrm{th}}(\vartheta_n)\,\mathrm{d}x\mathrm{d}t,$$

$$I_2 = -\int_0^T \eta \int_\Omega \partial_t\psi\varrho_n e_{\mathrm{th}}(\vartheta_n)\,\mathrm{d}x\mathrm{d}t,$$

$$I_3 = -\int_0^T \eta \int_\Omega \varrho_n e_{\mathrm{th}}(\vartheta_n)\mathbf{u}_n \cdot \nabla_x\psi\,\mathrm{d}x\mathrm{d}t,$$

$$I_4 = \int_0^T \eta \int_\Omega (\overline{\psi} - \psi)\mathbb{S}(\nabla_x\mathbf{u}_n) : \nabla_x\mathbf{u}_n\,\mathrm{d}x\mathrm{d}t,$$

$$I_5 = \int_0^T \eta \int_\Omega (\psi - \overline{\psi})\vartheta_n p_{\mathrm{th}}(\varrho_n)\mathrm{div}_x\mathbf{u}_n\,\mathrm{d}x\mathrm{d}t.$$

We deduce from Hölder's inequality and (251)–(252) that $|\{\varrho_n \geq 2\varepsilon\}| \geq \mathfrak{h}(\varepsilon) > 0$; whence thanks to the choice of function h, the left-hand side of the above inequality is bounded from below by expression $\frac{\mathfrak{h}(\varepsilon)}{|\Omega|}\int_{\{\varrho_n<\varepsilon\}} \eta\mathcal{K}(\vartheta_n)\mathrm{d}x\mathrm{d}t$. Each of integrals I_1-I_5 can be estimated by Hölder's inequality; the Lebesgue norms involving ψ and $\partial_t\psi$ will be estimated by means of standard elliptic L^p estimates for the Neumann problem applied to (265) and to

$$\Delta\partial_t\psi = -\operatorname{div}_x(h(\varrho_n)\mathbf{u}_n) + (h(\varrho_n) - \varrho_n h'(\varrho_n))\operatorname{div}_x\mathbf{u}_n - \frac{1}{|\Omega|}\int_\Omega (h(\varrho_n) - \varrho_n h'(\varrho_n))\operatorname{div}_x\mathbf{u}_n \,\mathrm{d}x,$$

$$\nabla_x\partial_t\psi\cdot\mathbf{n}|_{\partial\Omega} = 0,$$

where the renormalized continuity equation (247) has been used to calculate the right-hand side of the latter Laplace equation.

Performing this program we arrive, with the help of estimates (251)–(263) and assumption (230) (translated to $\mathcal{K}$ – see (212)) to the bound

$$\|(\vartheta_n)^{\alpha+1}\|_{L^1(\{\varrho_n<\varepsilon\})} \leq c(M_0, E_0, \underline{S}, F_0, T, \varepsilon);$$

whence

$$\|(\vartheta_n)^{\alpha+1}\|_{L^1(Q_T)} \leq c(M_0, E_0, \underline{S}, F_0, T) \tag{266}$$

by virtue of the last estimate and (264).

8.3 Weak Limits in the Momentum and Renormalized Continuity Equations

Bounds (252), (258), (260) imply existence of a subsequence (denoted again $(\varrho_n, \vartheta_n, \mathbf{u}_n)$) and of a trio $(\varrho, \tilde{\vartheta}, \mathbf{u})$ such that

$$\begin{aligned} \varrho_n &\rightharpoonup^* \varrho \text{ in } L^\infty(0,T;L^\gamma(\Omega)),\\ \vartheta_n &\rightharpoonup \tilde{\vartheta} \text{ in } L^2(0,T;W^{1,2}(\Omega)),\\ \mathbf{u}_n &\rightharpoonup \mathbf{u} \text{ in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)). \end{aligned} \tag{267}$$

In what follows we systematically denote by $\overline{g(\varrho,\vartheta,\mathbf{u})}$ a weak limit in $L^1(Q_T)$ of the sequence $g(\varrho_n,\vartheta_n,\mathbf{u}_n)$ in $L^1((0,T)\times\Omega))$.

Due to (263),

$$p(\varrho_n,\vartheta_n) \rightharpoonup \overline{p(\varrho,\vartheta)} \text{ in } L^p(Q_T) \text{ with some } p>1. \tag{268}$$

Next, we can use continuity equation (242), renormalized continuity equation (247), and momentum equation (243) to show the equi-continuity of functions

$$t \mapsto \int_\Omega \varrho_n \varphi \, \mathrm{d}x, \quad t \mapsto \int_\Omega b(\varrho_n)\varphi \, \mathrm{d}x, \quad t \mapsto \int_\Omega \varrho_n \mathbf{u}_n \varphi \, \mathrm{d}x,$$

on $[0, T]$, where $\varphi \in C_c^\infty(\Omega)$. This fact makes possible to use the Arzela-Ascoli compactness argument which in combination with the density argument yields convergence of the sequences ϱ_n, $b(\varrho_n)$, and $\varrho_n \mathbf{u}_n$ in $C_{\mathrm{weak}}([0, T]; L^q(\Omega))$ with some $q > 6/5$. Employing moreover the compact imbedding $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$, we get the convergence of these quantities in $L^2(0, T; W^{-1,2}(\Omega))$. Summarizing the above, we have

$$\begin{gathered}
\varrho_n \to \varrho \ \text{ in } C_{\mathrm{weak}}([0, T]; L^\gamma(\Omega)) \text{ and in } L^2(0, T; W^{-1,2}(\Omega)), \\
b(\varrho_n) \to \overline{b(\varrho)} \ \text{ in } C_{\mathrm{weak}}([0, T]; L^q(\Omega)), \text{ and in } L^2(0, T; W^{-1,2}(\Omega)), \\
\text{provided } b \in C[0, \infty) \cap C^1(0, \infty), b(\varrho_n) \text{ bounded in } L^\infty(0, T; L^q(\Omega)), \\
\varrho_n b'(\varrho_n) - b(\varrho_n) \text{ bounded in } L^2(Q_T)), \\
\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \ \text{ in } C_{\mathrm{weak}}([0, T]; L^{2\gamma/(\gamma+1)}(\Omega; \mathbb{R}^3)) \text{ and in } L^2(0, T; W^{-1,2}(\Omega, \mathbb{R}^3)), \\
\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \to \varrho \mathbf{u} \otimes \mathbf{u} \ \text{ in } L^2(0, T; L^{6\gamma/(4\gamma+3)}(\Omega, \mathbb{R}^9)).
\end{gathered} \tag{269}$$

The second relation in (269) employed with $b = p_{\mathrm{th}}$ in combination with the second relation in (267) yields, in particular,

$$\vartheta_n p_{\mathrm{th}}(\varrho_n) \rightharpoonup \tilde{\vartheta} \ \overline{p_{\mathrm{th}}(\varrho)} \text{ in } L^2(Q_T), \tag{270}$$

$$\vartheta_n p_{\mathrm{th}}(\varrho_n) T_k(\varrho_n) \rightharpoonup \tilde{\vartheta} \ \overline{p_{\mathrm{th}}(\varrho) T_k(\varrho)} \text{ in } L^2(Q_T).$$

Now, we are ready to let $n \to \infty$ in equations (242), (243), and (247). We get, in particular,

$$\int_\Omega \varrho(\tau, x)\varphi(\tau, x) \, \mathrm{d}x - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x = \int_0^T \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big) \, \mathrm{d}x \mathrm{d}t \tag{271}$$

for all $\tau \in [0, T]$ and any $\varphi \in C_c^1([0, T] \times \overline{\Omega})$;

$$\int_\Omega \varrho(\tau, x)\varphi(\tau, x) \, \mathrm{d}x - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \, \mathrm{d}x \tag{272}$$

$$= \int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \overline{p(\varrho, \vartheta)} \mathrm{div}_x \varphi - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi + \varrho \mathbf{f} \cdot \varphi \Big) \, \mathrm{d}x \mathrm{d}t$$

for all $\tau \in [0, T]$ and for any $\varphi \in C_c^1([0, T] \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi|_{\partial\Omega} = 0$;

$$\int_\Omega \overline{\varrho L_k(\varrho)}(\tau, x) \varphi(\tau, x) \mathrm{d}x - \int_\Omega \varrho_0 L_k(\varrho_0) \varphi(0, \cdot) \, \mathrm{d}x \tag{273}$$

$$- \int_0^\tau \int_\Omega \overline{\varrho L_k(\varrho)} \Big(\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi \Big) \mathrm{d}x \mathrm{d}t = - \int_0^T \int_\Omega \overline{T_k(\varrho) \mathrm{div} \mathbf{u}} \varphi \, \mathrm{d}x \mathrm{d}t,$$

and

$$\int_\Omega \overline{T_k(\varrho)}(\tau, x) \varphi(\tau, x) \mathrm{d}x - \int_\Omega T_k(\varrho_0) \varphi(0, \cdot) \mathrm{d}x - \int_0^\tau \int_\Omega \overline{T_k(\varrho)} \Big(\partial_t \varphi + \mathbf{u} \cdot \nabla_x \varphi \Big) \, \mathrm{d}x \mathrm{d}t$$
$$= - \int_0^\tau \int_\Omega \overline{(\varrho T_k'(\varrho) - T_k(\varrho)) \mathrm{div} \mathbf{u}} \, \varphi \, \mathrm{d}x \mathrm{d}t, \tag{274}$$

where $\tau \in [0, T]$ and $\varphi \in C_c^1([0, T] \times \overline{\Omega})$ and functions T_k, L_k are defined in (250).

8.4 Effective Viscous Flux Identity

The quantity

$$p(\varrho, \vartheta) - \Big(\frac{4}{3} \mu + \eta \Big) \mathrm{div}_x \mathbf{u}$$

called *effective viscous flux* or *effective pressure* satisfies a certain weak continuity property discovered by P.L. Lions [77] in the context of barotropic model. This property, in our situation, is formulated in the following lemma.

Lemma 10 (See [30, Proposition 6.1]). *Let* $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ *be the trio investigated in Lemma 9. Then for any* $k > 1$, *there holds*

$$\Big(\frac{4}{3} \mu + \eta \Big) \Big(\overline{T_k(\varrho) \mathrm{div}_x \mathbf{u}} - \overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} \Big) = \Big(\overline{p(\varrho, \vartheta) T_k(\varrho)} - \overline{p(\varrho, \vartheta)} \, \overline{T_k(\varrho)} \Big), \tag{275}$$

with functions T_k *defined in* (250).

In order to get the statement of Lemma 10, we proceed in several steps.
Step 1.
First, we multiply the momentum equation (243) by test function

$$\psi(t) \zeta(x) \nabla_x \Delta^{-1} [\tilde{\zeta} T_k(\varrho_n)], \text{ where } \psi \in C_c^1(0, T), \zeta, \tilde{\zeta} \in C_c^\infty(\Omega)$$

and integrate over the space-time cylinder Q_T. The (pseudodifferential) operator $\nabla_x \Delta^{-1}$ is defined in (111). Employing notation introduced in (111) and in Theorem 13, we get identity

$$\int_0^T \psi \int_\Omega \zeta p(\varrho_n, \vartheta_n) T_k(\varrho_n) \, \mathrm{d}x\mathrm{d}t - \int_0^T \psi \int_\Omega \zeta \mathbb{S}(\nabla_x \mathbf{u}_n) : \mathcal{R}[\tilde{\zeta} T_k(\varrho_n)] \, \mathrm{d}x\mathrm{d}t = \sum_{i=1}^7 I_n^i + J_n, \tag{276}$$

where

$$J_n = \int_0^T \psi \int_\Omega \mathbf{u}_n \cdot \Big(\tilde{\zeta} T_k(\varrho_n) \mathcal{R}[\zeta \varrho_n \mathbf{u}_n] - \zeta \varrho_n \mathbf{u}_n \cdot \mathcal{R}[\tilde{\zeta} T_k(\varrho_n)] \Big) \, \mathrm{d}x\mathrm{d}t,$$

and

$$I_n^1 = \int_0^T \psi \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \Big(\nabla_x \zeta \otimes \mathcal{A}[\tilde{\zeta} T_k(\varrho_n)] \Big) \, \mathrm{d}x\mathrm{d}t,$$

$$I_n^2 = -\int_0^T \psi \int_\Omega p(\varrho_n, \vartheta_n) \nabla_x \zeta \cdot \mathcal{A}[\tilde{\zeta} T_k(\varrho_n)] \, \mathrm{d}x\mathrm{d}t,$$

$$I_n^3 = -\int_0^T \psi \int_\Omega \zeta \varrho_n \mathbf{f} \cdot \mathcal{A}[\tilde{\zeta} T_k(\varrho_n)] \, \mathrm{d}x\mathrm{d}t,$$

$$I_n^4 = -\int_0^T \psi \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \Big(\nabla_x \zeta \otimes \mathcal{A}[\tilde{\zeta} T_k(\varrho_n)] \Big) \, \mathrm{d}x\mathrm{d}t,$$

$$I_n^5 = -\int_0^T \psi \int_\Omega \zeta \varrho_n \mathbf{u}_n \cdot \mathcal{A}[T_k(\varrho_n) \nabla_x \tilde{\zeta} \cdot \mathbf{u}_n] \, \mathrm{d}x\mathrm{d}t,$$

$$I_n^6 = -\int_0^T \psi' \int_\Omega \zeta \varrho_n \mathbf{u}_n \cdot \mathcal{A}[\tilde{\zeta} T_k(\varrho_n)] \, \mathrm{d}x\mathrm{d}t,$$

$$I_n^7 = \int_0^T \psi \int_\Omega \zeta \varrho_n \mathbf{u}_n \cdot \mathcal{A}[\tilde{\zeta}(\varrho_n T_k'(\varrho_n) - \varrho_n)] \, \mathrm{d}x\mathrm{d}t.$$

When deriving identity (276), we have used several times integration by parts, renormalized continuity equation (248), and *item* (iii) in Theorem 13.
Step 2.

Employing in the limiting momentum equation (272) test function

$$\psi(t)\zeta(x)\nabla_x \Delta^{-1}[\tilde{\zeta} \; \overline{T_k(\varrho)}], \text{ where } \psi \in C_c^1(0,T), \zeta, \tilde{\zeta} \in C_c^\infty(\Omega),$$

we get identity

$$\int_0^T \psi \int_\Omega \zeta \overline{p(\varrho, \vartheta)} \; \overline{T_k(\varrho)} \, \mathrm{d}x\mathrm{d}t - \int_0^T \psi \int_\Omega \zeta \mathbb{S}(\nabla_x \mathbf{u}) : \mathcal{R}[\tilde{\zeta} \; \overline{T_k(\varrho)}] \, \mathrm{d}x\mathrm{d}t = \sum_{i=1}^7 I^i + J, \tag{277}$$

where

$$J = \int_0^T \psi \int_\Omega \mathbf{u}_n \cdot \Big(\tilde{\zeta}\ \overline{T_k(\varrho)}\mathcal{R}[\zeta\varrho\mathbf{u}] - \zeta\varrho\mathbf{u}\cdot\mathcal{R}[\tilde{\zeta}\ \overline{T_k(\varrho)}]\Big)\,\mathrm{d}x\mathrm{d}t,$$

and

$$I^1 = \int_0^T \psi \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \Big(\nabla_x\zeta \otimes \mathcal{A}[\tilde{\zeta}\ \overline{T_k(\varrho)}]\Big)\,\mathrm{d}x\mathrm{d}t,$$

$$I^2 = -\int_0^T \psi \int_\Omega \overline{p(\varrho,\vartheta)}\nabla_x\zeta\cdot\mathcal{A}[\tilde{\zeta}\ \overline{T_k(\varrho)}]\,\mathrm{d}x\mathrm{d}t,$$

$$I^3 = -\int_0^T \psi \int_\Omega \zeta\varrho\mathbf{f}\cdot\mathcal{A}[\tilde{\zeta}\ \overline{T_k(\varrho)}]\,\mathrm{d}x\mathrm{d}t,$$

$$I^4 = -\int_0^T \psi \int_\Omega \varrho\mathbf{u}\otimes\mathbf{u} : \Big(\nabla_x\zeta \otimes \mathcal{A}[\tilde{\zeta}\ \overline{T_k(\varrho)}]\Big)\,\mathrm{d}x\mathrm{d}t,$$

$$I^5 = -\int_0^T \psi \int_\Omega \zeta\varrho\mathbf{u}\cdot\mathcal{A}[\overline{T_k(\varrho)}\nabla_x\tilde{\zeta}\cdot\mathbf{u}]\,\mathrm{d}x\mathrm{d}t,$$

$$I^6 = -\int_0^T \psi' \int_\Omega \zeta\varrho\mathbf{u}\cdot\mathcal{A}[\tilde{\zeta}\ \overline{T_k(\varrho)}]\,\mathrm{d}x\mathrm{d}t,$$

$$I^7 = \int_0^T \psi \int_\Omega \zeta\varrho\mathbf{u}\cdot\mathcal{A}[\tilde{\zeta}\ \overline{(\varrho T_k'(\varrho) - \varrho)}]\,\mathrm{d}x\mathrm{d}t.$$

When deriving identity (277), we have used several times integration by parts, renormalized continuity equation (274), and *item* (iii) in Theorem 13.

Step 3.

In view of estimates and induced convergence relations established in previous two sections together with the continuity properties of operators $\mathcal{A}$ and $\mathcal{R}$ reported in first two items of Theorem 13, it is a relatively easy task to verify that

$$I_n^i \to I^i \ (\text{as } n \to \infty). \tag{278}$$

Step 4.

Now, we shall establish relation

$$J^n \to J \ (\text{as } n \to \infty). \tag{279}$$

This relation is the key point in the proof of Lemma 10. In fact, this property does not follow by employing the "standard" compactness argument. Instead, we must use the compensated compactness. Indeed, combining the commutator lemma reported in Theorem 15 with the convergence established in the first two lines of (269), we get

$$\Big(\tilde{\zeta}\, T_k(\varrho_n)\mathcal{R}[\zeta\varrho_n\mathbf{u}_n] - \zeta\varrho_n\mathbf{u}_n\cdot\mathcal{R}[\tilde{\zeta}T_k(\varrho_n)]\Big)(t) \rightharpoonup \Big(\tilde{\zeta}\ \overline{T_k(\varrho)}\mathcal{R}[\zeta\varrho\mathbf{u}] - \zeta\varrho\mathbf{u}\cdot\mathcal{R}[\tilde{\zeta}\ \overline{T_k(\varrho)}]\Big)(t)$$

(weakly) in $L^r(\Omega;\mathbb{R}^3)$ with some $r > 6/5$ (in fact $r = \frac{2\gamma}{\gamma+1}$) for all $t \in [0,T]$. The compact imbedding $L^r(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$ now yields that the above convergence is strong in $W^{-1,2}(\Omega)$ for every $t \in [0,T]$. We use this fact together with the weak convergence of the sequence $\mathbf{u}_n$ established in (267) and the Lebsegue dominated convergence theorem used over $(0,T)$ in order to conclude

$$\int_0^T \psi \int_\Omega \mathbf{u}_n\cdot\Big(\tilde{\zeta}T_k(\varrho_n)\mathcal{R}[\zeta\varrho_n\mathbf{u}_n] - \zeta\varrho_n\mathbf{u}_n\cdot\mathcal{R}[\tilde{\zeta}T_k(\varrho_n)]\Big)\,\mathrm{d}x$$

$$\to \int_0^T \psi \int_\Omega \mathbf{u}\cdot\Big(\tilde{\zeta}\ \overline{T_k(\varrho)}\mathcal{R}[\zeta\varrho\mathbf{u}] - \zeta\varrho\mathbf{u}\cdot\mathcal{R}[\tilde{\zeta}\ \overline{T_k(\varrho)}]\Big)\,\mathrm{d}x.$$

This is exactly statement (279).
Step 5.

Integrating twice by parts and employing the property of the Riesz operator listed in *item* (iii) of Theorem 13, we get identities

$$\lim_{n\to\infty}\int_0^T\psi\int_\Omega\zeta\mathbb{S}(\nabla_x\mathbf{u}^n):\mathcal{R}[\tilde{\zeta}T_k(\varrho_n)]\,\mathrm{d}x\mathrm{d}t = \Big(\frac{4}{3}\mu+\eta\Big)\int_0^T\psi\int_\Omega\zeta\tilde{\zeta}\ \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}}\,\mathrm{d}x\mathrm{d}t,$$
$$\int_0^T\psi\int_\Omega\zeta\mathbb{S}(\nabla_x\mathbf{u}):\mathcal{R}[\tilde{\zeta}\ \overline{T_k(\varrho)}]\,\mathrm{d}x\mathrm{d}t = \Big(\frac{4}{3}\mu+\eta\Big)\int_0^T\psi\int_\Omega\zeta\tilde{\zeta}\ \overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u}\,\mathrm{d}x\mathrm{d}t. \tag{280}$$

Step 6.

At the point of conclusion, we perform $\lim_{n\to\infty}$ in the identity (276) and subtract from its identity (277). We obtain the statement of Lemma 10 in view of (278)–(280).

8.5 Oscillations Defect Measure

Let ϱ_n be a sequence and ϱ its weak limit in $L^1(Q_T)$. We introduce the oscillations defect measure of the sequence ϱ_n,

$$\mathrm{osc}_p[\varrho_n \rightharpoonup \varrho](Q_T) \equiv \sup_{k\geq 1}\Big(\limsup_{n\to\infty}\int_{Q_T}\Big|T_k(\varrho_n)-T_k(\varrho)\Big|^p\mathrm{d}x\mathrm{d}t\Big),\quad p\geq 1, \tag{281}$$

where function T_k is defined in (250).

The main achievement of the present section is the following lemma.

Lemma 11 (see [Proposition 6.2][30]). *Let* $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ *be the trio investigated in Lemma 9. Then*

$$\mathrm{osc}_{\gamma+1}[\varrho_n \rightharpoonup \varrho](Q_T) < \infty. \tag{282}$$

Step 1.

In view of (225),

$$p_{\rm el}(\varrho) = \frac{a_2}{\gamma}\varrho^\gamma + p_m(\varrho) + p_b(\varrho), \tag{283}$$

with $\varrho \mapsto p_m(\varrho) = p_{\rm el}(\varrho) - \frac{a_2}{2\gamma}\varrho^\gamma + b\min\{\underline{\mathfrak{r}}, \varrho\}$ is nondecreasing function on $[0,\infty)$ and $p_b(\varrho) = -\min\{\underline{\mathfrak{r}}, \varrho\}$, where $a_2\underline{\mathfrak{r}}^{\gamma-1} = 2b$. With this decomposition and with relation (270) at hand, effective viscous flux identity (275) implies

$$\begin{aligned}\frac{a_2}{\gamma}\int_0^T\int_\Omega \Big(\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)}\Big){\rm d}x{\rm d}t + \int_0^T\int_\Omega \Big(\overline{p_m(\varrho)T_k(\varrho)} - \overline{p_m(\varrho)}\,\overline{T_k(\varrho)}\Big)\,{\rm d}x{\rm d}t +\\ + \int_0^T\int_\Omega \tilde\vartheta\Big(\overline{p_{\rm th}(\varrho)T_k(\varrho)} - \overline{p_{\rm th}(\varrho)}\,\overline{T_k(\varrho)}\Big)\,{\rm d}x{\rm d}t = \limsup_{n\to\infty}\sum_{i=1}^3 I_n^i,\end{aligned} \tag{284}$$

where

$$I_n^1 = \Big(\frac{4}{3}\mu + \eta\Big)\int_0^T\int_\Omega \Big(T_k(\varrho_n) - T_k(\varrho)\Big){\rm div}_x\mathbf{u}_n\,{\rm d}x{\rm d}t,$$

$$I_n^2 = \Big(\frac{4}{3}\mu + \eta\Big)\int_0^T\int_\Omega \Big(T_k(\varrho) - \overline{T_k(\varrho)}\Big){\rm div}_x\mathbf{u}_n\,{\rm d}x{\rm d}t,$$

$$I_n^3 = -\int_0^T\int_\Omega \Big(p_b(\varrho_n)T_k(\varrho_n) - p_b(\varrho_n)\overline{T_k(\varrho)}\Big)\,{\rm d}x{\rm d}t.$$

Step 2.

By Hölder's inequality, lower weak semi-continuity of Lebesgue norms and interpolation

$$\begin{aligned}|I_n^1| + |I_n^2| &\le 2\limsup_{n\to\infty}\|{\rm div}_x\mathbf{u}_n\|_{L^2(Q_T)}\|T_k(\varrho_n) - T_k(\varrho)\|_{L^1(Q_T)}^{\frac{\gamma-1}{2\gamma}}\|T_k(\varrho_n) - T_k(\varrho)\|_{L^{\gamma+1}(Q_T)}^{\frac{\gamma+1}{2\gamma}}\\ &\le c(M_0, E_0, \underline{S}, F_0, T)\Big[{\rm osc}_{\gamma+1}[\varrho_n \rightharpoonup \varrho](Q_T)\Big]^{\frac{\gamma+1}{2\gamma}},\end{aligned}$$

where we have used bounds (251), (260). Similarly,

$$|I_n^3| \le c(M_0, E_0, \underline{S}, F_0, T).$$

Step 3.

We write

$$\begin{aligned}\int_0^T\int_\Omega \Big(\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\,\overline{T_k(\varrho)}\Big){\rm d}x{\rm d}t &= \limsup_{n\to\infty}\int_0^T\int_\Omega \Big(\varrho_n^\gamma - \varrho^\gamma\Big)\Big(T_k(\varrho_n) - T_k(\varrho)\Big)\,{\rm d}x{\rm d}t\\ &+ \int_0^T\int_\Omega \Big(\varrho^\gamma - \overline{\varrho^\gamma}\Big)\Big(\overline{T_k(\varrho)} - T_k(\varrho)\Big)\,{\rm d}x{\rm d}t.\end{aligned} \tag{285}$$

Since $\varrho \mapsto \varrho^\gamma$ is convex on $[0,\infty)$ and $\varrho \mapsto T_k(\varrho)$ is concave, the second integral at the right-hand side of (285) is nonnegative by virtue of Theorem 3. Next, by using the definition (250) of functions T_k and elementary properties of function $\varrho \mapsto \varrho^\gamma$, we easily verify algebraic relations

$$|a-b|^\gamma \le |a^\gamma - b^\gamma| \text{ and } |a-b| \ge |T_k(a) - T_k(b)|, \quad (a,b) \in [0,\infty)^2.$$

Consequently, formula (285) yields

$$\int_0^T \int_\Omega \Big(\overline{\varrho^\gamma T_k(\varrho)} - \overline{\varrho^\gamma}\ \overline{T_k(\varrho)}\Big) \mathrm{d}x\mathrm{d}t \ge \limsup_{n\to\infty} \int_0^T \int_\Omega \Big| T_k(\varrho_n) - T_k(\varrho) \Big|^{\gamma+1} \mathrm{d}x\mathrm{d}t. \tag{286}$$

Step 4.

Finally, according to Theorem 4, the second and third terms at the left-hand side of relation (284) are nonnegative. Coming back with this information, with relation (286) and with all estimates established in Step 3 to relations (284), we deduce inequality

$$\limsup_{n\to\infty} \int_0^T \int_\Omega \Big| T_k(\varrho_n) - T_k(\varrho) \Big|^{\gamma+1} \mathrm{d}x\mathrm{d}t$$
$$\le c(M_0, E_0, \underline{S}, F_0, T)\Big(1 + \Big[\mathrm{osc}_{\gamma+1}[\varrho_n \rightharpoonup \varrho](Q_T)\Big]^{\frac{\gamma+1}{2\gamma}}\Big).$$

The latter formula yields the statement of Lemma 11.

8.6 Renormalized Continuity Equation

Relation (282) implies that the limit quantities ϱ, $\mathbf{u}$ satisfy the renormalized continuity equation. The exact statement reads:

Lemma 12 (see [30, Proposition 6.3] and [33, Lemma 3.8]). *Let*

$$\begin{aligned} \varrho_n &\rightharpoonup \varrho && \text{in } L^p((0,T)\times\mathbb{R}^3), \quad p>1, \\ \mathbf{u}_n &\rightharpoonup \mathbf{u} && \text{in } L^r((0,T)\times\mathbb{R}^3;\mathbb{R}^3), \\ \nabla\mathbf{u}_n &\rightharpoonup \nabla\mathbf{u} && \text{in } L^r((0,T)\times\mathbb{R}^3;\mathbb{R}^9), \quad r>1. \end{aligned}$$

Let

$$osc_q[\varrho_n \rightharpoonup \varrho]((0,T)\times\mathbb{R}^3) < \infty \tag{287}$$

for $\frac{1}{q} < 1 - \frac{1}{r}$, where $(\varrho_n, \mathbf{u}_n)$ solve the renormalized continuity equation (247) (with any b belonging to (118)). Then the limit functions ϱ, $\mathbf{u}$ solve the renormalized continuity equation

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + \Big(\varrho b'(\varrho) - b(\varrho)\Big)\mathrm{div}_x u = 0 \ \ in\ \mathcal{D}'((0,T)\times\mathbb{R}^3) \tag{288}$$

for any b belonging to the same class (118).

We shall outline the proof of this lemma in several steps.
Step 1.
Passing to the limit in (248), we get

$$\partial_t \overline{T_k(\varrho)} + \mathrm{div}_x\Big(\overline{T_k(\varrho)}\mathbf{u}\Big) = -\overline{(\varrho T_k'(\varrho) - \varrho)\mathrm{div}_x\mathbf{u}} \ \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3).$$

Since for fixed $k > 0$, $\overline{T_k(\varrho)} \in L^\infty((0,T)\times\mathbb{R}^3)$, we can employ Theorem 18 in order to infer that

$$\begin{aligned}\partial_t b_M(\overline{T_k(\varrho)}) + \mathrm{div}_x\Big(b_M(\overline{T_k(\varrho)})\mathbf{u}\Big) + \Big(\overline{T_k(\varrho)}b_M'(\overline{T_k(\varrho)}) - b_M(\overline{T_k(\varrho)})\Big)\mathrm{div}_x\mathbf{u}\\ = -\overline{(\varrho T_k'(\varrho) - \varrho)\mathrm{div}_x\mathbf{u}}\, b_M'(\overline{T_k(\varrho)}) \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3)\end{aligned} \tag{289}$$

holds with any b_M in class (118) with compact support in $[0,M)$.
Step 2.
Seeing that by lower weak semi-continuity of L^1 norms,

$$\overline{T_k(\varrho)} \to \varrho \text{ in } L^1((0,T)\times\mathbb{R}^3) \text{ as } k\to\infty,$$

we obtain from equation (289) by using the Lebesgue dominated convergence theorem

$$\partial_t b_M(\varrho) + \mathrm{div}_x\Big(b_M(\varrho)\mathbf{u}\Big) + \Big(\varrho b_M'(\varrho) - b(\varrho)\Big)\mathrm{div}_x\mathbf{u} = 0 \ \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^3), \tag{290}$$

provided we show that

$$\Big\|\overline{(\varrho T_k'(\varrho) - \varrho)\mathrm{div}_x\mathbf{u})}b_M'(\overline{T_k(\varrho)})\Big\|_{L^1((0,T)\times\mathbb{R}^3)} \to 0 \ \text{ as } k\to\infty. \tag{291}$$

To show the latter relation, we use lower weak semi-continuity of L^1 norm, Hölder's inequality, uniform bound of $\mathbf{u}_n$ in $L^r(0,T;W^{1,r}(\mathbb{R}^3))$, and interpolation of $L^{r'}$ between Lebesgue spaces L^1 and L^q to get

$$\begin{aligned}&\Big\|\overline{(\varrho T_k'(\varrho) - \varrho)\mathrm{div}_x\mathbf{u})}b_M'(\overline{T_k(\varrho)})\Big\|_{L^1((0,T)\times\mathbb{R}^3)}\\ &\le \max_{z\in[0,M]}|b_M'(z)|\int_{\{\overline{T_k(\varrho)}\le M\}}|\overline{(\varrho T_k'(\varrho) - \varrho)\mathrm{div}_x\mathbf{u})}|\mathrm{d}x\mathrm{d}t\\ &\le c\ \sup_{n>0}\|\varrho_n T_k'(\varrho_n) - \varrho_n)\|_{L^1((0,T)\times\mathbb{R}^3)}^{\frac{q(r-1)-r}{r(q-1)}}\ \liminf_{n\to\infty}\|\varrho_n T_k'(\varrho_n) - \varrho_n)\|_{L^q(\{\overline{T_k(\varrho)}\le M\})}^{\frac{q}{r(q-1)}}.\end{aligned}$$

We have

$$\|\varrho_n T_k'(\varrho_n) - \varrho_n)\|_{L^1((0,T)\times\mathbb{R}^3)} \leq 2\mathrm{sup}_{n>0}\|\varrho_n\|_{L^1(\{\varrho_n\geq k\})} \to 0 \text{ as } k \to \infty$$

by virtue of the uniform bound of ϱ_n in $L^p((0,T)\times\mathbb{R}^3)$ (in the above we have also used algebraic relation $zT_k'(z) - T_k(z) \leq 2z1_{\{z\geq k\}}$), while

$$\|\varrho_n T_k'(\varrho_n) - \varrho_n)\|_{L^q(\{\overline{T_k(\varrho)}\leq M\})} \leq 2\|T_k(\varrho_n)\|_{L^1(\{\overline{T_k(\varrho)}\leq M\})}$$

$$\leq 2\Big(\|T_k(\varrho_n) - T_k(\varrho)\|_{L^q((0,T)\times\mathbb{R}^3)} + \|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^q((0,T)\times\mathbb{R}^3)} + \|\overline{T_k(\varrho)}\|_{L^q(\{\overline{T_k(\varrho)}\leq M\})}\Big),$$

where we have used algebraic relation $zT_k'(z) \leq 2T_k(z)$ and the Minkowski inequality. Since the latter expression remains bounded, relation (291) is proved. We have thus shown (290). Equation (290) with $b = b_M$ however implies (288) with any b in class (118) by virtue of the Lebesgue dominated convergence theorem. Lemma 12 is proved.

8.7 Strong Convergence of the Density Sequence

We deduce from (247) using Lemma 12 with $r = 2$, $p = \gamma$, $q = \gamma + 1$ that

$$\int_\Omega \varrho L_k(\varrho)(\tau,x)\varphi(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_0 L_k(\varrho_0)\varphi(0,\cdot)\,\mathrm{d}x - \int_0^\tau\int_\Omega \varrho L_k(\varrho)\Big(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\mathrm{d}t$$
$$= -\int_0^\tau\int_\Omega T_k(\varrho)\mathrm{div}\mathbf{u}\varphi\,\mathrm{d}x\mathrm{d}t, \tag{292}$$

with any $\tau \in [0,T]$ and $\varphi \in C_c^1([0,T]\times\overline{\Omega})$, where $L_k(\varrho)$ is defined in (250).

Next, we write (273) and (292) with test function $\varphi = 1$ and deduce

$$\int_\Omega \Big(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\Big)(\tau)\,\mathrm{d}x = -\int_0^\tau\int_\Omega g_k\,\mathrm{d}x\mathrm{d}t, \text{ where } g_k = \overline{T_k(\varrho)\mathrm{div}\mathbf{u}} - T_k(\varrho)\mathrm{div}\mathbf{u}. \tag{293}$$

We evaluate function g_k by using the effective viscous flux identity (275) with the decomposition of elastic pressure

$$p_{\mathrm{el}} = \mathfrak{p}_m(\varrho) - \mathfrak{p}_b(\varrho),\ \mathfrak{p}_b \in C_c^2[0,\infty),\ \mathfrak{p}_b \geq 0,$$

where $\mathfrak{p}_b(z) = 0$ whenever $z > \overline{\mathfrak{r}}$ with some $\overline{\mathfrak{r}} > 0$, and $\mathfrak{p}_m$ is an increasing function on $[0,\infty)$:

$$g_k = g_k^1 + g_k^2 + g_k^3, \quad g_k^1 = \Big(\overline{T_k(\varrho)}\mathrm{div}\mathbf{u} - T_k(\varrho)\mathrm{div}\mathbf{u}\Big),$$

$$g_k^2 = \frac{1}{\frac{4}{3}\mu + \eta}\Big[\Big(\overline{\mathfrak{p}_m(\varrho)T_k(\varrho)} - \overline{\mathfrak{p}_m(\varrho)}\ \overline{T_k(\varrho)}\Big) + \tilde{\vartheta}\Big(\overline{p_{\mathrm{th}}(\varrho)T_k(\varrho)} - \overline{p_{\mathrm{th}}(\varrho)}\ \overline{T_k(\varrho)}\Big)\Big],$$

$$g_k^3 = \frac{1}{\frac{4}{3}\mu + \eta}\Big(\overline{\mathfrak{p}_b(\varrho)T_k(\varrho)} - \overline{\mathfrak{p}_b(\varrho)}\ \overline{T_k(\varrho)}\Big).$$

Writing

$$\Big| \int_0^T \int_\Omega \left(\overline{T_k(\varrho)}\mathrm{div}\mathbf{u} - T_k(\varrho)\mathrm{div}\mathbf{u} \right) \, \mathrm{d}x\mathrm{d}t \Big| \le \| \overline{T_k(\varrho)} - T_k(\varrho) \|_{L^2(Q_T)} \| \mathrm{div}\mathbf{u} \|_{L^2(Q_T)}, \tag{294}$$

and realizing that

$$\| \overline{T_k(\varrho)} - T_k(\varrho) \|_{L^1(Q_T)} \le \| \overline{T_k(\varrho)} - \varrho \|_{L^1(Q_T)} + \| \varrho - T_k(\varrho) \|_{L^1(Q_T)} \to 0 \text{ as } k \to \infty, \tag{295}$$

we may use interpolation of L^2 between L^1 and $L^{\gamma+1}$ together with the boundedness of the oscillations defect measure established in Lemma 282 to show

$$\int_{Q_T} g_k^1 \, \mathrm{d}x\mathrm{d}t \to 0 \text{ as } k \to \infty. \tag{296}$$

On the other hand, by virtue of Theorem 4,

$$g_k^2 \ge 0. \tag{297}$$

Finally, we observe that there is $\Lambda = \Lambda(\mathfrak{p}_b) > 0$ such that

$$\varrho \mapsto \Lambda \varrho \log \varrho - \varrho \mathfrak{p}_b(\varrho) \text{ and } \varrho \mapsto \Lambda \varrho \log \varrho + \mathfrak{p}_b(\varrho) \tag{298}$$

are convex functions on $[0, \infty)$. We have, by employing several times Theorem 2,

$$\left(\frac{4}{3}\mu + \eta \right) \lim_{k\to\infty} \int_0^\tau \int_\Omega g_k^3 \, \mathrm{d}x\mathrm{d}t$$

$$= \lim_{k\to\infty} \int_0^\tau \int_\Omega \left(\overline{\mathfrak{p}_b(\varrho) T_k(\varrho)} - \overline{\mathfrak{p}_b(\varrho)} \, \overline{T_k(\varrho)} \right) \mathrm{d}x\mathrm{d}t = \lim_{k\to\infty} \int_0^\tau \int_\Omega \left(\overline{\mathfrak{p}_b(\varrho)\varrho} - \overline{\mathfrak{p}_b(\varrho)} \, \varrho \right) \mathrm{d}x\mathrm{d}t$$

$$\le \lim_{k\to\infty} \left[\Lambda \int_0^\tau \int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right) \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \left(\mathfrak{p}_b(\varrho) - \overline{\mathfrak{p}_b(\varrho)} \right) \varrho \, \mathrm{d}x\mathrm{d}t \right.$$

$$\le \lim_{k\to\infty} \left[\Lambda \int_0^\tau \int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right) \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\varrho < \bar{\mathfrak{r}}} \left(\mathfrak{p}_b(\varrho) - \overline{\mathfrak{p}_b(\varrho)} \right) \varrho \, \mathrm{d}x\mathrm{d}t \right]$$

$$\le (1 + \bar{\mathfrak{r}}) \Lambda \int_0^\tau \int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right) \mathrm{d}x\mathrm{d}t,$$

where we have used relation (295) in the first line, convexity of $\varrho \mapsto \Lambda \varrho \log \varrho - \varrho \mathfrak{p}_b(\varrho)$ in the second line, convexity of $\varrho \mapsto \Lambda \varrho \log \varrho + \mathfrak{p}_b(\varrho)$ in the last line, and the fact that $\mathfrak{p}_b$ is nonnegative and vanishing at large arguments in the third line. Coming with this information back to (293), we infer

$$\int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right)(\tau) \, \mathrm{d}x \le c \int_0^\tau \int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right)(\tau) \, \mathrm{d}x\mathrm{d}t \tag{299}$$

with some $c > 0$. Now the Gronwall lemma (cf. Theorem 21) says that necessarily

$$\int_\Omega \left(\overline{\varrho\log\varrho} - \varrho\log\varrho\right)(\tau)\,\mathrm{d}x \le 0.$$

Finally, since the function $z \to z\log z$ is strictly convex on $(0,\infty)$, we have

$$\overline{\varrho\log\varrho} - \varrho\log\varrho = 0 \text{ a. e. in } (0,T)\times\Omega \tag{300}$$

and

$$\varrho_n \to \varrho \text{ a.e. in } (0,T)\times\Omega \tag{301}$$

according to Theorem 3. With relation (301) at hand, we easily establish that

$$\overline{p_{\mathrm{el}}(\varrho)} = p_{\mathrm{el}}(\varrho),\ \overline{p_{\mathrm{th}}(\varrho)} = p_{\mathrm{th}}(\varrho),\ \overline{b(\varrho)} = b(\varrho),\ \overline{B(\varrho)} = B(\varrho), \tag{302}$$

where b, B are defined in (134).

The reader will notice that in the case of elastic pressure, one can deduce (301) immediately after (297). The analysis between formulas (298) and (299) is needed in order to accommodate the locally compactly non monotone elastic pressure. At this place the analysis hits the limits of the Lions-Feireisl method. The reader can consult [30, Section 6.6] for more details and proofs.

8.8 Limit in the Thermal Energy Equation

Step 1: Strong convergence of the temperature outside vacua

By virtue of (259),

$$e_{\mathrm{th}}(\vartheta_n) \rightharpoonup \overline{e_{\mathrm{th}}(\vartheta)} \ \text{ in } L^2(0,T;W^{1,2}(\Omega)), \tag{303}$$

consequently,

$$\varrho_n e_{\mathrm{th}}(\vartheta_n) \rightharpoonup \varrho\overline{e_{\mathrm{th}}(\vartheta)} \ \text{ in } L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega)), \tag{304}$$

where we have used the strong convergence of ϱ_n in $L^2(0,T;W^{-1,2}(\Omega))$ established in (269). Next, we evaluate the time derivative $\partial_t(\varrho_n e_{\mathrm{th}}(\vartheta_n))$ from the thermal energy equation (244) in order to be able to employ Feiresl's version of Lions-Aubin theorem (see Theorem 12), and we establish

$$\varrho_n e_{\mathrm{th}}(\vartheta_n) \to \varrho\overline{e_{\mathrm{th}}(\vartheta)} \text{ (strongly) in } L^2(0,T;W^{-1,2}(\Omega)). \tag{305}$$

The latter convergence in combination with (303) yields

$$\varrho_n(e_{\mathrm{th}}(\vartheta_n))^2 \rightharpoonup \varrho\left[\overline{e_{\mathrm{th}}(\vartheta)}\right]^2 \ \text{ in } L^1(Q_T). \tag{306}$$

Writing

$$\int_{\Omega} \varrho (e_{\rm th}(\vartheta_n))^2 \, {\rm d}x = \int_{\Omega} (\varrho - \varrho_n)(e_{\rm th}(\vartheta_n))^2 \, {\rm d}x + \int_{\Omega} \varrho_n (e_{\rm th}(\vartheta_n))^2 \, {\rm d}x,$$

and employing estimate (259) together with (301) and (306), we deduce

$$\int_0^T \int_{\Omega} \varrho \Big(e_{\rm th}(\vartheta_n) - \overline{e_{\rm th}(\varrho)} \Big)^2 \, {\rm d}x{\rm d}t \to 0,$$

which implies

$$e_{\rm th}(\vartheta_n) \to \overline{e_{\rm th}(\vartheta)} \ \text{ a.e. in } \{(t,x) | \varrho(t,x) > 0\}. \tag{307}$$

As function $e_{\rm th}$ admits an inverse function $e_{\rm th}^{-1}$ (since it is increasing according to assumption (227)), we get

$$\vartheta_n \to \tilde{\vartheta} \ \text{ a.e. in } \{(t,x)|\varrho(t,x) > 0\}, \ e_{\rm th}(\tilde{\vartheta}) = \overline{e_{\rm th}(\vartheta)} \ \text{ a.e. in } \{(t,x)|\varrho(t,x) > 0\}, \tag{308}$$

where $\tilde{\vartheta}$ is the weak limit of the sequence ϑ_n established in (267).

Step 2: Renormalized thermal energy equation

The goal is now to pass to the limit in the thermal energy equation (244) and get the thermal energy inequality $(217)_{e_{\rm ra}=0}$. The standard argument to achieve this goal would be to multiply equation (244) by test function φ in class (217) and integrate conveniently by parts before passing to the limit by using the already established convergence. This procedure allows to pass to the limit $n \to \infty$ (letting appear eventually an inequality due to the lower weak semi-continuity of term $\int_0^T \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \varphi \, {\rm d}x{\rm d}t$, $\varphi \geq 0$) in all terms except term $\int_0^T \int_{\Omega} {\rm div}_x \mathbf{q}(\vartheta_n) \varphi \, {\rm d}x{\rm d}t = \int_0^T \int_{\Omega} \mathcal{K}(\vartheta_n) \Delta \varphi \, {\rm d}x{\rm d}t$ with $\mathcal{K}(\vartheta_n)$ being bounded solely in $L^1(Q_T)$. This bound is not enough to guarantee the limit $\overline{\mathcal{K}(\vartheta)}$ to be a function but merely a measure.

To get around this difficulty, we shall first investigate the renormalized version of the thermal energy equation: we multiply equation (244) by functions $h_\omega(\vartheta_n)$, $\omega \in (0,1)$ introduced in (241).

Denoting

$$e_{{\rm th},\omega}(\vartheta) = \int_0^{\vartheta} h_\omega(z) c_v(z) {\rm d}z, \ \mathcal{K}_\omega(\vartheta) = \int_0^{\vartheta} h_\omega(z) \kappa(z) {\rm d}z,$$

and testing by φ vanishing at $t = T$ in class (217), i.e.,

$$\varphi \in C_c^1([0,T); C^2(\overline{\Omega})), \ \nabla_x \varphi \cdot \mathbf{n}|_{(0,T)\times\partial\Omega} = 0, \ \varphi \geq 0, \tag{309}$$

we obtain,

$$\int_0^T\int_\Omega\Big(\varrho_n e_{\text{th},\omega}(\vartheta_n)\partial_t\varphi+\varrho_n e_{\text{th},\omega}(\vartheta_n)\mathbf{u}_n\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega\mathcal{K}_h(\vartheta_n)\Delta\varphi\,\mathrm{d}x\mathrm{d}t \tag{310}$$

$$=\int_0^T\int_\Omega h_\omega(\vartheta_n)\vartheta_n p_{\text{th}}(\varrho_n)\mathrm{div}_x\mathbf{u}_n\varphi\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega h_\omega(\vartheta_n)\mathbb{S}(\nabla_x\mathbf{u}_n):\nabla_x\mathbf{u}_n\varphi\,\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_\Omega h'_\omega(\vartheta_n)\kappa(\vartheta_n)|\nabla_x\vartheta_n|^2\varphi\,\mathrm{d}x\mathrm{d}t-\int_\Omega\varrho_{n,0}e_{\text{th},\omega}(\vartheta_{n,0})\varphi(0,\cdot)\,\mathrm{d}x.$$

We now pass to the limit $n\to\infty$ for fixed ω. Before starting, we observe that the family of functions h_ω, $\omega\in(0,1)$ verifies

$$h_\omega\in C^2[0,\infty),\ h_\omega(0)=1,\ h_\omega\text{ nonincreasing}, \tag{311}$$

$$\lim_{z\to\infty}h_\omega(z)=0,\ h''_\omega(z)\,h_\omega(z)\geq 2(h'_\omega(z))^2\text{ for all }z\geq 0,$$

$$h_\omega\nearrow 1\text{ as }\omega\to 0+.$$

Writing

$$\int_0^T\int_\Omega h_\omega(\vartheta_n)p_{\text{th}}(\varrho_n)\vartheta_n\mathrm{div}_x\mathbf{u}_n\varphi\,\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega\Big(p_{\text{th}}(\varrho_n)-p_{\text{th}}(\varrho)\Big)h_\omega(\vartheta_n)\vartheta_n\mathrm{div}_x\mathbf{u}_n\varphi\,\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_\Omega p_{\text{th}}(\varrho)h_\omega(\vartheta_n)\vartheta_n\mathrm{div}_x\mathbf{u}_n\varphi\,\mathrm{d}x\mathrm{d}t,$$

where $\varphi\in L^\infty(Q_T)$, we deduce from (301), (308), (267), assumptions $(226)_{\Gamma\leq\gamma/3}$, (311), and estimates (252), (259), (266) that

$$h_\omega(\vartheta_n)\vartheta_n p_{\text{th}}(\varrho_n)\mathrm{div}_x\mathbf{u}_n\rightharpoonup h_\omega(\tilde\vartheta)\tilde\vartheta\,p_{\text{th}}(\varrho)\mathrm{div}_x\mathbf{u}\ \text{ in }L^1(Q_T). \tag{312}$$

We proceed in a similar way, to get

$$\varrho_n e_{\text{th},\omega}(\vartheta_n)\rightharpoonup\varrho e_{\text{th},\omega}(\tilde\vartheta),\ \ \varrho_n e_{\text{th},\omega}(\vartheta_n)\mathbf{u}_n\rightharpoonup\varrho e_{\text{th},\omega}(\tilde\vartheta)\mathbf{u}\ \text{ in }L^1(Q_T). \tag{313}$$

Since h_ω verifies (311), the function

$$\mathbb{R}^2\ni(s,z)\mapsto\begin{cases}h_\omega(s)z^2 & \text{if } s\geq 0,\\ \infty & \text{if } s<0,\end{cases}$$

is convex lower semicontinuous; consequently, we deduce from Theorem 2 that

$$\int_0^\tau \int_\Omega h_\omega(\vartheta)\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\varphi \, \mathrm{d}x\mathrm{d}t \le \liminf_{n\to\infty} \int_0^\tau \int_\Omega h_\omega(\vartheta_n)\mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n\varphi \, \mathrm{d}x\mathrm{d}t \tag{314}$$

for any $\varphi \in L^\infty(Q_T)$, $\varphi \ge 0$.

The last term to be treated is the term containing $\mathcal{K}_\omega(\vartheta_n)$: First, according to (266), (212), $(230)_{\alpha\ge 2}$,

$$\|\mathcal{K}_\omega(\vartheta_n)\|_{L^1(Q_T)} \le \|\mathcal{K}(\vartheta_n)\|_{L^1(Q_T)} \le c(M_0, E_0, \underline{S}, F_0, T) \tag{315}$$

uniformly in n and ω. Second, since

$$\lim_{z\to\infty} \frac{\mathcal{K}_\omega(z)}{\mathcal{K}(z)} = 0,$$

we can use Theorem 1 to deduce that

$$\mathcal{K}_\omega(\vartheta_n) \rightharpoonup \overline{\mathcal{K}_\omega(\vartheta)} \text{ in } L^1(Q_T), \tag{316}$$

where by virtue of the almost everywhere convergence established in (308)

$$\overline{\mathcal{K}_\omega(\vartheta)}(t,x) = \mathcal{K}_\omega(\tilde\vartheta(t,x)) \text{ for a.a. } (t,x) \in \{(t,x)|\varrho(t,x) > 0\}. \tag{317}$$

With relations (312)–(316), we are ready to pass to the limit $n \to \infty$ in equation (310) and get

$$\int_0^T \int_\Omega \Big(\varrho e_{\mathrm{th},\omega}(\tilde\vartheta)\partial_t\varphi + \varrho e_{\mathrm{th},\omega}(\tilde\vartheta)\mathbf{u}\cdot\nabla_x\varphi\Big) \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \overline{\mathcal{K}_\omega(\vartheta)}\Delta\varphi \, \mathrm{d}x\mathrm{d}t \tag{318}$$

$$\le \int_0^T\int_\Omega h_\omega(\tilde\vartheta)\vartheta\, p_{\mathrm{th}}(\varrho)\mathrm{div}_x\mathbf{u}\varphi \, \mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega h_\omega(\tilde\vartheta)\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\varphi \, \mathrm{d}x\mathrm{d}t - \int_\Omega \varrho_0 e_{\mathrm{th},\omega}(\vartheta_0)\varphi(0,\cdot) \, \mathrm{d}x$$

with any φ in class (309), where we have used also the fact that $\int_0^\tau \int_\Omega h'_\omega(\vartheta_n)\kappa(\vartheta_n)$ $|\nabla_x\vartheta_n|^2\varphi \, \mathrm{d}x\mathrm{d}t$ has negative sign.

Step 3: Thermal energy inequality

The goal now is to pass to the limit $\omega \to 0$ in (318). As $h_\omega \nearrow 1$, we have

$$\varrho e_{\mathrm{th},\omega}(\tilde\vartheta) \rightharpoonup \varrho e_{\mathrm{th}}(\tilde\vartheta), \quad \varrho e_{\mathrm{th},\omega}(\tilde\vartheta)\mathbf{u} \rightharpoonup \varrho e_{\mathrm{th}}(\tilde\vartheta)\mathbf{u},$$

$$h_\omega(\tilde\vartheta)\tilde\vartheta\, p_{\mathrm{th}}(\varrho)\mathrm{div}_x\mathbf{u} \rightharpoonup \tilde\vartheta\, p_{\mathrm{th}}(\varrho)\mathrm{div}_x\mathbf{u}, \quad h_\omega(\tilde\vartheta)\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} \rightharpoonup \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}$$

weakly in $L^1(Q_T)$ as $\omega \to 0$ by the Lebesgue dominated convergence theorem.

The most difficult term in this limit passage is term $\int_0^T \int_\Omega \overline{\mathcal{K}_\omega(\vartheta)}\Delta\varphi \, \mathrm{d}x\mathrm{d}t$. We observe that sequence $\{\overline{\mathcal{K}_\omega(\vartheta(t,x))}\}_{\omega\to 0}$ is increasing as $\omega \to 0$ and uniformly bounded in $L^1(Q_T)$ by virtue of (266). Consequently, by virtue of the monotone convergence theorem and by (318), there is $\Theta \in L^1(Q_T)$ such that

$$\overline{\mathcal{K}_\omega(\vartheta(t,x))} \nearrow \Theta(t,x) \text{ for a.a. } (t,x) \in Q_T \; as \; \omega \to 0. \tag{319}$$

On the other hand, due to (317) and definition of $\mathcal{K}_\omega$, the value of Θ is directly calculable outside vacua, namely,

$$\Theta(t,x) = \mathcal{K}(\tilde{\vartheta}(t,x)) \text{ for a.a. } (t,x) \in \{(t,x)|\varrho(t,x) > 0\}. \tag{320}$$

At this stage we set

$$\vartheta(t,x) = \mathcal{K}^{-1}\Big(\Theta(t,x)\Big). \tag{321}$$

By virtue of (320)

$$\vartheta(t,x) = \tilde{\vartheta}(t,x) \text{ for a.a. } (t,x) \in \{(t,x)|\varrho(t,x) > 0\}.$$

Passing to $\omega \to 0$ with these observations at hand in (318), we get the required weak formulation (217) of the thermal energy balance.

Step 4: Positivity of temperature and total energy balance

By virtue of Theorem 2,

$$-\ln \overline{\mathcal{K}_\omega(\vartheta)} \leq -\overline{\ln \mathcal{K}_\omega(\vartheta)}. \tag{322}$$

Moreover according to (230) and (241), $\mathcal{K}_\omega(\vartheta_n)$ is equivalent to ϑ_n near 0, and the same is true for $\ln \mathcal{K}_\omega(\vartheta_n)$ and $\ln \vartheta_n$. Therefore, relation (322) together with (257) and (266) implies

$$\| \ln \overline{\mathcal{K}_\omega(\vartheta)} \|_{L^2(Q_T)} \leq c \text{ uniformly with respect to } \omega \in (0,1).$$

Moreover, $\Theta(t,x) \geq \overline{K_\omega(\vartheta)}(t,x)$ for a.a. $(t,x) \in Q_T$ and at the same time $\Theta \in L^1(Q_T)$. Consequently,

$$\ln \Theta \in L^2(Q_T).$$

After the analysis of behavior of $\mathcal{K}^{-1}$ near 0 and near ∞ obtained with the help of $(230)_{\alpha \geq 2}$, we get from (321)

$$\ln \vartheta \in L^2(Q_T), \; \vartheta \in L^{\alpha+1}(Q_T).$$

Finally, we obtain from the total energy balance (246) in the limit its weak formulation (218) by virtue of (313), (301), and (263) and the last line in (269).

The procedure described above is very much related to the notion of biting limits of bounded sequences in L^1. The reader may consult [30, Sections 6.7.2–6.8.2] for more details of the proofs and on these problems.

9 Navier-Stokes-Fourier System in the Entropy Formulation

In this section we shall deal with the Navier-Stokes-Fourier system in the entropy formulation, where the internal energy balance is replaced by the entropy balance:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{323}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) + \varrho \mathbf{f}, \tag{324}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta)}{\vartheta}\right) = \sigma. \tag{325}$$

We recall that the specific entropy s is related to the internal energy e and pressure p by the Gibbs relation

$$\mathrm{d}s = \frac{1}{\vartheta}\left(\mathrm{d}e - \frac{p}{\varrho^2}\mathrm{d}\varrho\right), \tag{326}$$

where the pressure and internal energy obey (34). Entropy production rate σ is given by formula (10); recall

$$\sigma = \frac{1}{\vartheta}\left(\mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta}\right). \tag{327}$$

We consider Newtonian fluids (12) with the heat flux given by Fourier's law (13), specifically,

$$\mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) = \mu(\varrho, \vartheta)\mathbb{T}(\nabla_x \mathbf{u}) + \eta(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \quad \mathbb{T}(\nabla_x \mathbf{u}) = \nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{328}$$

$$\mathbf{q} = -\kappa(\varrho, \vartheta)\nabla_x \vartheta, \tag{329}$$

where μ, η, κ obey (14)–(15). Equations (323)–(327) are supplemented with initial conditions

$$\varrho(0, \cdot) = \varrho_0,\ \varrho \mathbf{u}(0, \cdot) = \varrho_0 \mathbf{u}_0,\ \varrho s(\varrho, \vartheta)(0, \cdot) = \varrho_0 s(\varrho_0, \vartheta_0), \quad \varrho_0 \geq 0,\ \vartheta_0 > 0, \tag{330}$$

and *no-slip* boundary conditions for velocity (20) and zero heat transfer conditions (21) on the boundary, recall

$$\mathbf{q} \cdot \mathbf{n}|_{(0,T)\times\partial\Omega} = 0, \tag{331}$$

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{332}$$

In [33], the authors have introduced a concept of *weak solution* to the Navier-Stokes-Fourier system (323)–(332). This concept postulates, in agreement with the second law of thermodynamics, that the entropy production rate σ is a nonnegative measure,

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta}\right). \tag{333}$$

With this postulate, equation (325) becomes inequality. In order to compensate the loss of information, we may postulate that the total energy of the system in the volume Ω is conserved, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\right)\,\mathrm{d}x = \int_\Omega \varrho \mathbf{f}\cdot \mathbf{u}\,\mathrm{d}x. \tag{334}$$

Putting together equations (325) and (334), we obtain the so-called dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_{\overline{\vartheta}}(\varrho, \vartheta) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})\right)\mathrm{d}x \tag{335}$$

$$+ \int_\Omega \frac{\overline{\vartheta}}{\vartheta}\left(\mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta)\cdot \nabla_x \vartheta}{\vartheta}\right)\mathrm{d}x\mathrm{d}t \leq \int_0^T \int_\Omega \varrho \mathbf{f}\cdot \mathbf{u}\,\mathrm{d}x\mathrm{d}t,$$

where we have taken into account inequality (333) and conservation of mass (323). In this inequality $\overline{\varrho}$ and $\overline{\vartheta}$ are positive constants and $H_{\overline{\vartheta}} = \varrho e - \overline{\vartheta} s$ is the *Helmoholtz function* introduced in (29).

On the other hand, if $(\varrho, \vartheta, \mathbf{u})$ $\varrho > 0$, $\vartheta > 0$ is a trio of smooth functions satisfying (323)–(332), one may derive, at least formally, the so-called *relative energy identity*,

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + E(\varrho, \vartheta | r, \Theta)\right)(\tau, \cdot)\,\mathrm{d}x \tag{336}$$

$$+ \int_0^\tau \int_\Omega \Theta \frac{\mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u})}{\vartheta} : \nabla_x \mathbf{u}\,\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega \Theta \frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta)}{\vartheta^2} : \nabla_x \vartheta\,\mathrm{d}x\mathrm{d}t$$

$$= \int_\Omega \left(\frac{1}{2}\varrho_0 |\mathbf{u}_0 - \mathbf{U}(0, \cdot)|^2 + E(\varrho_0, \vartheta_0 | r(0, \cdot), \Theta(0, \cdot))\right)\,\mathrm{d}x$$

$$+ \int_0^\tau \int_\Omega \mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U}\,\mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega \frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta)}{\vartheta}\cdot \nabla_x \Theta\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \varrho\Big(\partial_t \mathbf{U} + \mathbf{u}\cdot \nabla_x \mathbf{U}\Big)\cdot(\mathbf{U} - \mathbf{u})\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \varrho\Big(s(r, \Theta) - s(\varrho, \vartheta)\Big)\Big(\partial_t \Theta + \mathbf{u}\cdot \nabla_x \Theta\Big)\,\mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \Big[\Big(1 - \frac{\varrho}{r}\Big) \partial_t p(r, \Theta) - \varrho \mathbf{u} \cdot \frac{\nabla_x p(r, \Theta)}{r} \Big] \,\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau \int_\Omega p(\varrho, \vartheta) \mathrm{div}_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \varrho \mathbf{f} \cdot (\mathbf{u} - \mathbf{U}) \,\mathrm{d}x\mathrm{d}t,$$

where we have denoted

$$E(\varrho, \vartheta \mid r, \Theta) = H_\Theta(\varrho, \vartheta) - \partial_\varrho H_\Theta(r, \Theta)(\varrho - r) - H_\Theta(r, \Theta),$$

$$H_\Theta(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \Theta \varrho s(\varrho, \vartheta).$$

In (336), (r, Θ) is a couple of positive sufficiently‘ smooth functions on $[0, T] \times \overline{\Omega}$, and $\mathbf{U}$ is a sufficiently smooth vector field with compact support in $[0, T] \times \Omega$.

Conformably to (333), for a weak solution $(\varrho, \vartheta, \mathbf{u})$, the identity (336) has to be replaced by an inequality with the inequality sign $\leq$. This inequality is usually called the *relative energy inequality*. Notice that the dissipation balance (335) is a particular case of the relative energy inequality, where $r = \overline{\varrho}$, $\Theta = \overline{\vartheta}$, and $\mathbf{U} = 0$.

The material of this section is based on the monograph [33, Chapters 1–3] for the notion of (finite energy) weak solutions and on papers [34] for the notion of relative energy functional and dissipative solutions [70] for the notion of bounded energy weak solutions.

9.1 Definition of Finite Energy Weak Solutions

Definition 8. Let Ω be a bounded domain, and let the initial functions $(\varrho_0, \mathbf{u}_0, \vartheta_0)$ satisfy condition

$$\varrho_0 : \Omega \to [0, +\infty),\ \mathbf{u}_0 : \Omega \to \mathbb{R}^3,\ \vartheta_0 : \Omega \to (0, \infty), \tag{337}$$

where

$$\varrho_0 \mathbf{u}_0 = 0 \text{ and } \varrho_0 \mathbf{u}_0^2 = 0 \text{ a.e. in the set } \{x \in \Omega | \varrho_0(x) = 0\},$$

with finite total energy $E_0 = \int_\Omega (\frac{1}{2}\varrho_0 \mathbf{u}_0^2 + \varrho_0 e(\varrho_0, \vartheta_0)) \,\mathrm{d}x$, finite mass $0 < M_0 = \int_\Omega \varrho_0 \,\mathrm{d}x$, and $\int_\Omega \varrho_0 |s(\varrho_0, \vartheta_0)| \,\mathrm{d}x \equiv S_0 < \infty$.

We shall say that the trio $(\varrho, \vartheta, \mathbf{u})$ is a *finite energy weak solution* to the Navier-Stokes-Fourier system (323)–(332) emanating from the initial data $(\varrho_0, \vartheta_0, \mathbf{u}_0,)$ if:

(a)

$$\varrho, \vartheta \in L^\infty(0, T; L^1(\Omega)),\ \varrho \geq 0,\ \vartheta > 0 \ \text{ a.e. in } (0, T) \times \Omega,\ p(\varrho, \vartheta) \in L^1(Q_T),$$
$$\mathbf{u} \in L^2(0, T; W_0^{1,q}(\Omega));\ \ \varrho \mathbf{u},\ \frac{1}{2}\varrho \mathbf{u}^2,\ \varrho e(\varrho, \vartheta),\ \varrho s(\varrho, \vartheta) \in L^\infty(0, T; L^1(\Omega)),\ q > 1, \tag{338}$$

$$\varrho s(\varrho,\vartheta)\mathbf{u},\ \mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}),\ \frac{1}{\vartheta}\Big(\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big)\in L^1(Q_T);$$

(b) $\varrho\in C_{\rm weak}([0,T];L^1(\Omega))$ and equation (323) is replaced by a family of integral identities

$$\int_\Omega \varrho\varphi\,\mathrm{d}x\Big|_0^\tau=\int_0^\tau\int_\Omega\Big(\varrho\partial_t\varphi+\varrho\mathbf{u}\cdot\nabla_x\varphi\Big)\,\mathrm{d}x\mathrm{d}t \tag{339}$$

for all $\tau\in[0,T]$ and for any $\varphi\in C_c^1([0,T]\times\overline{\Omega})$;

(c) $\varrho\mathbf{u}\in C_{\rm weak}([0,T];L^1(\Omega;\mathbb{R}^3))$ and momentum equation (324) is satisfied in the sense of distributions, specifically

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi\,\mathrm{d}x\Big|_0^\tau= \tag{340}$$

$$\int_0^\tau\int_\Omega\Big(\varrho\mathbf{u}\cdot\partial_t\varphi+\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\varphi+p(\varrho,\vartheta)\mathrm{div}_x\varphi-\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}):\nabla_x\varphi+\varrho\mathbf{f}\cdot\varphi\,\mathrm{d}x\Big)\mathrm{d}t$$

for all $\tau\in[0,T]$ and for any $\varphi\in C_c^1([0,T]\times\Omega;\mathbb{R}^3)$;

(d) the entropy balance (325), (333) is replaced by a family of integral inequalities

$$\begin{aligned}&-\int_\Omega\varrho s(\varrho,\vartheta)\varphi\,\mathrm{d}x\Big|_0^\tau+\int_0^\tau\int_\Omega\frac{\varphi}{\vartheta}\left(\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right)\,\mathrm{d}x\mathrm{d}t\\ &\le-\int_0^\tau\int_\Omega\left(\varrho s(\varrho,\vartheta)\partial_t\varphi+\varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi+\frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\varphi}{\vartheta}\right)\,\mathrm{d}x\mathrm{d}t\end{aligned} \tag{341}$$

for a.a. $\tau\in(0,T)$ and for any $\varphi\in C^1([0,T]\times\overline{\Omega})$, $\varphi\ge0$;

(e) the balance of total energy (334) in the volume Ω is verified in the weak sense

$$\begin{aligned}-\int_0^T\int_\Omega\left(\frac12\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)\right)\psi'(t)\,\mathrm{d}x\mathrm{d}t&=\int_\Omega\psi(0)\left(\frac12\varrho_0|\mathbf{u}_0|^2+\varrho_0e(\varrho_0,\vartheta_0)\right)\,\mathrm{d}x\\ &+\int_0^T\psi(t)\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t\quad\text{for all }\psi\in C_c^1[0,T).\end{aligned} \tag{342}$$

Definition 9. Weak solution whose density-velocity component $(\varrho,\mathbf{u})$ satisfies the continuity equation in the renormalized sense (116)–(117) with $f=0$, with any test function b belonging to (118), is called renormalized weak solution.

Remark 14. 1. We deduce from (89) and (88) that the total energy balance (342) is equivalent with the formulation

$$\int_\Omega\left(\frac12\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)\right)\,\mathrm{d}x\Big|_0^\tau=\int_0^\tau\int_\Omega\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t\quad\text{for a.a. }\tau\in(0,T). \tag{343}$$

2. We deduce from (341) and (77), (85) that

$$-\int_{\Omega}[\varrho s(\varrho,\vartheta)\eta](\tau,x)\,\mathrm{d}x+\int_{\Omega}\varrho_0 s(\varrho_0,\vartheta_0)\eta\,\mathrm{d}x+\sigma_\eta\Big[[0,\tau]\Big] \tag{344}$$

$$=-\int_0^\tau\int_{\Omega}\left(\varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\eta+\frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\eta}{\vartheta}\right)\,\mathrm{d}x\mathrm{d}t \quad \text{for a.a. } \tau\in(0,T),$$

where $\eta\in C^1(\overline{\Omega})$, $\eta\geq 0$, and σ_η is a nonnegative Radon measure on Borel sets of $[0,T]$. Likewise, we deduce from (90), (95), and (341) that

$$-\int_{\Omega}[\varrho s(\varrho,\vartheta)](\tau,x)\,\mathrm{d}x+\int_{\Omega}\varrho_0 s(\varrho_0,\vartheta_0)\,\mathrm{d}x+\sigma\Big[[0,\tau]\times\overline{\Omega}\Big]=0 \quad \text{for a.a. } \tau\in(0,T), \tag{345}$$

where σ is a nonnegative Radon measure on Borel sets of $[0,T]\times\overline{\Omega}$ satisfying

$$\sigma\Big[[0,\tau]\times\overline{\Omega}\Big]\geq\int_0^\tau\int_{\Omega}\frac{1}{\vartheta}\Big(\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big)\,\mathrm{d}x\mathrm{d}t.$$

3. Putting together (343) and (345) we get the so-called dissipation identity in the form

$$\int_{\Omega}\Big(\varrho|\mathbf{u}|^2+H_{\overline{\vartheta}}(\varrho,\vartheta)\Big)\,\mathrm{d}x\Big|_0^\tau+\sigma\Big[[0,\tau]\times\overline{\Omega}\Big]=\int_0^\tau\int_{\Omega}\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t \tag{346}$$

for a.a. $\tau\in(0,T)$ and $\overline{\vartheta}=const>0$. Similarly, by the same token involving (96),

$$\int_{\Omega}\Big(\varrho|\mathbf{u}|^2+H_{\overline{\vartheta}}(\varrho,\vartheta)\Big)\,\mathrm{d}x\Big|_z^\tau+\sigma\Big[[z,\tau]\times\overline{\Omega}\Big]=\int_z^\tau\int_{\Omega}\varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t \tag{347}$$

for a.a. $0<z<\tau\in(0,T)$ and $\overline{\vartheta}=const>0$, where

$$\sigma\Big[[z,\tau]\times\overline{\Omega}\Big]\geq\int_z^\tau\int_{\Omega}\frac{1}{\vartheta}\Big(\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}):\nabla_x\mathbf{u}-\frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big)\,\mathrm{d}x\mathrm{d}t.$$

4. According to (94) applied to the entropy balance (341), the right and left instantaneous values $[\varrho s(\varrho,\vartheta)](\tau+)$ and $[\varrho s(\varrho,\vartheta)](\tau-)$ defined in (68)–(69) are continuous linear functionals on $C(\overline{\Omega})$ satisfying

$$[\varrho s(\varrho,\vartheta)](\tau+)\geq[\varrho s(\varrho,\vartheta)](\tau-). \tag{348}$$

5. We deduce from (94) (with $\varphi=1$) applied to the entropy balance (341) that the function of instantaneous values of global entropy

$$[0,T]\ni\tau\mapsto \mathrm{inst}\Big[\int_{\Omega}\varrho(\cdot,x)s(\vartheta(\cdot,x),\varrho(\cdot,x))\,\mathrm{d}x\Big](\tau) \tag{349}$$

is a nondecreasing function (with a countable number of jumps). Likewise we deduce from (72)–(73) that the instantaneous values of the total energy

$$\tau \ni [0,T] = E(\tau) = \text{inst}\Big[\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2(\cdot,x) + [\varrho e(\varrho,\vartheta)](\cdot,x)\right)\,\mathrm{d}x\Big](\tau) \tag{350}$$

yield an absolutely continuous function.

6. In the important case of the potential forces $\mathbf{f}(t,x) = \nabla_x F(t,x)$, it is convenient to replace in the definition of finite energy weak solutions the total energy balance (342) with

$$-\int_0^T\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)-\varrho F\right)\psi'(t)\,\mathrm{d}x\mathrm{d}t = \int_\Omega\psi(0)\left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2+\varrho_0 e(\varrho_0,\vartheta_0)-\varrho_0 F\right)\mathrm{d}x \tag{351}$$

for all $\psi \in C_c^1[0,T)$ which is equivalent to

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F\right)\,\mathrm{d}x\Big|_0^\tau = 0 \text{ for a.a. } \tau \in (0,T). \tag{352}$$

If $F \in L^\infty(0,T;W^{1,\infty}(\Omega))$ and $\varrho\mathbf{u} \in C_{\mathrm{weak}}([0,T];L^q(\Omega;\mathbb{R}^3))$ with some $q > 1$, all formulations (342), (351), (343), and (352) are equivalent.

7. If one considers problem (323)–(331) with *slip boundary condition* (22), one must modify adequately the Definition 8 of finite energy weak solutions at two points: (1) In the function spaces (338), velocity $\mathbf{u}$ must belong to the space $L^2(0,T;W^{1,q}(\Omega;\mathbb{R}^3))$, $q > 1$ with the normal trace $\mathbf{u}\cdot\mathbf{n}|_{(0,T)\times\partial\Omega} = 0$ (and not to $L^2(0,T;W_0^{1,q}(\Omega;\mathbb{R}^3))$). (2) Test functions in the weak formulation of the momentum equation must belong to class

$$\varphi \in C_c^1([0,T]\times\overline{\Omega};\mathbb{R}^3),\ \varphi\cdot\mathbf{n}|_{(0,T)\times\partial\Omega} = 0. \tag{353}$$

Other items in the definition remain without changes.

Considering the entropy production rate as a nonnegative measure satisfying (333) transforms the *balance of entropy identity* (325) into the *variational inequality* (341). It may considerably extend the number of weak solution. To compensate this loss of information, we require that the weak solution obeys the global energy conservation (342). This makes from Definition 8 admissible definition. Indeed, any sufficiently regular weak solution is a classical solution as stated in the following lemma whose proof can be found in [33, Section 2].

Lemma 13. *Let the trio* $(\varrho,\vartheta,\mathbf{u})$ *be a finite energy weak solution to problem* (323)–(332) *in class*

$$(\varrho,\vartheta,\mathbf{u}) \in C^1(\overline{Q}_T)\times C^1(\overline{Q}_T)\cap C([0,T];C^2(\overline{\Omega})) \times C^1(\overline{Q}_T;\mathbb{R}^3)\cap C([0,T];C^2(\overline{\Omega};\mathbb{R}^3)),\quad \varrho>0,\ \vartheta>0.$$

Then $(\varrho,\vartheta,\mathbf{u})$ *is a classical solution to the Navier-Stokes-Fourier system. In particular, it satisfies all variants of energy balance laws* (5), (7), (9)–(10) *as identities on* Q_T.

Lemma 13 remains valid if we replace the homogenous Dirichlet boundary conditions (332) with the slip or Navier's slip boundary conditions (22) or (23).

9.2 Relative Energy Functional

We shall now define dissipative solutions. This definition is inspired by identity (336). We introduce *relative energy* as a function of four variables as follows:

$$[0,\infty)\times(0,\infty)^3 \ni (\varrho,\vartheta,r,\Theta)\mapsto E(\varrho,\vartheta\,|r,\Theta)\in\mathbb{R}, \tag{354}$$

$$E(\varrho,\vartheta\,|r,\Theta) = H_\Theta(\varrho,\vartheta) - \partial_\varrho H_\Theta(r,\Theta)(\varrho - r) - H_\Theta(r,\Theta),$$

where

$$H_\Theta(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \Theta\varrho s(\varrho,\vartheta).$$

If the *thermodynamic stability conditions* (30) are satisfied, then the function $E(\cdot|\cdot)$ has a remarkable property of a "quasi-distance"

$$E(\varrho,\vartheta|r,\Theta)\geq 0 \text{ and } E(\varrho,\vartheta|r,\Theta) = 0 \;\Leftrightarrow\; (\varrho,\vartheta) = (r,\Theta). \tag{355}$$

Indeed, we deduce this property from the splitting

$$E(\varrho,\vartheta|r,\Theta) = [H_\Theta(\varrho,\vartheta) - H_\Theta(\varrho,\Theta)] + [H_\Theta(\varrho,\Theta) - \partial_\varrho H_\Theta(r,\Theta)(\varrho - r) - H_\Theta(r,\Theta)], \tag{356}$$

by virtue of relations (32)–(33). We may introduce functional

$$\mathcal{E}(\varrho,\vartheta,\mathbf{u}\,|\,r,\Theta,\mathbf{U}) = \int_\Omega \Big(\frac{1}{2}(\varrho|\mathbf{u}-\mathbf{U}|^2 + E(\varrho,\vartheta|r,\Theta)\Big)\,\mathrm{d}x, \tag{357}$$

where $(\varrho,\vartheta,\mathbf{u})$, $\varrho\geq 0$, $\vartheta>0$ are integrable functions on Q_T representing the state of the gas and $(r,\Theta,\mathbf{U})$ are arbitrary integrable functions with positive r and Θ a.e. in Q_T. According to property (355), if the thermodynamic stability conditions (378) are satisfied, then

$$\mathcal{E}(\varrho,\vartheta,\mathbf{u}\,|\,r,\Theta,\mathbf{U})\geq 0 \text{ and } \mathcal{E}(\varrho,\vartheta,\mathbf{u}\,|\,r,\Theta,\mathbf{U}) = 0 \Leftrightarrow\; (\varrho,\vartheta,\mathbf{u}) = (r,\Theta,\mathbf{U}). \tag{358}$$

Consequently the functional $\mathcal{E}$ is able to measure the "distance" between a state $(\varrho,\vartheta,\mathbf{u})$ of the gas and arbitrary trio $(r,\Theta,\mathbf{U})$ with positive r and Θ. In fact, under the hypotheses of the thermodynamic stability, the relative energy function $E(\cdot|\cdot)$ obeys stronger coercivity properties than (355). They are described in the following lemma:

Lemma 14 (see [33, Proposition 3.2 and Lemma 5.1], [44, Lemma 4.1]). *Let the constitutive relations for e, p, s obey regularity* (34)*, Gibbs relation* (326)*, and thermodynamic stability conditions* (30)*. Let*

$$0 < \underline{r} < \overline{r}, \;\; 0 < \underline{\Theta} < \overline{\Theta}$$

be given constants.

Then there exists $c = c(\underline{r}, \overline{r}, \underline{\Theta}, \overline{\Theta}) > 0$ such that for all $(\varrho, \vartheta) \in [0, \infty) \times (0, \infty)$ and all $(r, \Theta) \in [\underline{r}, \overline{r}] \times [\underline{\Theta}, \overline{\Theta}]$

$$E(\varrho, \vartheta | r, \Theta) \geq c \begin{cases} |\varrho - r|^2 + |\vartheta - \Theta|^2 \; \textit{if} \; (\varrho, \vartheta) \in \mathcal{O}_{\text{ess}} \\ \\ \varrho e(\varrho, \vartheta) + \Theta |s(\varrho, \vartheta)| + 1 \; \textit{if} \; (\varrho, \vartheta) \in \mathcal{O}_{\text{res}}, \end{cases} \tag{359}$$

where $\mathcal{O}_{\text{ess}}$, $\mathcal{O}_{\text{res}}$ are essential and residual subsets in the density-temperature two-dimensional phase space defined by

$$\mathcal{O}_{\text{ess}} = [\underline{r}/2, 2\overline{r}] \times [\underline{\Theta}/2, 2\overline{\Theta}], \quad \mathcal{O}_{\text{res}} = [0, \infty) \times (0, \infty) \setminus \mathcal{O}_{\text{ess}}.$$

Proof of Lemma 14 is based on the thermodynamic stability conditions expressed in the form (32), (33) and on the definition of function H_Θ (see 354).

9.3 Bounded Energy Weak Solutions

The concept of finite energy weak solutions is not convenient for investigation of weak solutions on unbounded domains. In fact, the finite energy weak solutions are not able to track the conditions at infinity (24). If the thermodynamic stability conditions are satisfied, then $E(\varrho, \vartheta | \varrho_\infty, \vartheta_\infty) = 0$ if and only if $(\varrho, \vartheta) = (\varrho_\infty, \vartheta_\infty)$ according to the property (355) (at least provided $\varrho_\infty > 0$, $\vartheta_\infty > 0$). The conditions (24) will be then verified in the sense that $[\mathcal{E}(\varrho, \vartheta, \mathbf{u} | \varrho_\infty, \vartheta_\infty, \mathbf{u}_\infty)](\tau)$ is bounded for a.a. $\tau \in (0, T)$. We shall incorporate this property into the definition of weak solutions. Such weak solution will be called *bounded energy weak solution*.

Definition 10. Let Ω be a bounded or an unbounded domain, and let conditions at infinity $(\varrho_\infty, \vartheta_\infty, \mathbf{u}_\infty)$ specified in (24) be given in the case of unbounded Ω. Suppose that initial data verify

$$\varrho_0 : \Omega \to [0, +\infty), \; \mathbf{u}_0 : \Omega \to \mathbb{R}^3, \; \vartheta_0 : \Omega \to (0, \infty),$$

with

$$\varrho_0 \mathbf{u}_0 \in L^1_{\text{loc}}(\overline{\Omega}), \; \varrho_0 \mathbf{u}_0 = 0 \text{ and } \varrho_0 \mathbf{u}_0^2 = 0 \text{ a.e. in the set } \{x \in \Omega | \varrho_0(x) = 0\},$$

$$\varrho_0 \in L^1_{\text{loc}}(\overline{\Omega}), \; \varrho_0 s(\varrho_0, \vartheta_0) \in L^1_{\text{loc}}(\overline{\Omega})),$$

$$\int_\Omega \Big(\varrho_0 |\mathbf{u}_0 - \overline{\mathbf{u}}|^2 + H_{\overline{\vartheta}}(\varrho_0, \vartheta_0) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big(\varrho_0 - \overline{\varrho} \Big) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \Big) \, \mathrm{d}x < \infty,$$

where we have set $\overline{\varrho} = \varrho_\infty$, $\overline{\vartheta} = \vartheta_\infty$, $\overline{\mathbf{u}} = \mathbf{u}_\infty$ if Ω is unbounded, and $\overline{\varrho}$, $\overline{\vartheta}$ positive numbers, $\overline{\mathbf{u}} = 0$ in the case of a bounded domain.

The trio $(\varrho, \vartheta, \mathbf{u})$ is a *bounded energy weak solution* to problem (323)–(332) – with conditions at infinity (24), if Ω is unbounded – provided:

(a)

$$\varrho, \vartheta \in L^\infty(0,T; L^1_{\mathrm{loc}}(\overline{\Omega})),\ \varrho \geq 0,\ \vartheta > 0 \text{ a.e. in } (0,T)\times\Omega,\ p(\varrho,\vartheta) \in L^1(0,T; L^1_{\mathrm{loc}}(\overline{\Omega})), \tag{360}$$

$$\mathbf{u} \in L^2(0,T; W^{1,q}_{0,\mathrm{loc}}(\overline{\Omega}));\ \varrho\mathbf{u},\ \frac{1}{2}\varrho\mathbf{u}^2,\ \varrho e(\varrho,\vartheta),\ \varrho s(\varrho,\vartheta) \in L^\infty(0,T; L^1_{\mathrm{loc}}(\Omega)),\ q > 1,$$

$$\varrho s(\varrho,\vartheta)\mathbf{u},\ \mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}) \in L^1(0,T; L^1_{\mathrm{loc}}(\overline{\Omega})),$$
$$\frac{1}{\vartheta}\Big(\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big) \in L^1(Q_T),$$

$$H_{\overline{\vartheta}}(\varrho,\vartheta) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\Big(\varrho - \overline{\varrho}\Big) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) \in L^\infty(0,T; L^1(\Omega)),\ \varrho|\mathbf{u}-\overline{\mathbf{u}}|^2 \in L^\infty(0,T; L^1(\Omega));$$

(b) $\varrho \in C_{\mathrm{weak}}([0,T]; L^1(K))$ for any compact subset $K \subset \overline{\Omega}$, and weak formulation (339) of the continuity equation holds;
(c) $\varrho\mathbf{u} \in C_{\mathrm{weak}}([0,T]; L^1(K;\mathbb{R}^3))$ for any compact subset $K \subset \overline{\Omega}$, and weak formulation (340) of the momentum equation is verified;
(d) the weak formulation (341) of the entropy balance is satisfied;
(e) the balance of total energy is replaced by the weak formulation of the dissipation inequality (335) in the integral form,

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}-\overline{\mathbf{u}}|^2 + H_{\overline{\vartheta}}(\varrho,\vartheta) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\Big(\varrho-\overline{\varrho}\Big) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\Big)\,\mathrm{d}x\Big|_0^\tau \tag{361}$$

$$+\int_0^\tau\int_\Omega \frac{1}{\vartheta}\Big(\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big)\,\mathrm{d}x\mathrm{d}t \leq \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t$$

for a.a. $\tau \in (0,T)$.

Remark 15. 1. In view of the dissipation balance (346) and continuity equation (339), any finite energy weak solution is a bounded energy weak solution for bounded domains. It is not known whether the opposite statement is true.
2. If one considers the slip boundary conditions (22), one has to modify accordingly the definition: Condition $\mathbf{u} \in L^2(0,T; W^{1,q}_{0,\mathrm{loc}}(\overline{\Omega}))$ in (360) must be replaced by $\mathbf{u} \in L^2(0,T; W^{1,q}_{\mathrm{loc}}(\overline{\Omega}))$, $\mathbf{u}\cdot\mathbf{n}|_{(0,T)\times\partial\Omega} = 0$, and the test function φ in the weak formulation of the momentum equation must be taken in class (353).

9.4 Dissipative Solutions

Definition 11. We say that the triplet $(\varrho, \vartheta, \mathbf{u})$ is a *dissipative solution* to the Navier-Stokes-Fourier system (323)–(332) if it belongs to class (338) and if it satisfies *relative energy inequality*

$$\mathcal{E}(\varrho, \vartheta, \mathbf{u} \,|\, r, \Theta, \mathbf{U})\Big|_0^\tau + \int_0^\tau \int_\Omega \Theta \frac{\mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u})}{\vartheta} : \nabla_x \mathbf{u} \, \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega \Theta \frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta)}{\vartheta^2} : \nabla_x \vartheta \, \mathrm{d}x\mathrm{d}t \tag{362}$$
$$\leq \int_0^\tau \int_\Omega \mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_\Omega \frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \, \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \varrho \Big(s(r, \Theta) - s(\varrho, \vartheta) \Big) \Big(\partial_t \Theta + \mathbf{u} \cdot \nabla_x \Theta \Big) \, \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_\Omega \Big[\Big(1 - \frac{\varrho}{r} \Big) \partial_t p(r, \Theta) - \varrho \mathbf{u} \cdot \frac{\nabla_x p(r, \Theta)}{r} \Big] \, \mathrm{d}x\mathrm{d}t$$
$$- \int_0^\tau \int_\Omega p(\varrho, \vartheta) \mathrm{div}_x \mathbf{U} \, \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \varrho \mathbf{f} \cdot (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x\mathrm{d}t$$

for a.a. $\tau \in (0, T)$ with any

$$(r, \Theta, \mathbf{U}) \in C_c^1([0, T] \times \overline{\Omega}; \mathbb{R}^5), \; r > 0, \; \Theta > 0, \; \mathbf{U}|_{(0,T) \times \partial\Omega} = 0. \tag{363}$$

Remark 16. 1. If one considers the slip boundary conditions (22) in place of the homogenous Dirichlet boundary conditions (20), the definition must be modified: we must replace condition $\mathbf{U}|_{(0,T)\times\partial\Omega} = 0$ in (363) with the condition $\mathbf{U} \cdot \mathbf{n}|_{(0,T)\times\partial\Omega} = 0$.

2. If one considers unbounded domains with conditions (24) at infinity with $\mathbf{u}_\infty = 0$ (for simplicity) and with homogenous boundary conditions (332), it is necessary to modify the definition as follows: Inequality (362) remains as it stays, but one must replace (363) by

$$r - \varrho_\infty, \; \Theta - \vartheta_\infty \in C_c^1([0, T] \times \overline{\Omega}), \; r > 0, \; \Theta > 0, \; \mathbf{U}|_{\partial\Omega} = 0.$$

The reader can consult [70] to find more details about the dissipative solutions and relative energy inequality in the situations described in *items* 1. and 2. above.

Bounded energy weak solutions in the sense of Definition 8 are dissipative solutions under mild assumptions on constitutive laws and transport coefficients. This is subject of the following theorem:

Theorem 39. *Let Ω be a bounded domain, and let $(\varrho, \vartheta, \mathbf{u})$ be a bounded energy weak solution to the Navier-Stokes-Fourier system* (323)–(332) *in the sense of Definition 10. Then $(\varrho, \vartheta, \mathbf{u})$ is a dissipative solution; in particular it satisfies relative energy inequality* (362).

Remark 17. 1. The reader has noticed that relations (323)–(332) include implicitly certain regularity assumptions and sign assumptions on the constitutive laws for p, e (namely, (34)) and transport coefficients μ, η, κ (namely, (14)–(15)).

2. Theorem 39 holds true regardless whether thermodynamic stability conditions are satisfied. However, it becomes a useful and powerful tool of analysis especially in the case when the thermodynamic stability conditions are satisfied. Indeed, relative energy inequality (362) governs the evolution of the relative energy functional $\mathcal{E}(\varrho, \vartheta, \mathbf{u}|r, \Theta, \mathbf{U})$. If the thermodynamic stability conditions are satisfied, the functional $\mathcal{E}(\varrho, \vartheta, \mathbf{u}|r, \Theta, \mathbf{U})$ measures the "distance" between the weak solution $(\varrho, \vartheta, \mathbf{u})$ and other state $(r, \Theta, \mathbf{U})$ of the fluid by means of Lemma 14. Due to this fact, Theorem 39 has many potential applications. In this chapter of the handbook, we will mention two of them that are directly related to the existence theory: (1) stability and weak-strong uniqueness and (2) longtime behavior. There are other applications, e.g., investigation of various singular limits to the complete Navier-Stokes-Fourier system that goes far beyond the scope of this chapter (see, e.g., [35, 38] and monograph [33]).
3. According to Theorem 39, if Ω is a bounded domain, then any bounded energy weak solution is a dissipative solution. This statement is not known to be true for the bounded energy weak solution on unbounded domains. However, under certain additional structural assumptions on the constitutive laws, one can construct bounded energy weak solutions that are dissipative. This questions will be discussed later in more details.

Proof of Theorem 39. If we take in the continuity equation (339) as test function $\varphi = \frac{|\mathbf{U}|^2}{2}$, we obtain the identity

$$\int_\Omega \varrho \frac{|\mathbf{U}|^2}{2} dx \Big|_0^\tau = \int_0^\tau \int_\Omega \varrho \mathbf{U} \cdot \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla \mathbf{U}\Big) \mathrm{d}x \mathrm{d}t. \tag{364}$$

Momentum equation (340) with the test function $\varphi = \mathbf{U}$ reads

$$-\int_\Omega \varrho \mathbf{u} \cdot \mathbf{U} \mathrm{d}x \Big|_0^\tau = -\int_0^\tau \int_\Omega \Big[\varrho \mathbf{u} \cdot (\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla \mathbf{U}) + p(\varrho, \vartheta) \mathrm{div}_x \mathbf{U} - \mathbb{S}(\varrho, \vartheta, \nabla \mathbf{u}) : \nabla \mathbf{U} + \varrho \mathbf{U} \cdot \mathbf{f}\Big] \mathrm{d}x \mathrm{d}t. \tag{365}$$

Taking in the entropy inequality (341) $\varphi = \Theta$ as test function, we obtain

$$\begin{aligned} &-\int_\Omega \varrho s(\varrho, \vartheta) \Theta dx \Big|_0^\tau + \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \Big(\mathbb{S}(\varrho, \vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\varrho, \vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta}\Big) \mathrm{d}x \mathrm{d}t \\ &\quad \le -\int_0^\tau \int_\Omega \Big[\varrho s(\varrho, \vartheta)(\partial_t \Theta + \mathbf{u} \cdot \nabla \Theta) + \frac{\mathbf{q}(\varrho, \vartheta, \nabla \vartheta) \cdot \nabla \Theta}{\vartheta}\Big] \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{366}$$

Summing up energy identity (343) with identities (364), (365) and with the inequality (366), we arrive at the inequality

$$\int_\Omega \Big[\frac{\varrho}{2}|\mathbf{u}-\mathbf{U}|^2 + H_\Theta(\varrho,\vartheta)\Big]\mathrm{d}x\Big|_0^\tau \tag{367}$$

$$+\int_0^\tau\int_\Omega \frac{\Theta}{\vartheta}\Big(\mathbb{S}(\varrho,\vartheta,\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$\le \int_0^\tau\int_\Omega \mathbb{S}(\varrho,\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{U} - \frac{\mathbf{q}(\varrho,\vartheta,\nabla\vartheta)\cdot\nabla\Theta}{\vartheta}\,\mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \varrho\Big(\partial_t\mathbf{U} + \mathbf{u}\cdot\nabla\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega \varrho s(\varrho,\vartheta)(\partial_t\Theta + \mathbf{u}\cdot\nabla\Theta)\,\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau\int_\Omega p(\varrho,\vartheta)\mathrm{div}_x\mathbf{U}\mathrm{d}x\mathrm{d}t + \int_0^\tau\int_\Omega \varrho\mathbf{f}\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\mathrm{d}t.$$

Due to the Gibbs relation (326),

$$a\partial_\varrho H_b(a,b) - H_b(a,b) = p(a,b).$$

Consequently,

$$\int_\Omega \Big(r\partial_\varrho H_\Theta(r,\Theta) - H_\Theta(r,\Theta)\Big)\mathrm{d}x\Big|_0^\tau = \int_0^\tau\int_\Omega \partial_t p(r,\Theta)\,\mathrm{d}x\mathrm{d}t$$

for a.a. $\tau\in(0,T)$.

Further, continuity equation (339) with test function $-\partial_\varrho H_\Theta(r,\Theta)$ yields

$$-\int_\Omega \varrho\partial_\varrho H_\Theta(r,\Theta)\mathrm{d}x\Big|_0^\tau =$$

$$-\int_0^\tau\int_\Omega \varrho\Big(\partial_t\partial_\varrho H_\Theta(r,\Theta) + \mathbf{u}\cdot\nabla_x\partial_\varrho H_\Theta(r,\Theta)\Big)\mathrm{d}x\mathrm{d}t,$$

where, by the Gibbs relation (326),

$$\partial_y\partial_\varrho H_b(a,b) = \frac{1}{a}\partial_y p(a,b) - s(a,b)\partial_y b.$$

Whence adding to the left-hand side of (367) the term

$$\int_\Omega \Big[-\varrho\partial_\varrho H_\Theta(r,\Theta) + r\partial_\varrho H_\Theta(r,\Theta) - H_\Theta(r,\Theta)\Big]\mathrm{d}x\Big|_0^\tau,$$

we arrive at the inequality

$$\int_{\Omega} \left[\frac{\varrho}{2}|\mathbf{u}-\mathbf{U}|^2 + H_{\Theta}(\varrho,\vartheta) - \partial_{\varrho} H_{\Theta}(r,\Theta)(\varrho - r) - H_{\Theta}(r,\Theta)\right] \mathrm{d}x \Big|_0^{\tau} \tag{368}$$

$$+ \int_0^{\tau}\int_{\Omega} \frac{\Theta}{\vartheta}\Big(\mathbb{S}(\varrho,\vartheta,\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\varrho,\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^{\tau}\int_{\Omega} \mathbb{S}(\varrho,\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{U} - \frac{\mathbf{q}(\varrho,\vartheta,\nabla\vartheta)\cdot\nabla\Theta}{\vartheta}\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^{\tau}\int_{\Omega} \varrho\Big(\partial_t\mathbf{U} + \mathbf{u}\cdot\nabla\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^{\tau}\int_{\Omega} \varrho\Big(s(r,\Theta) - s(\varrho,\vartheta)\Big)(\partial_t\Theta + \mathbf{u}\cdot\nabla\Theta)\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^{\tau}\int_{\Omega}\Big(1-\frac{\varrho}{r}\Big)\partial_t p(r,\Theta)\mathrm{d}x\mathrm{d}t - \int_0^{\tau}\int_{\Omega} p(\varrho,\vartheta)\mathrm{div}\mathbf{U}\,\mathrm{d}x\mathrm{d}t$$

$$- \int_0^{\tau}\int_{\Omega} \varrho\mathbf{u}\cdot\frac{\nabla_x p(r,\Theta)}{r}\mathrm{d}x\mathrm{d}t + \int_0^{\tau}\int_{\Omega}\varrho\mathbf{f}\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\mathrm{d}t.$$

Theorem 39 is proved.

9.5 Constitutive Relations and Transport Coefficients for the Existence Theory

In the above setting, we will be able to build up existence theory under certain assumptions on constitutive laws on pressure, internal energy, and transport coefficients that are listed in the sequel. The reader is advised to confront these conditions with the physically motivated constraints due to statistical mechanics exposed in (47)–(50), (51), due to thermodynamic stability conditions exposed in (58)–(59), and due to the physical transport properties of the fluid exposed in (16)–(18).

(i) *Pressure, internal energy, and specific entropy*

$$p(\varrho,\vartheta) = \vartheta^{\gamma/(\gamma-1)} P\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right) + \frac{a}{3}\vartheta^4,\ a>0,\ \gamma>1, \tag{369}$$

where

$$P \in C^1[0,\infty),\ P(0)=0,\ P'(Z)>0 \text{ for all } Z\geq 0 \tag{370}$$

$$0 < \frac{\gamma P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z>0, \tag{371}$$

$$\lim_{Z\to\infty}\frac{P(Z)}{Z^\gamma}=P_\infty>0. \tag{372}$$

The internal energy must write

$$e(\varrho,\vartheta)=\frac{1}{\gamma-1}\frac{\vartheta^{\gamma/(\gamma-1)}}{\varrho}P\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right)+a\frac{\vartheta^4}{\varrho}, \tag{373}$$

and the formula for (specific) entropy reads

$$s(\varrho,\vartheta)=S\left(\frac{\varrho}{\vartheta^{1/(\gamma-1)}}\right)+\frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{374}$$

where

$$S'(Z)=-\frac{1}{\gamma-1}\frac{\gamma P(Z)-P'(Z)Z}{Z^2}<0. \tag{375}$$

(ii) Transport coefficients

$$\mu,\eta\in C^1[0,\infty)\cap L^\infty(0,\infty),\ \mu'\in L^\infty(0,\infty), \tag{376}$$

$$\underline{\mu}(1+\vartheta^\beta)\le\overline{\mu}(1+\vartheta^\beta),\ 0\le\eta(\vartheta)\le\overline{\eta}(1+\vartheta^\beta),$$

$$\kappa\in C^1[0,\infty),\quad \underline{\kappa}(1+\vartheta^3)\le\kappa(\vartheta)\le\overline{\kappa}(1+\vartheta^3), \tag{377}$$

where $\underline{\mu},\overline{\mu},\overline{\eta},\underline{\kappa},\overline{\kappa}$ are positive constants.

It should be underlined that pressure and internal energy defined through formulas (369)–(375) verify the *thermodynamic stability conditions,* namely,

$$\frac{\partial p(\varrho,\vartheta)}{\partial\varrho}>0,\ \frac{\partial e(\varrho,\vartheta)}{\partial\vartheta}>0 \text{ for all } \varrho,\vartheta>0. \tag{378}$$

Recall that these conditions can be rewritten in terms of the Helmoltz function via formulas

$$\partial^2_\varrho H_{\overline{\vartheta}}(\varrho,\overline{\vartheta})=\frac{1}{\varrho}\frac{\partial p(\varrho,\vartheta)}{\partial\varrho},\quad \partial_\vartheta H_{\overline{\vartheta}}(\varrho,\vartheta)=\varrho\frac{\vartheta-\overline{\vartheta}}{\vartheta}\frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} \tag{379}$$

with any $\overline{\vartheta}>0$, meaning that

$$\varrho\mapsto H_{\overline{\vartheta}}(\varrho,\overline{\vartheta}) \text{ is strictly convex}, \tag{380}$$

$$\vartheta\mapsto H_{\overline{\vartheta}}(\varrho,\vartheta) \text{ attains its global minimum at } \vartheta=\overline{\vartheta}. \tag{381}$$

9.6 Existence of Weak Solutions

We shall present two existence theorems for weak solutions. The first one deals with $\gamma = 5/3$ (this case corresponds to the monoatomic gas) and β is allowed to vary in a certain range:

Theorem 40 (see [33, Theorems 3.1 and 3.2] reproved in Feireisl, Pražák [44, Theorem 4.3]). *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1)$ and let $\mathbf{f} \in L^\infty(Q_T;\mathbb{R}^3)$. Suppose that the thermodynamic functions p, e, and s satisfy hypotheses (369)–(375) and that the transport coefficients μ, η, and κ obey (376), (377), where*

$$\gamma = 5/3, \quad \beta \in (2/5, 1].$$

Finally, assume that the initial data (330) verify (337). Then the complete Navier-Stokes-Fourier system (323)–(332) admits at least one renormalized finite energy weak solution with the following additional properties:

$$\mathbf{u} \in L^q(0,T;W^{1,p}(\Omega;\mathbb{R}^3)) \text{ with } q = \frac{6}{4-\beta},\, p = \frac{18}{10-\beta}, \tag{382}$$

$$\varrho \in C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^\gamma(\Omega)) \cap L^q(Q_T) \text{ with some } q > \gamma, \tag{383}$$

$$\varrho\mathbf{u} \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)) \cap C_{\mathrm{weak}}([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega)), \tag{384}$$

$$\ln\vartheta,\ \vartheta^{\underline{\beta}} \in L^2(0,T;W^{1,2}(\Omega)),\ \vartheta \in L^\infty(0,T;L^4(\Omega)),\ \underline{\beta} \in [0,3/2], \tag{385}$$

$$\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) \in L^q(Q_T;\mathbb{R}^9) \text{ with some } q > 1. \tag{386}$$

There holds

$$\operatorname{ess}\lim_{t\to 0+}\int_\Omega \varrho s(\varrho,\vartheta)\eta(x)\,\mathrm{d}x \geq \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\eta\,\mathrm{d}x,\ \ \eta \in C_c^1(\Omega), \eta \geq 0. \tag{387}$$

If, moreover, $\vartheta_0 \in W^{1,\infty}(\Omega)$, then

$$\operatorname{ess}\lim_{t\to 0+}\int_\Omega \varrho s(\varrho,\vartheta)\eta(x)\,\mathrm{d}x = \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\eta\,\mathrm{d}x,\ \ \eta \in C_c^1(\Omega).$$

In the second variant of the existence theorem, we allow $\gamma > 3/2$ and fix $\beta = 1$.

Theorem 41 (see [70, Theorems 2.1 and 2.2]). *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1)$ and let $\mathbf{f} \in L^\infty(Q_T;\mathbb{R}^3)$. Suppose that the thermodynamic functions p, e, s satisfy hypotheses (369)–(375) and that the transport coefficients μ, η, and κ obey (376), (377), where*

$$\gamma > 3/2, \quad \beta = 1.$$

Finally, assume that the initial data (330) *verify* (337)*. Then the complete Navier-Stokes-Fourier system* (323)–(332) *admits at least one renormalized bounded and finite entropy weak solution with further properties* (383)–(387) *and with*

$$\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega)). \tag{388}$$

Remark 18. 1. The conclusion of Theorems 40 and 41 is valid under the same assumptions also for bounded Lipschitz domains as one can verify by using the techniques introduced for this purpose by Kukucka [73] and Poul [94].

2. One can consider the same problem (323)–(332) with the *complete slip* (22) boundary conditions for the velocity (instead of $\mathbf{u}|_{(0,T)\times\partial\Omega} = 0$) on a bounded domain. After the necessary appropriate modifications in the definition of weak solutions exposed in *item* 7 of Remark 14, one can prove their existence under the same assumptions on the regularity of the domain, initial data, external force, constitutive relations, and transport coefficients as in Theorems 40 and 41. The solutions constructed in this way enjoy all additional properties mentioned in Theorems 40, resp., 41, according to the case. The reader can consult [33, Chapter 3] for the details.
3. Definition of weak solutions introduced through (337)–(342) and investigated in Theorems 40 and 41 relies essentially on the fact that the fluid system must be mechanically and thermally isolated (meaning that $\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}, \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0$). If in the boundary conditions one of these assumptions is violated, the theory cannot be applied.
4. The system (323)–(332) on an unbounded domain with condition (24) admits under certain circumstances a *bounded energy weak solution*. For example, if in (24), $\varrho_\infty > 0, \vartheta_\infty > 0, and\ \mathbf{u}_\infty = 0$, it is known that the system (323)–(332) admits on any unbounded uniformly Lipschitz domain a bounded energy weak and dissipative solution under the same assumptions on constitutive laws p, e and transport coefficients μ, η , κ, as in Theorems 40, resp.,41 provided $\mathbf{f} = \nabla_x F$, $F \in L^\infty(0,T;W^{1,\infty} \cap W^{1,1}(\Omega;\mathbb{R}^3))$ (see [50, Theorem 2.5 and Remarks 2.5, 2.6]). The same problem on unbounded domains with the complete slip conditions is investigated in the same paper in Sect. 6.

Remark 19. 1. Existence theorems of type Theorem 40 and Theorem 41 are known also to be true for the phenomenological constitutive laws of real gasses (compressible fluids) of general form (36), (37). Indeed, the compactness result established in [32, Theorem 3.1] in combination with the construction of weak solutions suggested in [33, Chapter 3] and existence theorem proved in [58, Theorem 3.1] can be summarized in the following way.

Assumptions on the constitutive equations and the transport coefficients are the following:

(1) *Pressure and internal anergy take form*

$$p(\varrho,\vartheta) = p_F(\varrho,\vartheta) + \frac{a}{3}\vartheta^4,\ a > 0, \tag{389}$$

$$e(\varrho,\vartheta) = e_F(\varrho,\vartheta) + a\frac{\vartheta^4}{\varrho}, \tag{390}$$

where p_F, e_F satisfy Gibbs' relation (8) for a certain entropy s_F. Moreover, we impose the hypothesis of thermodynamic stability

$$\frac{\partial p_F(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e_F(\varrho,\vartheta)}{\partial \vartheta} > 0 \text{ for all } \varrho,\vartheta > 0. \tag{391}$$

Further, we suppose $p_F \in C^2((0,\infty)^2) \cap C^1([0,\infty)^2)$,

$$\lim_{\varrho\to 0+} p_F(\varrho,\vartheta) = 0 \text{ for any } \vartheta > 0, \ \lim_{\vartheta\to 0+} p_F(\varrho,\vartheta) = p_c(\varrho) \text{ for any } \varrho > 0, \tag{392}$$

with the "cold pressure" p_c satisfying

$$\underline{p}\varrho^\gamma \le p_c(\varrho) \le \overline{p}(1+\varrho)^\gamma, \ \underline{p} > 0. \tag{393}$$

In addition, we suppose

$$\left|\frac{\partial p_F(\varrho,\vartheta)}{\partial \vartheta}\right| \le c\left(1+\varrho^{\gamma/3}+\vartheta^3\right) \text{ for all } 0 < \vartheta < \Theta_c(\varrho), \tag{394}$$

where $\varrho \mapsto \Theta_c(\varrho)$ is a continuous curve satisfying

$$\Theta_c(\varrho) \ge c\varrho^{\gamma/4} - 1 \text{ for a certain } c > 0. \tag{395}$$

As for the internal energy e_F, we assume

$$e_F(\varrho,\vartheta) \ge 0, \ \lim_{[\varrho,\vartheta]\to[0,0]} e_F(\varrho,\vartheta) = 0, \tag{396}$$

$$c_v(\varrho,\vartheta) \equiv \frac{\partial e_F(\varrho,\vartheta)}{\partial \vartheta} \in C([0,\infty)^2), \tag{397}$$

$$0 < \underline{c}(1+\vartheta)^\omega \le c_v(\varrho,\vartheta) \le \overline{c}(1+\vartheta)^\omega \text{ for all } \varrho,\vartheta > 0. \tag{398}$$

(2) *Transport coefficients*
The viscous stress $\mathbb{S}(\vartheta, \nabla_x \mathbf{u})$ is given by Newton's rheological law (12), where $\mu = \mu(\vartheta), \ \eta = \eta(\vartheta) \in W^{1,\infty}[0,\infty)$,

$$0 < \underline{\mu}(1+\vartheta)^\beta \le \mu(\vartheta) \le \overline{\mu}(1+\vartheta)^\beta, \tag{399}$$

$$0 < \underline{\eta}(1+\vartheta)^\beta \le \eta(\vartheta) \le \overline{\eta}(1+\vartheta)^\beta, \tag{400}$$

$$|\mu'(\vartheta)|\overline{\mu} \le (1+\vartheta)^{\beta-1}, \ \ |\eta'(\vartheta)|\overline{\eta} \le (1+\vartheta)^{\beta-1},$$

for all $\vartheta \in [0,\infty)$. The heat flux $\mathbf{q}(\vartheta, \nabla_x \vartheta)$ is given by Fourrier's law (13) where $\kappa \in C^1[0,\infty)$ verifies

$$\underline{\kappa}(1+\vartheta)^\alpha \le \kappa(\vartheta) \le \overline{\kappa}(1+\vartheta)^\alpha. \tag{401}$$

In the above, $\underline{\mu}, \overline{\mu}, \underline{\eta}, \overline{\eta}, \underline{\kappa}, \overline{\kappa}$ are positive constants.

Under assumptions (389)–(401) with

$$\gamma > 3/2,\ 0 \le \beta \le 4/3,\ \alpha \ge \frac{16}{3} - \beta,\ 0 \le \omega \le 1/2,$$

or

$$\gamma > 3,\ -4 \le \beta \le 0,\ \alpha \ge \frac{16}{3} - \beta,\ 0 \le \omega \le 1/2,$$

there is a finite energy weak solution in the sense of Definition 8 on a bounded sufficiently smooth domain.

2. We notice that the bulk viscosity coefficient η is supposed to be strictly positive. This assumption can be relaxed in the case $\beta \ge 0$, for which the lower bound in (400) can be replaced by $\eta \ge 0$.

9.7 Construction of Weak Solutions

Proof of Theorems 40 and 41 can be done via several levels of approximations:

(i) *Continuity equation*

The equation of continuity (323) is regularized by means of an artificial viscosity term:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = \varepsilon \Delta \varrho \text{ in } (0,T) \times \Omega, \tag{402}$$

and supplemented with the homogeneous Neumann boundary condition

$$\nabla_x \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{403}$$

and the initial condition

$$\varrho(0,\cdot) = \varrho_{0,\delta}, \tag{404}$$

where

$$\varrho_{0,\delta} \in C^{2,\nu}(\overline{\Omega}),\ \inf_{x\in\Omega} \varrho_{0,\delta}(x) > 0,\ \nabla_x \varrho_{0,\delta} \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{405}$$

is a convenient approximation of the initial density ϱ_0.

(ii) *Momentum equation*

The momentum balance (324) expressed through the integral identity (340) is replaced by a *Faedo-Galerkin approximation*:

$$\int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + \Big(p(\varrho,\vartheta) + \delta(\varrho^\Gamma + \varrho^2) \Big) \mathrm{div}_x \boldsymbol{\varphi} \Big) \, \mathrm{d}x \mathrm{d}t \tag{406}$$

$$= \int_0^T \int_\Omega \Big(\varepsilon (\nabla_x \varrho \nabla_x \mathbf{u}) \cdot \boldsymbol{\varphi} + \mathbb{S}_\delta : \nabla_x \boldsymbol{\varphi} - \varrho \mathbf{f}_\delta \cdot \boldsymbol{\varphi} \Big) \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\varrho \mathbf{u})_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x,$$

to be satisfied for any test function $\boldsymbol{\varphi} \in C_c^1([0,T); X_n)$, where

$$X_n \subset C^{2,\nu}(\overline{\Omega}; \mathbb{R}^3) \subset L^2(\Omega; \mathbb{R}^3) \tag{407}$$

is a finite-dimensional (n-dimensional) vector space of functions satisfying

$$\boldsymbol{\varphi}|_{\partial\Omega} = 0 \text{ in the case of the no-slip boundary conditions.} \tag{408}$$

The spaces $X_n \subset X_{n+1}$ are endowed with the Hilbert structure induced by the scalar product of the Lebesgue space $L^2(\Omega; \mathbb{R}^3)$, and the linear hull of $\cup_{n\in N} X_n$ is dense in $L^2(\Omega; \mathbb{R}^3)$.

Furthermore, we set

$$\mathbb{S}_\delta = \mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) \ = (\mu(\vartheta) + \delta\vartheta)\Big(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\, \mathbb{I}\Big) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\, \mathbb{I}, \tag{409}$$

while the function

$$\mathbf{f}_\delta \in C^1([0,T] \times \overline{\Omega}; \mathbb{R}^3) \tag{410}$$

is a suitable approximation of the driving force $\mathbf{f}$.

(iii) *Entropy balance*

Instead of the entropy balance (325), we consider a modified internal energy equation in the form

$$\partial_t(\varrho e_\delta(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e_\delta(\varrho, \vartheta)\mathbf{u}) - \mathrm{div}_x \nabla_x \mathcal{K}_\delta(\vartheta) \tag{411}$$

$$= \mathbb{S}_\delta(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u} + \varepsilon\delta(\Gamma\varrho^{\Gamma-2} + 2)|\nabla_x \varrho|^2 + \delta\frac{1}{\vartheta^2} - \varepsilon\vartheta^5,$$

supplemented with the Neumann boundary condition

$$\nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{412}$$

and the initial condition

$$\vartheta(0, \cdot) = \vartheta_{0,\delta}, \tag{413}$$

$$\vartheta_{0,\delta} \in W^{1,2}(\Omega) \cap L^\infty(\Omega), \ \operatorname*{ess\,inf}_{x\in\Omega} \vartheta_{0,\delta}(x) > 0, \tag{414}$$

where $\vartheta_{0,\delta}$ is a convenient approximation of ϑ_0. Here

$$e_\delta(\varrho, \vartheta) = e_{\mathrm{mo},\delta}(\varrho, \vartheta) + a\vartheta^4, \quad e_{\mathrm{mo},\delta}(\varrho, \vartheta) = e_{\mathrm{mo}}(\varrho, \vartheta) + \delta\vartheta, \tag{415}$$

$$\mathcal{K}_\delta(\vartheta) = \int_1^\vartheta \kappa_\delta(z)\,\mathrm{d}z, \ \kappa_\delta(\vartheta) = \kappa(\vartheta) + \delta\Big(\vartheta^\Gamma + \frac{1}{\vartheta}\Big).$$

In problem (402)–(415), the quantities ε, δ are small positive parameters, while $\Gamma > 0$ is a sufficiently large fixed number. Loosely speaking, the ε–dependent quantities provide more regularity of the approximate solutions modifying the type of the field equations, while the δ–dependent quantities prevent concentrations yielding better estimates on the amplitude of the approximate solutions. For technical reasons, the limit passage must be split up in two steps letting first $\varepsilon \to 0$ and then $\delta \to 0$.

The complete existence proof goes far behind the scope of the handbook. The reader can find all details in [33, Chapter 3]. In the handbook, we shall show solely the weak compactness property of the set of weak solutions. This property already contains the main ingredients of the existence proof. Note, however, that the compressible models are very much "approximation sensitive," and the way from the weak compactness to the real existence is always a delicate task.

10 Weak Compactness of the Set of Weak Solutions

In this section we show weak compactness of the (hypothetical) set of weak solutions emanating from initial data $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ in the situation corresponding to assumptions of Theorem 41. This exercise follows main ideas exposed in [33, Chapter 3] and illustrates all essential difficulties that one faces during the existence proof.

Theorem 42. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $\mathbf{f} \in L^\infty(Q_T, \mathbb{R}^3)$. Suppose that the thermodynamic functions p, e, s satisfy hypotheses (369)–(375) and that the transport coefficients μ, η, and κ obey (376), (377) with $\gamma > 3/2$, $\beta = 1$. Finally assume that the initial data $(\varrho_{n,0}, \vartheta_{n,0}, \mathbf{u}_{n,0})$ satisfy*

$$\varrho_{n,0} \rightharpoonup \varrho_0 \ \text{in } L^1(\Omega), \ \varrho_{n,0}\mathbf{u}_{n,0} \rightharpoonup \varrho_0\mathbf{u}_0 \ \text{in } L^1(\Omega; \mathbb{R}^3), \tag{416}$$

$$\varrho_{n,0} s(\vartheta_{n,0}) \rightharpoonup \varrho_0 s(\vartheta_0) \ \text{in } L^1(\Omega),$$

$$\int_\Omega \left(\frac{1}{2}\varrho_{n,0}|\mathbf{u}_{n,0}|^2 + \varrho_{n,0} e(\varrho_{n,0}, \vartheta_{n,0})\right) \mathrm{d}x \to \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0)\right) \mathrm{d}x,$$

where $\varrho_{n,0}, \vartheta_{n,0}, \mathbf{u}_{n,0}$ and $\varrho_0, \vartheta_0, \mathbf{u}_0$ verify (337) with $M_{n,0} > 0$, $E_{n,0} > 0$, $S_{n,0} > 0$, and $M_0 > 0$, $E_0 > 0$, $S_0 > 0$, respectively. Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ be a sequence of renormalized finite energy weak solutions to the complete Navier-Stokes-Fourier system (323)–(332) with initial data $(\varrho_{n,0}, \vartheta_{n,0}, \mathbf{u}_{n,0})$. Then there exists a subsequence (denoted again $(\varrho_n, \vartheta_n, \mathbf{u}_n)$) such that

$$\varrho_n \rightharpoonup^* \varrho \ \text{in } L^\infty(0, T; L^\gamma(\Omega)),$$

$$\vartheta_n \rightharpoonup \vartheta \ \text{in } L^2(0, T; W^{1,2}(\Omega)),$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \ \text{in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3)),$$

and the trio $(\varrho, \vartheta, \mathbf{u})$ *is a weak solution of the complete Navier-Stokes-Fourier system* (323)–(332) *with initial data* $(\varrho_0, \vartheta_0, \mathbf{u}_0)$.

Remark 20. 1. It is to be noticed that Theorem 42 can be proved with less restrictive conditions on the heat conductivity κ: One can admit heat conductivity is dependent on both density and temperature, namely, $C^1([0,\infty) \times [0,\infty)) \ni \kappa = \kappa(\varrho, \vartheta)$ enjoying bounds (377) (see [32]). However, in spite of the available compactness result in this situation, and in contrast with the case $\kappa = \kappa(\vartheta)$, the construction of weak solutions under condition $\kappa = \kappa(\varrho, \vartheta)$ remains an open problem.

10.1 Estimates and Weak Limits

10.1.1 Estimates

Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ be a sequence of weak solutions of the problem (323)–(332) on $(0, T) \times \Omega$. Any trio of this sequence satisfies, in particular, the dissipation inequality $(361)_{\overline{\mathbf{u}}=0}$. The dissipation inequality will produce most of a priori estimates that are available in this problem. It will be convenient to split $H_{\overline{\vartheta}}(\varrho, \vartheta) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$ according to (356). Employing (31), (369), (373), and (419), we obtain

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) = \int_{\overline{\vartheta}}^{\vartheta} \partial_\vartheta H_{\overline{\vartheta}}(\varrho, z) \mathrm{d}z \geq 4a \int_{\overline{\vartheta}}^{\vartheta} z^2 (z - \overline{\vartheta}) \mathrm{d}x, \tag{417}$$

and

$$H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})(\varrho - \tilde{r}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) = \int_{\overline{\varrho}}^{\varrho} \Big(\int_{\tilde{r}}^{z} \partial_\varrho^2 H(w, \overline{\vartheta}) \mathrm{d}w \Big) \mathrm{d}z \tag{418}$$

$$\sim \Big[\varrho \log(\varrho/\overline{\varrho}) - \Big(\varrho - \overline{\varrho}\Big) \Big] + \Big[\varrho^\gamma - \gamma \overline{\varrho}^{\gamma-1}(\varrho - \overline{\varrho}) - \overline{\varrho}^\gamma \Big],$$

where we have used the equivalence

$$P'(Z) \sim 1 + Z^{\gamma-1}, \quad Z > 0, \tag{419}$$

that can be derived from (371)–(372). With observations (417)–(418) at hand, and using the conservation of mass

$$\int_\Omega \varrho_n \, \mathrm{d}x = M_0, \tag{420}$$

we deduce from the dissipation balance (361) the following estimates:

$$\mathrm{esssup}_{(0,T)} \int_\Omega \varrho_n \mathbf{u}_n^2 \mathrm{d}x \leq c(M_0, E_0, S_0, T), \tag{421}$$

$$\mathrm{esssup}_{(0,T)} \int_\Omega \varrho_n^\gamma \mathrm{d}x \le c(M_0, E_0, S_0, T), \tag{422}$$

$$\mathrm{esssup}_{(0,T)} \int_\Omega \vartheta_n^4 \mathrm{d}x \le c(M_0, E_0, S_0, T). \tag{423}$$

By virtue of (421)–(422), we deduce for the momentum,

$$\|\varrho_n \mathbf{u}_n\|_{L^\infty(0,T;L^{2\gamma/(\gamma+1)}(\Omega;\mathbb{R}^3))} \le c(M_0, E_0, S_0, T). \tag{424}$$

The "velocity part" of the entropy production yields bounds

$$\|\mathbb{T}(\nabla_x \mathbf{u}_n)\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^{3\times 3}))} + \int_0^T \int_\Omega \frac{1}{\vartheta_n} |\mathbb{T}(\nabla_x \mathbf{u}_n)|^2 \mathrm{d}x\mathrm{d}t \le c(M_0, E_0, S_0, T); \tag{425}$$

whence employing first the Korn type theorem (see Theorem 9) and then the standard Poincaré inequality, we get

$$\|\mathbf{u}_n\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \le c(M_0, E_0, S_0, T). \tag{426}$$

The "temperature part" of the entropy production rate gives

$$\begin{gathered} \|\nabla_x \vartheta_n^\beta\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \le c(M_0, E_0, S_0, T), \quad \beta \in [1, 3/2], \\ \|\nabla_x \log \vartheta_n\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \le c(M_0, E_0, S_0, T). \end{gathered} \tag{427}$$

In agreement with (374)–(375),

$$|\varrho s(\varrho, \vartheta)| \le c(\varrho + \varrho|\log \varrho| + \varrho|\log \vartheta| + \vartheta^3). \tag{428}$$

With this observation at hand, we verify that assumptions of Lemma 1 are satisfied with some $3 < p < 4$. Therefore, we deduce from (427) and the Poincaré-type inequality from Theorem 6,

$$\|\log \vartheta_n - \log \overline{\vartheta}\|_{L^2(0,T;W^{1,2}(\Omega)} + \|\vartheta_n^\beta - \overline{\vartheta}^\beta\|_{L^2(0,T;W^{1,2}(\Omega)} \le c(M_0, E_0, S_0, T), \ \beta \in [1, 3/2]. \tag{429}$$

We get by the Sobolev imbedding and by interpolation from (421)–(429) using eventually (369)–(377)

$$\|\vartheta_n\|_{L^3(0,T;L^9(\Omega))} \le c(M_0, E_0, S_0, T), \quad \|\vartheta_n\|_{L^{17/3}((0,T)\times\Omega)} \le c(M_0, E_0, S_0, T), \tag{430}$$

$$\|\mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n)\|_{L^2(0,T;L^{4/3}(\Omega;\mathbb{R}^{3\times 3}))} \le c(M_0, E_0, S_0, T), \tag{431}$$

$$\|\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n)/\vartheta_n\|_{L^2(0,T;L^{8/7}(\Omega;\mathbb{R}^3))} \le c(M_0, E_0, S_0, T), \tag{432}$$

$$\|\varrho_n s(\varrho_n, \vartheta_n)\|_{L^\infty(0,T;L^q(\Omega))} \le c(M_0, E_0, S_0, T) \text{ with some } q > 1, \tag{433}$$

$$\|\varrho_n s(\varrho_n, \vartheta_n)\mathbf{u}_n\|_{L^q((0,T)\times\Omega;\mathbb{R}^3)} \le c(M_0, E_0, S_0, T) \text{ with some } q > 1. \tag{434}$$

Under assumptions (369)–(372)

$$|p(\varrho,\vartheta)| \leq c(\varrho\vartheta + \varrho^{\gamma} + \vartheta^{4}). \tag{435}$$

Consequently, we can deduce from (422)–(423) only $\|p(\varrho_n,\vartheta_n\|_{L^\infty(0,T;L^1(\Omega))} \leq c(M_0, E_0, S_0, T)$. We however need for the pressure better estimate than an estimate in $L^1(\Omega)$. To improve this estimate, we use in the momentum equation (340) (written with $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ on Ω) the test function $\varphi = \eta(t)\mathcal{B}_\Omega[\varrho_n^\omega - \frac{1}{|\Omega|}\int_\Omega \varrho_n^\omega \,\mathrm{d}x]$, where $\omega > 0$, $\eta \in C^1_c(0,T)$ and $\mathcal{B}$ is the Bogovskii operator introduced in Theorem 5. A straightforward but laborious calculation (the same as exposed in (262)) leads to the conclusion that

$$\begin{aligned} &\int_0^T\int_\Omega p(\varrho_n,\vartheta_n)\varrho_n^\omega \mathrm{d}x\mathrm{d}t \leq c(M_0, E_0, S_0, T, \omega) \text{ with some } \omega > 0,\\ &\int_0^T\int_\Omega |p(\varrho_n,\vartheta_n)|^q \mathrm{d}x\mathrm{d}t \leq c(M_0, E_0, S_0, T, q) \text{ with some } q > 1. \end{aligned} \tag{436}$$

10.1.2 Weak Limits

Estimates derived in the previous sections together with equations (339), (340), (133)–(134) written with $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ give rise to the following convergence relations for a chosen subsequence denoted again $(\varrho_n, \vartheta_n, \mathbf{u}_n)$:

$$\begin{aligned} \varrho_n &\rightharpoonup^* \varrho \text{ in } L^\infty(0,T;L^\gamma(\Omega)),\\ \vartheta_n &\rightharpoonup^* \vartheta \text{ in } L^\infty(0,T;L^4(\Omega)),\\ \mathbf{u}_n &\rightharpoonup \mathbf{u} \text{ in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)),\\ \vartheta_n &\rightharpoonup \vartheta \text{ in } L^2(0,T;W^{1,2}(\Omega)), \end{aligned} \tag{437}$$

and

$$\text{sequences } \varrho_n, b(\varrho_n), \varrho_n\mathbf{u}_n, \varrho_n\mathbf{u}_n\otimes\mathbf{u}_n \text{ verify convergence relations (269)} \tag{438}$$

(see relations (267)–(269) for the similar reasoning). Moreover, if we denote by $\overline{g(\varrho,\vartheta,\mathbf{u})}$ weak limit of the sequence $g(\varrho_n,\vartheta_n,\mathbf{u}_n)$ in $L^1((0,T)\times\Omega))$, we have for the nonlinear quantities

$$\begin{aligned} \log\vartheta_n &\rightharpoonup \overline{\log\vartheta} \text{ in } L^2(0,T;W^{1,2}(\Omega)),\\ p(\varrho_n,\vartheta_n) &\rightharpoonup \overline{p(\varrho,\vartheta)} \text{ in } L^q((0,T)\times\Omega) \text{ with some } q > 1,\\ \mathbb{S}(\vartheta_n,\nabla_x\mathbf{u}_n) &\rightharpoonup \overline{\mathbb{S}(\vartheta,\nabla_x\mathbf{u})} \text{ in } L^{4/3}((0,T)\times\Omega;\mathbb{R}^{3\times3}), \end{aligned} \tag{439}$$

$$\varrho_n s(\varrho_n, \vartheta_n) \rightharpoonup \overline{\varrho s(\varrho, \vartheta)} \text{ in } L^q((0,T)\times\Omega) \text{ with some } q>1,$$

$$\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n)/\vartheta_n \rightharpoonup \overline{\mathbf{q}(\vartheta, \nabla_x\vartheta)/\vartheta} \text{ in } L^{8/7}((0,T)\times\Omega;\mathbb{R}^3).$$

The main goal in what follows is to "remove" bars over all nonlinear quantities in the weak limits (439). This will be done if we show convergence almost everywhere in Q_T of the sequences ϱ_n and ϑ_n.

10.1.3 Limiting Momentum, Continuity, and Renormalized Continuity Equations

Now, we are ready to let $n\to\infty$ in the weak formulation of the momentum equation, continuity equation, and the renormalized continuity equation. We have, similarly as in (271)–(274), in particular:

(1) Limiting momentum equation

$$\int_\Omega \varrho(\tau,x)\varphi(\tau,x)\,\mathrm{d}x - \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\,\mathrm{d}x \tag{440}$$

$$= \int_0^T\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + \overline{p(\varrho,\vartheta)}\mathrm{div}_x\varphi - \overline{\mathbb{S}(\vartheta,\nabla_x\mathbf{u})} : \nabla_x\varphi\Big)\,\mathrm{d}x\mathrm{d}t$$

for all $\tau\in[0,T]$ and for any $\varphi\in C^1_c([0,T]\times\overline{\Omega};\mathbb{R}^3)$, $\varphi|_{\partial\Omega}=0$;

(2) Limiting continuity and renormalized continuity equations in form

$$\text{equations (271), (273), (274) hold,} \tag{441}$$

where functions T_k, L_k are defined in (250), cf. formulas (248)–(249).

10.2 Strong Convergence of Temperature

10.2.1 Entropy Production Rate as a Nonnegative Radon Measure

Entropy balance (341) can be rewritten as identity

$$\int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\varphi(0,\cdot)\mathrm{d}x + <\sigma_n,\varphi> \tag{442}$$

$$= -\int_0^T\int_\Omega\left(\varrho_n s(\varrho_n,\vartheta_n)\partial_t\varphi + \varrho_n s(\varrho_n,\vartheta_n)\mathbf{u}_n\cdot\nabla_x\varphi + \frac{\mathbf{q}(\vartheta_n,\nabla_x\vartheta_n)\cdot\nabla_x\varphi}{\vartheta_n}\right)\,\mathrm{d}x\mathrm{d}t,$$

where σ_n is a nonnegative linear functional on the space $C^1_c([0,T)\times\overline{\Omega})$ defined by the above equation. According to (90), (92), there is a sequence of continuous linear functionals $\Sigma_n\in(C([0,T]\times\overline{\Omega}))^*$,

$$\|\Sigma_n\|_{(C[0,T]\times\overline{\Omega}))^*} \le c(M_0,E_0,S_0,T), \tag{443}$$

such that

$$< \Sigma_n, \varphi >_{C([0,T]\times\overline{\Omega})} = < \sigma_n, \varphi > \quad \text{for all } \varphi \in C_c^1([0,T)\times\overline{\Omega}).$$

10.2.2 A Consequence of Div-Curl Lemma

We may now apply the Div-Curl lemma (see Theorem 14) to the four-dimensional vector fields

$$\mathbf{V}_n = (\varrho_n s(\varrho_n, \vartheta_n), \varrho_n s(\varrho_n, \vartheta_n)\mathbf{u}_n + \mathbf{q}(\vartheta_n, \nabla_x \vartheta_n)/\vartheta_n), \qquad \mathbf{W}_n = (\mathcal{T}_k(\vartheta_n), 0, 0, 0).$$

Since $\mathrm{div}\mathbf{V}_n = \Sigma_n$ and since the imbedding $(C([0,T]\times\overline{\Omega}))^* \hookrightarrow W^{-1,q}((0,T)\times\Omega)$ is compact for any $q \in (1, 4/3)$, the assumptions of the lemma on $(0,T)\times\Omega$ are satisfied. Therefore,

$$\overline{\mathcal{T}_k(\vartheta)\varrho s_{\mathrm{mo}}(\varrho,\vartheta)} + \frac{4}{3}a\overline{\mathcal{T}_k(\vartheta)\vartheta^3} = \overline{\mathcal{T}_k(\vartheta)}\;\overline{\varrho s_{\mathrm{mo}}(\varrho,\vartheta)} + \frac{4}{3}a\overline{\mathcal{T}_k(\vartheta)}\;\overline{\vartheta^3}, \tag{444}$$

where $s_{\mathrm{mo}}(\varrho,\vartheta) = S(\varrho/\vartheta^{\frac{1}{\gamma-1}})$.

We shall first prove that

$$\overline{\mathcal{T}_k(\vartheta)\varrho s_{\mathrm{mo}}(\varrho,\vartheta)} \geq \overline{\mathcal{T}_k(\vartheta)}\;\overline{\varrho s_{\mathrm{mo}}(\varrho,\vartheta)}, \tag{445}$$

where

$$\mathcal{T}_k(z) = k\mathcal{T}(z/k), \quad C[0,\infty) \ni \mathcal{T} = \left\{ \begin{array}{c} z \text{ if } z \in [0,1], \\ \mathcal{T} \text{ strictly increasing on } [0,\infty), \\ \lim_{z\to\infty}\mathcal{T}(z) = 2. \end{array} \right\}$$

To this end we write

$$\begin{aligned} &\varrho_n s_{\mathrm{mo}}(\varrho_n,\vartheta_n)\Big(\mathcal{T}_k(\vartheta_n) - \overline{\mathcal{T}_k(\vartheta)}\Big) = \\ &\varrho_n\Big[s_{\mathrm{mo}}\Big(\varrho_n, \mathcal{T}_k^{-1}\Big(\mathcal{T}_k(\vartheta_n)\Big)\Big) - s_{\mathrm{mo}}\Big(\varrho_n, \mathcal{T}_k^{-1}\Big(\overline{\mathcal{T}_k(\vartheta)}\Big)\Big)\Big]\Big(\mathcal{T}_k(\vartheta_n) - \overline{\mathcal{T}_k(\vartheta)}\Big) \\ &+\varrho_n s_{\mathrm{mo}}\Big(\varrho_n, \mathcal{T}_k^{-1}\Big(\overline{\mathcal{T}_k(\vartheta)}\Big)\Big)\Big(\mathcal{T}_k(\vartheta_n) - \overline{\mathcal{T}_k(\vartheta)}\Big). \end{aligned}$$

Therefore, inequality (445) will be shown if we prove that

$$\varrho_n s_{\mathrm{mo}}\Big(\varrho_n, \mathcal{T}_k^{-1}\Big(\overline{\mathcal{T}_k(\vartheta)}\Big)\Big)\Big(\mathcal{T}_k(\vartheta_n) - \overline{\mathcal{T}_k(\vartheta)}\Big) \rightharpoonup 0 \text{ weakly in } L^1((0,T)\times\Omega) \text{ as } n \to \infty. \tag{446}$$

10.2.3 Application of Theorem on Parametrized Young Measures

The quantity

$$\varrho_n s_{\rm mo}\Big(\varrho_n, \mathcal{T}_k^{-1}\Big(\overline{\mathcal{T}_k(\vartheta)}\Big)(t,x)\Big) = \psi(t,x,\varrho_n) \tag{447}$$

can be regarded as a composition of a Carathéodory function with a weakly convergent sequence ϱ_n.

Since according to (437), (438)

$$b(\varrho_n) \to \overline{b(\varrho)} \text{ in } L^2(0,T;W^{-1,2}(\Omega)),$$
$$G(\vartheta_n) \to \overline{G(\vartheta)} \text{ in } L^2(0,T;W^{1,2}(\Omega)),$$

we have

$$\overline{b(\varrho)G(\vartheta)} = \overline{b(\varrho)}\;\overline{G(\vartheta)} \tag{448}$$

for any b and $G \in W^{1,\infty}((0,\infty))$, $zb' - b \in L^\infty(0,\infty)$. This implies (446) by virtue of the fundamental theorem on parametrized Young measures (see Theorem 17).

Indeed, denote $\nu_{(t,x)}^{\varrho,\vartheta}$, $\nu_{(t,x)}^{\varrho}$, and $\nu_{(t,x)}^{\vartheta}$ the parametrized Young measures corresponding, in accordance with Theorem 17, to the sequences (ϱ_n, ϑ_n), ϱ_n, and ϑ_n, respectively. Then we have, due to (448) and in agreement with Theorem 17,

$$\int_{\mathbb{R}^2} h(\lambda)G(\mu)\mathrm{d}\nu^{(\varrho,\vartheta)}(\lambda,\mu) = \int_{\mathbb{R}} h(\lambda)\mathrm{d}\nu^{\varrho}(\lambda) \times \int_{\mathbb{R}} G(\mu)\mathrm{d}\nu^{\vartheta}(\mu).$$

Consequently,

$$\overline{\psi(t,x,\varrho)G(\vartheta)}(t,x) = \int_{\mathbb{R}^2} \psi(t,x,\lambda)\, G(\mu)\,\mathrm{d}\nu_{(t,x)}^{\varrho}(\lambda)\,\mathrm{d}\nu_{(t,x)}^{\vartheta}(\mu) = \Big(\overline{\psi(t,x,\varrho)}\;\overline{G(\vartheta)}\Big)(t,x).$$

10.2.4 Monotone Functions Versus Weak Convergence

Now we shall use the properties of monotone operators with respect to the weak convergence reported in Theorem 4. Theorem 4 implies, in particular,

$$\overline{\mathcal{T}_k(\vartheta)\vartheta^3} \ge \overline{\mathcal{T}_k(\vartheta)}\;\overline{\vartheta^3},$$

that in turn with (444)–(445) yields

$$\overline{\mathcal{T}_k(\vartheta)\vartheta^3} = \overline{\mathcal{T}_k(\vartheta)}\;\overline{\vartheta^3},$$

and finally, by monotone convergence, as $k \to \infty$,

$$\overline{\vartheta^4} = \vartheta\overline{\vartheta^3}. \tag{449}$$

The last identity implies

$$\vartheta_n \to \vartheta \text{ a.e. in } (0, T) \times \Omega \tag{450}$$

by virtue of (105) in Theorem 4 and Theorem 3.

10.2.5 Weak Limits of the Momentum Equation and Entropy Balance

Coming back with (450) to the momentum equation (440), we obtain

$$- \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot)\, \mathrm{d}x \tag{451}$$

$$= \int_0^T \int_\Omega \Big(\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \overline{p(\varrho, \vartheta)} \mathrm{div}_x \varphi - \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \varphi + \varrho \mathbf{f} \cdot \varphi \Big)\, \mathrm{d}x \mathrm{d}t$$

for any $\varphi \in C_c^1([0, T) \times \overline{\Omega}; \mathbb{R}^3)$, $\varphi|_{\partial\Omega} = 0$.

Moreover, estimates (425), (427) yield boundedness of the sequences

$$\sqrt{\frac{\mu(\vartheta_n)}{\vartheta_n}} \Big(\nabla_x \mathbf{u}_n + (\nabla_x \mathbf{u}_n)^T - \frac{2}{3} \mathrm{div} \mathbf{u}_n \Big), \ \sqrt{\frac{\eta(\vartheta_n)}{\vartheta_n}} \mathrm{div} \mathbf{u}_n, \ \frac{\sqrt{\kappa(\vartheta_n)}}{\vartheta_n} \nabla_x \vartheta_n \tag{452}$$

in $L^2((0, T) \times \Omega))$; whence by the lower weak continuity combined with (450) and (437), one gets

$$\int_0^T \int_\Omega \frac{\varphi}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \mathrm{d}x \mathrm{d}t \tag{453}$$

$$\leq \liminf_{n \to \infty} \int_0^T \int_\Omega \frac{\varphi}{\vartheta} \left(\mathbb{S}(\vartheta_n, \nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n - \frac{\mathbf{q}(\vartheta_n, \nabla_x \vartheta_n) \cdot \nabla_x \vartheta_n}{\vartheta_n} \right) \mathrm{d}x \mathrm{d}t,$$

for any $0 \leq \varphi \in C_c([0, T] \times \overline{\Omega})$.

Thus effectuating the limit $n \to \infty$ in (341) (with $\varrho_n, \vartheta_n, \mathbf{u}_n$ in place of $\varrho, \vartheta, \mathbf{u}$), we get

$$\begin{aligned} &\int_\Omega \varrho_0 s(\varrho_0, \vartheta_0) \varphi(0, \cdot)\, \mathrm{d}x + \int_0^T \int_\Omega \frac{\varphi}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \mathrm{d}x \mathrm{d}t \\ &\leq - \int_0^T \int_\Omega \left(\overline{\varrho s(\varrho, \vartheta)} \partial_t \varphi + \overline{\varrho s(\varrho, \vartheta) \mathbf{u}} \cdot \nabla_x \varphi + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \varphi \right) \mathrm{d}x \mathrm{d}t \end{aligned} \tag{454}$$

for any $\varphi \in C_c^1([0, T) \times \overline{\Omega})$, $\varphi \geq 0$.

10.3 Strong Convergence of Density

10.3.1 Effective Viscous Flux Identity

The main result of this section is the following lemma.

Lemma 15 (see [33, formula (3.324) and its proof in Section 3.7.4]). *Let* $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ *be the sequence investigated in Theorem 42. Then for any* $k > 1$, *there holds*

$$\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} = \Big(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\Big)\Big(\overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x\mathbf{u}\Big) \tag{455}$$

with functions T_k *defined in* (250).

Proof. Repeating step-by-step proof of Lemma 10, we arrive at identity

$$\begin{aligned}&\int_0^T\int_\Omega \zeta\tilde{\zeta}\Big(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\Big)\,\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\int_\Omega \zeta(t,x)\Big(\overline{\mathbb{S}(\vartheta,\mathbf{u}):\mathcal{R}[\tilde{\zeta}T_k(\varrho)]} - \mathbb{S}(\vartheta,\mathbf{u}):\mathcal{R}[\tilde{\zeta}\overline{T_k(\varrho)}]\Big)\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{456}$$

where $\zeta, \tilde{\zeta} \in C_c^\infty((0,T)\times\Omega)$.

In order to write the right-hand side of formula (456) in the form of the right-hand side of formula (455), we use properties listed in *item* (iii) of Theorem 13 yielding identity

$$\begin{aligned}\int_0^T\int_\Omega \zeta\mathbb{S}(\vartheta,\mathbf{u}):\mathcal{R}[\tilde{\zeta}T_k(\varrho)]\,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\Omega \zeta\tilde{\zeta}\Big(-\frac{2}{3}\mu(\vartheta)+\eta(\vartheta)\Big)T_k(\varrho)\,\mathrm{div}_x\mathbf{u}\,\mathrm{d}x\mathrm{d}t\\ +\int_0^T\int_\Omega \tilde{\zeta}T_k(\varrho)\Big\{\mathcal{R}:\Big[\zeta\mu(\vartheta)\Big(\nabla_x\mathbf{u}+(\nabla_x\mathbf{u})^T\Big)\Big] &- \zeta\mu(\vartheta)\mathcal{R}:\Big[\nabla_x\mathbf{u}+(\nabla_x\mathbf{u})^T\Big]\Big\}\,\mathrm{d}x\mathrm{d}t\\ +\int_0^T\int_\Omega \zeta\tilde{\zeta}T_k(\varrho)\mu(\vartheta)\mathcal{R}&:\Big[\nabla_x\mathbf{u}+(\nabla_x\mathbf{u})^T\Big]\,\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $\mathcal{R}:(\mathbb{Z}) = \sum_{i,j=1}^3 \mathcal{R}_{ij}(Z_{ij})$ and $\mathcal{R}:[\nabla_x\mathbf{u}+(\nabla_x\mathbf{u})^T] = 2\mathrm{div}_x\mathbf{u}$. Consequently,

$$\begin{aligned}\int_0^T\int_\Omega \overline{\zeta\mathbb{S}(\vartheta,\mathbf{u}):\mathcal{R}[\tilde{\zeta}T_k(\varrho)]}\,\mathrm{d}x\mathrm{d}t &= \lim_{n\to\infty}\int_0^T\int_\Omega \zeta\tilde{\zeta}\Big(\frac{4}{3}\mu(\vartheta_n)+\eta(\vartheta_n)\Big)T_k(\varrho_n)\,\mathrm{div}_x\mathbf{u}_n\,\mathrm{d}x\mathrm{d}t\\ &+\lim_{n\to\infty}\int_0^T\int_\Omega \tilde{\zeta}T_k(\varrho_n)\omega(\vartheta_n,\mathbf{u}_n)\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{457}$$

and

$$\begin{aligned}\int_0^T\int_\Omega \zeta\mathbb{S}(\vartheta,\mathbf{u}):\mathcal{R}[\tilde{\zeta}\overline{T_k(\varrho)}]\,\mathrm{d}x\mathrm{d}t &= \int_0^T\int_\Omega \zeta\tilde{\zeta}\Big(\frac{4}{3}\mu(\vartheta)+\eta(\vartheta)\Big)\overline{T_k(\varrho)}\,\mathrm{div}_x\mathbf{u}\,\mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_\Omega \tilde{\zeta}\,\overline{T_k(\varrho)}\omega(\vartheta,\mathbf{u})\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{458}$$

where

$$\omega(\vartheta_n, \mathbf{u}_n) = \Big(\mathcal{R}\Big[\zeta(t,x)\mu(\vartheta_n)\big(\nabla\mathbf{u}_n + (\nabla\mathbf{u}_n)^T\big)\Big] - \zeta(t,x)\mu(\vartheta_n)\mathcal{R} : \big[\nabla\mathbf{u}_n + (\nabla\mathbf{u}_n)^T\big]\Big).$$

In order to treat the difference between the last terms in formulas (457) and (458), we will need two compensated compactness results: Div-Curl lemma reported in Theorem 14 and a specific commutator lemma reported in Theorem 16. Thanks to Theorem 16, the sequence

$$\omega(\vartheta_n, \mathbf{u}_n) \text{ is bounded in } L^1(0,T; W^{\beta,q}(\Omega;\mathbb{R}^3)) \text{ with some } \beta \in (0,1), q > 1. \tag{459}$$

Now we consider the four-dimensional vector fields

$$\mathbf{V}_n \equiv [T_k(\varrho_n), T_k(\varrho_n)\mathbf{u}_n], \quad \mathbf{U}_n \equiv [\omega(\vartheta_n, \mathbf{u}_n), 0, 0, 0].$$

Seeing that $\mathrm{curl}\mathbf{U}_n$ is compact in $W^{-1,r}((0,T)\times\Omega;\mathbb{R}^{3\times 3})$ with some $r > 1$ by virtue of (459), (423), (430) (and of course $\mathrm{div}\mathbf{V}_n$ is compact in $W^{-1,r}((0,T)\times\Omega;\mathbb{R}^{3\times 3})$ because of the fact that $(\varrho_n, \mathbf{u}_n)$ satisfies renormalized continuity equation (133)–(134), with $b = T_k$) we may employ Div-Curl lemma reported in Theorem 14 to get

$$\omega(\vartheta_n, \mathbf{u}_n)\, T_k(\varrho_n) \rightharpoonup \overline{\omega(\vartheta, \mathbf{u})}\ \overline{T_k(\varrho)}, \text{ in } L^1((0,T)\times\Omega),$$

where, due to (450),

$$\overline{\omega(\vartheta, \mathbf{u})} = \omega(\vartheta, \mathbf{u}).$$

This result in combination with (456) and (457)–(458) yields the effective viscous flux identity (455).

10.3.2 Oscillations Defect Measure

Going back to (419), we deduce employing the hypotheses (369)–(372) that

$$p(\varrho, \vartheta) = d\varrho^\gamma + p_m(\varrho, \vartheta), \text{ for some } d > 0, \tag{460}$$

where $\partial_\varrho p_m(\varrho, \vartheta) \geq 0$. Reasoning as in (285), we get

$$d \limsup_{n\to 0} \int_0^T \int_\Omega \frac{\zeta}{1+\vartheta} |T_k(\varrho_n) - T_k(\varrho)|^{\gamma+1}\, \mathrm{d}x\mathrm{d}t \tag{461}$$

$$\leq d \limsup_{n\to\infty} \int_0^T \int_\Omega \frac{\zeta}{1+\vartheta} \Big((T_k(\varrho_n) - T_k(\varrho)\Big)(\varrho_n^\gamma - \varrho^\gamma)\, \mathrm{d}x\mathrm{d}t$$

$$\leq d\int_0^T\int_\Omega \frac{\zeta}{1+\vartheta}\Big(\overline{\varrho^\gamma\, T_k(\varrho)} - \overline{\varrho^\gamma}\ \overline{T_k(\varrho)}\Big)\mathrm{d}x\mathrm{d}t \leq \int_0^T\int_\Omega \frac{\zeta}{1+\vartheta}\Big(\overline{p(\varrho,\vartheta)\, T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\Big)\mathrm{d}x\mathrm{d}t,$$

with any $\zeta \in C_c^\infty((0,T)\times\Omega)$, $\zeta \geq 0$. To derive the last inequality in formula (461), we have employed decomposition (460), the fact that

$$weak - \lim_{n\to\infty} p(\varrho_n, \vartheta_n) g(\varrho_n) \equiv \overline{p(\varrho, \vartheta) g(\varrho)} = \overline{p(\cdot, \vartheta) g(\cdot)} \equiv weak - \lim_{n\to\infty} p(\varrho_n, \vartheta) g(\varrho_n)$$

for any bounded function g (that holds, thanks to the almost everywhere convergence of the sequence ϑ_n; see (450)), and the relation between the weak limits of monotone functions

$$\overline{p_m(\cdot, \vartheta) T_k(\cdot)} - \overline{p_m(\cdot, \vartheta)}\ \overline{T_k(\varrho)} \geq 0, \tag{462}$$

reported in Theorem 4.

Next, we verify that

$$\int_0^T \int_\Omega |T_k(\varrho_n) - T_k(\varrho)|^q \mathrm{d}x = \int_0^T \int_\Omega \frac{1}{(1+\vartheta_n)^\omega} |T_k(\varrho_n) - T_k(\varrho)|^q \Big(1 + \vartheta_n\Big)^\omega \,\mathrm{d}x\mathrm{d}t$$
$$\leq c \Big[\int_0^T \int_\Omega \frac{1}{1+\vartheta_n} |T_k(\varrho_n) - T_k(\varrho)|^{\gamma+1} \,\mathrm{d}x\mathrm{d}t \Big]^{q/(\gamma+1)},$$

where $q > 2$, provided $\omega(\gamma+1) = q$ and $\omega(\gamma+1)/(\gamma+1-q) \leq 17/3$, cf. (430).

According to (461), expression

$$\int_0^T \int_\Omega \frac{1}{1+\vartheta_n} |T_k(\varrho_n) - T_k(\varrho)|^{\gamma+1} \,\mathrm{d}x\mathrm{d}t,$$

which stays at the right-hand side of the last inequality, can be estimated by calculating the right-hand side of (461) from the effective viscous flux identity (455). Now, reasoning as in Step 2 of the proof of Lemma 11, we show the following lemma:

Lemma 16. *Let* $(\varrho_n, \mathbf{u}_n)$ *be the density-velocity component of the sequence investigated in Theorem 42. Then*

$$\mathrm{osc}_q[\varrho_n \to \varrho](Q_T) \leq c(M_0, E_0, F_0, T) \ \textit{with some } q > 2, \tag{463}$$

where $\mathrm{osc}_q[\varrho_n \to \varrho](Q_T)$ *is defined in* (281).

10.3.3 Renormalized Continuity Equation and Strong Convergence of Density

Lemma 16 guarantees satisfaction of all hypotheses of Lemma 12. Using the latter lemma, we deduce that the weak limit $(\varrho, \mathbf{u})$ constructed in (437) verifies the renormalized continuity equation (133)–(134), in particular equation (292) holds. Recall that also (273) holds in our setting according to (441). We deduce from (273) and (292) with the help of the effective viscous flux identity (455),

$$\int_\Omega \Big(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\Big)(\tau)\,\mathrm{d}x = -\int_0^\tau \int_\Omega g_k \,\mathrm{d}x\mathrm{d}t, \text{ where } g_k = \overline{T_k(\varrho)\mathrm{div}\mathbf{u}} - T_k(\varrho)\mathrm{div}\mathbf{u}, \tag{464}$$

in particular,

$$g_k = g_k^1 + g_k^2,\ g_k^1 = \overline{T_k(\varrho)\mathrm{div}_x\mathbf{u}} - \overline{T_k(\varrho)}\mathrm{div}_x u,$$

$$g_k^2 = \frac{1}{\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)}\Big(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)}\Big).$$

Reasoning as in (294), we find that $\lim_{k\to\infty}\int_0^\tau\int_\Omega g_k^1\,\mathrm{d}x\mathrm{d}t = 0$, while $\int_0^\tau\int_\Omega g_k^2\,\mathrm{d}x\mathrm{d}t \geq 0$ by virtue of (460) and (462). Now we get from (464) exactly by the same argument leading to (300), completed and modified by decomposition (460), and relation (462) that this formula holds also in the present case. Formula (300) implies

$$\varrho_n \to \varrho \text{ a.e. in } Q_T$$

by virtue of Theorem 3. This is the last convergence relation needed to conclude the proof of Theorem 42.

11 Stability Results and Weak-Strong Uniqueness

The results presented in this section will rely on the relative energy inequality. They are based on paper [34], where the relative energy method for the Navier-Stokes-Fourier system has been introduced. We have observed in Theorem 39 under very mild assumptions on constitutive laws and transport coefficients that any bounded energy weak solution verifies relative energy inequality (362). If the thermodynamic stability conditions are satisfied, some of various consequences of the relative energy inequality are theorems on the stability of weak solutions with respect to strong solutions and on the weak-strong uniqueness principle. We shall formulate these results in several settings. The least requirement on the constitutive relations is contained in the following result:

Theorem 43 (see [31]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e are twice continuously differentiable on $(0,\infty)^2$ and verify the thermodynamic stability conditions (378).*

Let $(\varrho, \vartheta, \mathbf{u})$ be a bounded energy weak solution to the Navier-Stokes-Fourier system (323)–(332) in space time cylinder Q_T, $T > 0$ in the sense specified in Definition 8, emanating from the initial data (337), verifying in addition

$$0 < \underline{\varrho} < \varrho(t,x) < \overline{\varrho} < \infty, \quad 0 < \underline{\vartheta} < \vartheta(t,x) < \overline{\vartheta} < \infty \text{ for a.a. } (t,x) \in Q_T. \tag{465}$$

Finally, suppose that the Navier-Stokes-Fourier system admits a strong solution $(r > 0, \Theta > 0, \mathbf{U})$ in class

$$X \equiv \{\partial_t r,\ \partial_t\Theta,\ \partial_t\mathbf{U},\ \nabla_x^m r,\ \nabla_x^m\Theta,\ \nabla_x^m\mathbf{U} \in L^\infty(Q_T), \quad m = 0,1,2\} \tag{466}$$

emanating from the same initial data. Then

$$(\varrho, \vartheta, \mathbf{u}) = (r, \Theta, \mathbf{U}).$$

Remark 21. 1. It is to be noticed that conditions (323)–(332) include implicitly requirements (8), (14)–(15), in particular, that $e, p, s, \mu, \eta, \kappa$ are continuously differentiable functions of density and temperature and that e, p verify the Gibbs relation.
2. The drawback of this theorem dwells in the fact that it is not known whether weak solutions satisfying (465) do exist on (arbitrary large) time interval $(0, T)$.

There are however situations (characterized by the constitutive laws and transport coefficients) when the weak-strong uniqueness principle holds unconditionally in the class of weak solutions whose existence is guaranteed by Theorems 40 and 41. We report the following result:

Theorem 44 (see [34] for the original version of the result with $\gamma = 5/3, \beta = 1$; see [31] for the case $\gamma = 5/3$, $\beta \in (2/5, 1]$; see [70] for the case $\gamma > 3/2$, $\beta = 1$). *Let Ω be a bounded Lipschitz domain. Let the constitutive laws for e, p and transport coefficients μ, η, κ satisfy all assumptions of existence Theorem 40 or of existence Theorem 41. Assume further that the Third law*

$$\lim_{Z \to \infty} S(Z) = 0 \tag{467}$$

is verified and that the function P is twice continuously differentiable on $(0, \infty)$.

Let $(\varrho, \vartheta, \mathbf{u})$ be a bounded energy weak solution to the Navier-Stokes-Fourier system (323)–(332) in space time cylinder Q_T, $T > 0$ emanating from initial data $(\varrho_0, \vartheta_0, \mathbf{u}_0)$ in the class (337) and external force $\mathbf{f} \in L^\infty(Q_T; \mathbb{R}^3)$ constructed in Theorem 40 or 41 according to the case.

Let $(r > 0, \Theta > 0, \mathbf{U})$ be a strong solution to the Navier-Stokes-Fourier system (323)–(332) in class (466) emanating from the $(r_0, \Theta_0, \mathbf{U}_0) \in$ (337) and external force $\mathbf{g} \in L^\infty(Q_T; \mathbb{R}^3)$. Then there exists a positive constant c depending on the parameters of constitutive laws, transport coefficients, Q_T, lower bounds of r and Θ, and the norms of the strong solution in class (466) (but independent on the weak solution, initial data, and external forces) such that

$$\mathcal{E}(\varrho, \vartheta, \mathbf{u} \,|\, r, \Theta, \mathbf{U}) \le c\Big(\mathcal{E}(\varrho_0, \vartheta_0, \mathbf{u}_0 \,|\, r_0, \Theta_0, \mathbf{U}_0) + \|\mathbf{f} - \mathbf{g}\|^2_{L^\infty(Q_T)}\Big),$$

where $\mathcal{E}$ is the relative energy functional introduced in (357). In particular, if $(\varrho_0, \vartheta_0, \mathbf{u}_0) = (r_0, \Theta_0, \mathbf{U}_0)$ and $\mathbf{f} = \mathbf{g}$, then

$$(\varrho, \vartheta, \mathbf{u}) = (r, \Theta, \mathbf{U}).$$

Remark 22. 1. Theorems 43 and 44 remain true for the bounded energy weak solutions with the *complete slip* (22) boundary conditions (see [70, Section 6] for more details).
2. Since on bounded domains the class of finite energy weak solutions is contained in the class of bounded energy weak solutions, Theorems 43 and 44 are true also for the finite energy weak solutions.

3. Theorems 43 and 44 are formulated in the class of bounded energy weak solutions. They are however true also in the seemingly larger class of *dissipative solutions* since the proof relies basically on the relative energy inequality.
4. On unbounded domains with boundary conditions (24) one cannot, in general, construct finite energy weak solutions. In some situations satisfied by the conditions at infinity, one can construct bounded energy weak solutions on unbounded domains with uniformly Lipschitz boundary, provided $e, p, s, \mu, \eta, \kappa$ verify assumptions of Theorem 40 or 41. These bounded energy weak solutions are not necessarily dissipative solutions, and they do not verify the weak-strong uniqueness principle. In the class of bounded energy weak solutions, there are however solutions that are dissipative. Then the weak-strong uniqueness principle holds in the class of bounded energy dissipative solutions. The reader is advised to consult [70], Theorem 2.5 (for the no-slip boundary conditions) and Theorem 6.5 (for the complete slip boundary conditions) to learn more about these problems.
5. Under certain additional hypotheses, a strong solution $(r, \Theta, \mathbf{U})$ exists at least locally in time. For example: If Ω is a bounded domain of class C^4, $\mathbf{f} \in C^1([0,T]; W^{2,2}(\Omega))$, $c_v, \mu, \eta, \kappa \in C^3(0,\infty)$, $c_v \geq \underline{c}_v > 0$, $\mu \geq \underline{\mu} > 0$, $\kappa \geq \underline{\kappa} > 0$, if the initial data verify $0 < r_0 \in W^{3,2}(\Omega)$, $0 < \Theta_0 \in W^{3,2}(\Omega)$, $\mathbf{U}_0 \in W^{3,2}(\Omega; \mathbb{R}^3)$ and satisfy the natural and classical compatibility conditions at the boundary, then there exists $T_M > 0$ such that the Navier-Stokes-Fourier system (323)–(332) admits a unique strong solution $(r > 0, \Theta > 0, \mathbf{U})$ on the interval $[0, T_M)$ in a subclass of (466) (see [104, Theorem A and Remark 3.3]).
6. The weak-strong uniqueness principle turns some of the blow up criteria for strong solutions of the Navier-Stokes-Fourier system to the criteria of regularity of weak solutions (see [53] and [45] for more details about this issue).
7. In some situations, the assumption that the constitutive equations must verify the third law (see assumption (467) in Theorem 44) can be disregarded. This is the case of constitutive laws and weak solutions mentioned in Remark 18 as shown in [58, Theorem 4.1].

11.1 Sketch of the Proof of Theorems 43 and 44

11.1.1 Relative Energy Inequality with a Strong Solution as a Test Function

We denote

$$A = \Big(r(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}) + \nabla p(r,\Theta) - r\mathbf{g}\Big)\cdot(\mathbf{u}-\mathbf{U}) + \mathbb{S}(\Theta,\nabla\mathbf{U}) : \nabla(\mathbf{u}-\mathbf{U}),$$

and

$$B=(\vartheta-\Theta)\Big(r(\partial_t s(r,\Theta)+\mathbf{U}\cdot\nabla s(r,\Theta))-\frac{\mathbb{S}(\Theta,\nabla\mathbf{U}):\nabla\mathbf{U}}{\Theta}+\frac{\mathbf{q}(\Theta,\nabla\Theta)\cdot\nabla\Theta}{\Theta^2}\Big)+\frac{\mathbf{q}(\Theta,\nabla\Theta)\cdot\nabla(\Theta-\vartheta)}{\Theta}.$$

Since the trio $(r, \Theta, \mathbf{U})$ verifies equations (324)–(325) with (327) and boundary conditions (331)–(332) in the classical sense, there holds

$$\int_0^\tau \int_\Omega (A+B)\,\mathrm{d}x\mathrm{d}t = 0.$$

We now add this identity to the right-hand side of the relative energy inequality (362). Employing several times conveniently the Gibbs relation (326) in the form $\frac{1}{r}\partial_\vartheta p(r,\Theta) = -r\partial_\rho s(r,\Theta)$ and the continuity equation (323) satisfied by $(r,\mathbf{U})$, we transform after a long and tedious computation relative energy inequality (362) to the form stated in the following lemma:

Lemma 17. *Let Ω be a bounded Lipschitz domain and $\mathbf{f},\mathbf{g}\in L^\infty(Q_T,\mathbb{R}^3)$. Let $(\varrho,\vartheta,\mathbf{u})$ be a bounded energy weak solution to the Navier-Stokes-Fourier system emanating from initial data $(\varrho_0,\vartheta_0,\mathbf{u}_0)$ specified in (337) and external force $\mathbf{f}$. Let $(r>0,\Theta>0,\mathbf{U})$ be a strong solution of the same system emanating from initial data $(r_0,\Theta_0,\mathbf{U}_0)$ in (337) and external force $\mathbf{g}$, in the class (466). Then,*

$$\mathcal{E}(\varrho,\vartheta,\mathbf{u}\,|\,r,\Theta,\mathbf{U})(\tau)-\mathcal{E}(\varrho_0,\vartheta_0,\mathbf{u}_0\,|\,r_0,\Theta_0,\mathbf{U}_0) \tag{468}$$

$$+\int_0^\tau\int_\Omega D_{\rm mech}(t,x)\,{\rm d}x{\rm d}t+\int_0^\tau\int_\Omega D_{\rm th}(t,x)\,{\rm d}x{\rm d}t\le\int_0^\tau\int_\Omega R(t,x)\,{\rm d}x{\rm d}t$$

for a.a. $\tau\in(0,T)$, where

$$D_{\rm mech}=\frac{\Theta}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}-\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{U}+\mathbb{S}(\Theta,\nabla\mathbf{U}):\nabla(\mathbf{U}-\mathbf{u})+\frac{\vartheta-\Theta}{\Theta}\mathbb{S}(\Theta,\nabla\mathbf{U}):\nabla\mathbf{U},$$

$$D_{\rm th}=-\Big(\frac{\Theta}{\vartheta}\frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\vartheta-\frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\Theta$$
$$+\frac{\mathbf{q}(\Theta,\nabla\Theta)}{\Theta}\cdot\nabla(\Theta-\vartheta)+\frac{\vartheta-\Theta}{\Theta}\frac{\mathbf{q}(\Theta,\nabla\Theta)}{\Theta}\cdot\nabla\Theta\Big),$$

$$R=\Big((\varrho-r)\partial_t\mathbf{U}+(\varrho\mathbf{u}-r\mathbf{U})\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})+(\varrho\mathbf{f}-r\mathbf{g})\cdot(\mathbf{u}-\mathbf{U})$$
$$-\Big(\mathcal{S}(\varrho,\vartheta)-(\varrho-r)\partial_\varrho\mathcal{S}(r,\Theta)-(\vartheta-\Theta)\partial_\vartheta\mathcal{S}(r,\Theta)-\mathcal{S}(r,\Theta)\Big)\Big(\partial_t\Theta+\mathbf{U}\cdot\nabla\Theta\Big)$$
$$-\Big(p(\varrho,\vartheta)-(\varrho-r)\partial_\varrho p(r,\Theta)-(\vartheta-\Theta)\partial_\vartheta p(r,\Theta)-p(r,\Theta)\Big){\rm div}\mathbf{U}$$
$$+\varrho\Big(s(r,\Theta)-s(\varrho,\vartheta)\Big)(\mathbf{u}-\mathbf{U})\cdot\nabla_x\Theta+(\mathbf{f}-\mathbf{g})\cdot(\mathbf{u}-\mathbf{U}),$$

and

$$\mathcal{S}(\varrho,\vartheta)=\varrho s(\varrho,\vartheta). \tag{469}$$

11.1.2 Relative Energy Inequality Rewritten

We shall investigate separately the cases $0<\vartheta<\Theta$ and $\vartheta\ge\Theta$. In the first case, we have

$$1_{\{0<\vartheta<\Theta\}}\Big(\frac{\Theta}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}-\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{U}$$
$$+\mathbb{S}(\Theta,\nabla\mathbf{U}):\nabla(\mathbf{U}-\mathbf{u})+\frac{\vartheta-\Theta}{\Theta}\mathbb{S}(\Theta,\nabla\mathbf{U}):\nabla\mathbf{U}\Big)$$

$$\geq 1_{\{0<\vartheta<\Theta\}} \frac{\mu(\vartheta)}{2} |\mathbb{T}(\nabla_x(\mathbf{u}-\mathbf{U}))|^2 - 1_{\{0<\vartheta<\Theta\}} \Big| \frac{\Theta-\vartheta}{\Theta} \Big(\mathbb{S}(\vartheta,\nabla_x\mathbf{U}) - \mathbb{S}(\Theta,\nabla_x\mathbf{U})\Big) : \nabla_x\mathbf{U}$$

$$+\Big(\mathbb{S}(\vartheta,\nabla\mathbf{U}) - \mathbb{S}(\Theta,\nabla\mathbf{U})\Big) : \nabla_x(\mathbf{u}-\mathbf{U}) + 2\frac{\Theta-\vartheta}{\Theta}\mathbb{S}(\vartheta,\nabla\mathbf{U}) : \nabla_x(\mathbf{u}-\mathbf{U})\Big)\Big| \equiv I_1 - R_1,$$

where we have used definition of $\mathbb{S}$ and the convexity of the quadratic form $\mathbb{Z} \to \mathbb{S}(\vartheta,\mathbb{Z}) : \mathbb{Z}$, namely, the inequality $\mathbb{S}(\vartheta,\mathbb{Z}+\mathbb{H}) : (\mathbb{Z}+\mathbb{H}) - \mathbb{S}(\vartheta,\mathbb{Z}) : \mathbb{Z} \geq 2\mathbb{S}(\vartheta,\mathbb{Z}) : \mathbb{H}$, and where

$$I_1 = 1_{\{0<\vartheta<\Theta\}} \frac{\mu(\vartheta)}{2} |\mathbb{T}(\nabla_x(\mathbf{u}-\mathbf{U}))|^2 \geq 0.$$

In the second case, we write

$$1_{\{\vartheta\geq\Theta\}}\Big(\frac{\Theta}{\vartheta}\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u} - \mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{U} + \mathbb{S}(\Theta,\nabla\mathbf{U}) : \nabla(\mathbf{U}-\mathbf{u}) + \frac{\vartheta-\Theta}{\Theta}\mathbb{S}(\Theta,\nabla\mathbf{U}) : \nabla\mathbf{U}\Big)$$

$$= 1_{\{\vartheta\geq\Theta\}}\Big[\frac{\Theta}{\vartheta}\Big(\mathbb{S}(\vartheta,\nabla\mathbf{u}) - \mathbb{S}(\vartheta,\nabla\mathbf{U})\Big) : \nabla_x(\mathbf{u}-\mathbf{U}) + \Theta\Big(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{U})}{\vartheta} - \frac{\mathbb{S}(\Theta,\nabla\mathbf{U})}{\Theta}\Big) : \nabla_x(\mathbf{u}-\mathbf{U})$$

$$+(\vartheta-\Theta)\Big(\frac{\mathbb{S}(\Theta,\nabla\mathbf{U})}{\Theta} - \frac{\mathbb{S}(\vartheta,\nabla\mathbf{u})}{\vartheta}\Big) : \nabla_x\mathbf{U}\Big] \geq 1_{\{\vartheta\geq\Theta\}}\frac{\Theta}{\vartheta}\frac{\mu(\vartheta)}{2}|\mathbb{T}(\nabla_x(\mathbf{u}-\mathbf{U})|^2$$

$$-1_{\{\vartheta\geq\Theta\}}\Big|\Theta\Big(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{U})}{\vartheta} - \frac{\mathbb{S}(\Theta,\nabla\mathbf{U})}{\Theta}\Big) : \nabla_x(\mathbf{u}-\mathbf{U})$$

$$+(\vartheta-\Theta)\Big(\frac{\mathbb{S}(\Theta,\nabla\mathbf{U})}{\Theta} - \frac{\mathbb{S}(\vartheta,\nabla\mathbf{u})}{\vartheta}\Big) : \nabla_x\mathbf{U}\Big| \equiv I_2 - R_2,$$

where

$$I_2 = 1_{\{\vartheta\geq\Theta\}}\frac{\Theta}{\vartheta}\frac{\mu(\vartheta)}{2}|\mathbb{T}(\nabla_x(\mathbf{u}-\mathbf{U})|^2 \geq 0.$$

In the same spirit, we write

$$-\frac{\Theta}{\vartheta}\frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta} + \frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\Theta}{\vartheta} - \frac{\mathbf{q}(\Theta,\nabla\Theta)}{\Theta}\cdot\nabla(\Theta-\vartheta) - \frac{\vartheta-\Theta}{\Theta}\frac{\mathbf{q}(\Theta,\nabla\Theta)\cdot\nabla\Theta}{\Theta}$$

$$= \Theta\kappa(\vartheta)|\nabla(\log\vartheta - \log\Theta)|^2 + \Theta\Big(\mathbf{q}(\Theta,\nabla\log\Theta) - \mathbf{q}(\vartheta,\nabla\log\Theta)\Big)\cdot\nabla(\log\vartheta - \log\Theta)$$

$$+(\vartheta-\Theta)\mathbf{q}(\Theta,\nabla\log\Theta)\cdot\nabla(\log\vartheta - \log\Theta) \equiv I_3 - R_3,$$

where

$$I_3 = \Theta\kappa(\vartheta)|\nabla(\log\vartheta - \log\Theta)|^2 \geq 0.$$

We are now able to rewrite the relative energy inequality (468) in the form

$$\mathcal{E}(\varrho,\vartheta,\mathbf{u}\,|\,r,\Theta,\mathbf{U})(\tau)-\mathcal{E}(\varrho_0,\vartheta_0,\mathbf{u}_0\,|\,r_0,\Theta_0,\mathbf{U}_0)+\int_0^\tau\int_\Omega (I_1+I_2+I_3)\,\mathrm{d}x\mathrm{d}t \\ \leq \int_0^\tau\int_\Omega (R+R_1+R_2+R_3)\,\mathrm{d}x\mathrm{d}t. \tag{470}$$

11.1.3 Essential and Residual Sets

We set in Lemma 14

$$\underline{r}=\inf_{(t,x)\in Q_T} r(t,x),\ \overline{r}=\sup_{(t,x)\in Q_T} r(t,x),\ \underline{\Theta}=\inf_{(t,x)\in Q_T}\Theta(t,x),\ \overline{\Theta}=\sup_{(t,x)\in Q_T}\Theta(t,x).$$

Recalling the definition of the relative energy functional (357), we obtain by integrating (359) over Ω

$$\mathcal{E}(\varrho,\vartheta,\mathbf{u}\,|\,r,\Theta,\mathbf{U})(\tau) \\ \geq c\int_\Omega \Big([1]_{\rm res}+[\varrho]^\gamma_{\rm res}+[\vartheta]^4_{\rm res}+|[\varrho-r]_{\rm ess}|^2+|[\vartheta-\Theta]_{\rm ess}|^2\Big)\,\mathrm{d}x \text{ for a.a. } \tau\in(0,T), \tag{471}$$

where we have used the properties (369)–(375) of constitutive relations in the same way as in (417)–(418) and where we have denoted for a function $(t,x)\mapsto h(t,x)$

$$[h]_{\rm ess}(t,x)=h(t,x)1_{\mathcal{O}_{\rm ess}}(\varrho(t,x),\vartheta(t,x)),\quad [h]_{\rm res}(t,x)=1-[h]_{\rm ess}(t,x). \tag{472}$$

11.1.4 Proof of Theorem 43

We split the right-hand side of inequality (470) on its essential and residual parts as follows: $\int_0^\tau\int_\Omega(R+R_1+R_2+R_3)\,\mathrm{d}x\mathrm{d}t=\int_0^\tau\int_{N_{\rm ess}(t)}(R+R_1+R_2+R_3)\,\mathrm{d}x\mathrm{d}t+\int_0^\tau\int_{N_{\rm res}(t)}(R+R_1+R_2+R_3)\,\mathrm{d}x\mathrm{d}t$. A quick glance at the form of R, R_1, R_2, R_3 yields the estimate of the essential part

$$\int_0^\tau\int_{N_{\rm ess}(t)}(R+R_1+R_2+R_3)\,\mathrm{d}x\mathrm{d}t\leq c\int_0^\tau\int_\Omega\Big(\varrho(\mathbf{u}-\mathbf{V})^2+|\varrho-r|^2+|\vartheta-\Theta|^2\Big)\,\mathrm{d}x\mathrm{d}t, \tag{473}$$

by virtue of the Taylor formula and Cauchy-Schwarz inequality. Moreover, due to Lemma 14 and Taylor's formula, there is $\underline{c}=\underline{c}(\underline{r},\overline{r},\underline{\Theta},\overline{\Theta})>0$ and $\overline{c}=\overline{c}(\underline{r},\overline{r},\underline{\Theta},\overline{\Theta})>0$ such that

$$\underline{c}E(\varrho,\vartheta|r,\Theta)\leq(r-\varrho)^2+(\Theta-\vartheta)^2\leq\overline{c}E(\varrho,\vartheta|r,\Theta)$$

for all $(\varrho,\vartheta)\in\mathcal{O}_{\rm ess}$ and $(r,\Theta)\in[\underline{r},\overline{r}]\times[\underline{\Theta},\overline{\Theta}]$. Finally, due assumptions (465),

$$\int_0^\tau\int_{N_{\rm res}(t)}(R+R_1+R_2+R_3)\,\mathrm{d}x\mathrm{d}t=0.$$

Coming back with the last three formulas to relative energy inequality (470), while recalling that dissipation $\int_0^\tau \int_\Omega (I_1 + I_2 + I_3)\,\mathrm{d}x\mathrm{d}t$ is nonnegative, yields

$$\mathcal{E}(\varrho, \vartheta, \mathbf{u} \,|\, r, \Theta, \mathbf{U})(\tau) - \mathcal{E}(\varrho_0, \vartheta_0, \mathbf{u}_0 \,|\, r_0, \Theta_0, \mathbf{U}_0) \le \int_0^\tau \mathcal{E}(\varrho, \vartheta, \mathbf{u} \,|\, r, \Theta, \mathbf{U})(t)\,\mathrm{d}t. \tag{474}$$

The latter inequality yields the statement of Theorem 43 by direct application of the Gronwall's inequality.

11.1.5 Proof of Theorem 44

Estimates from Below for the Viscous and Heat Dissipation

In the case of Theorem 44, the residual part of integrals at the right-hand side of the relative energy inequality (470) enters into the game. We need to get as much as possible information from the viscous and thermal dissipation in order to close the estimates for as large as possible class of constitutive laws. We see immediately that,

$$\int_\Omega I_1\,\mathrm{d}x \ge c \int_{\{0<\vartheta<\Theta\}} |\mathbb{T}(\nabla_x \mathbf{u})|^2 \mathrm{d}x.$$

We get by using the Hölder inequality that

$$\|v\|^2_{L^r(\{\vartheta\ge\Theta\})} = \Big(\int_{\{\vartheta\ge\Theta\}} \Big[\frac{\vartheta}{\mu(\vartheta)}\Big]^{r/2} \Big[\frac{\mu(\vartheta)}{\vartheta}\Big]^{r/2} |v|^r \mathrm{d}x\Big)^{2/r}$$

$$\le \Big\|\Big[\frac{\vartheta}{\mu(\vartheta)}\Big]\Big\|_{L^{r/(2-r)}(\{\vartheta\ge\Theta\})} \int_{\{\vartheta\ge\Theta\}} \frac{\mu(\vartheta)}{\vartheta} |v|^2 \mathrm{d}x \le c \int_{\{\vartheta\ge\Theta\}} \frac{\mu(\vartheta)}{\vartheta} |v|^2 \mathrm{d}x$$

provided $r = \frac{8}{5-\beta}$, where the last inequality holds since $\vartheta \in L^\infty(0, T; L^4(\Omega))$. Consequently,

$$\int_{\{\vartheta\ge\Theta\}} I_2 \mathrm{d}x \ge c\Big(\int_{\{\vartheta\ge\Theta\}} |\mathbb{T}(\nabla_x \mathbf{u})|^r \mathrm{d}x\Big)^{2/r}.$$

Putting together the estimates of I_1 and I_2 and applying the standard Korn's inequality reported in Theorem 8 together with the classical Poincaré inequality, we arrive at the estimate

$$\int_0^\tau \int_\Omega (I_1 + I_2)\,\mathrm{d}x\mathrm{d}t \ge a\|\mathbf{u} - \mathbf{U}\|^2_{L^2(0,\tau;W^{1,r}(\Omega;\mathbb{R}^3))} \tag{475}$$

where a is a positive constant. Similar but more straightforward calculation leads to estimate

$$\int_0^\tau \int_\Omega I_3\,\mathrm{d}x\mathrm{d}t \ge a\|\sqrt{\kappa(\vartheta)}\nabla_x(\log\vartheta - \log\Theta)\|^2_{L^2((0,\tau)\times\Omega;\mathbb{R}^3)} \tag{476}$$

with $a > 0$.

Estimates of the Right-Hand Side of the Relative Energy Inequality (470)

We split each integral term at the right-hand side of the relative energy inequality (470) to the essential and residual parts and estimate them separately. The essential part is already estimated in inequality (473) whose right-hand side can be further bounded from above by expression $c\int_0^\tau \mathcal{E}(\varrho, \mathbf{u}, \vartheta | r, \Theta, \mathbf{V})\ \mathrm{d}t$. The estimate of the residual part $\int_0^\tau \int_{N_{\mathrm{res}}(t)} (R + R_1 + R_2)\ \mathrm{d}t$ is technically more complicated. By using Hölder, Young, and Sobolev imbedding inequalities together with estimate (471) and all structural assumptions of p and s, one arrives after a long and tedious calculations to

$$c(\delta)\Big(\int_0^\tau \mathcal{E}(\varrho, \vartheta, \mathbf{u}\,|\,r, \Theta, \mathbf{U})\ \mathrm{d}t + \|\mathbf{f} - \mathbf{g}\|^2_{L^\infty(Q_\tau;\mathbb{R}^3)}\Big) + \delta\|\mathbf{u} - \mathbf{U}\|^2_{L^2(0,\tau;W^{1,r}(\Omega))} \tag{477}$$

with any $\delta > 0$ where the constant c depends in addition to δ on M_0, E_0, S_0, T, Ω; the physical characteristics of e, p, μ, η, κ; the lower bounds $\underline{r}$; $\underline{\Theta}$ of r and Θ; and on the norm $(r, \Theta, \mathbf{U})$ in the space X (cf. formula (466)).

The details of these estimates in the full generality can be found in papers [31] ($\gamma = 5/3$, $\beta \in (2/5, 1]$) and [70] ($\gamma > 3/2$, $\beta = 1$).

Application of the Gronwall Inequality

We now put together estimate (470) with (475), (476), and (477). Choosing $\delta > 0$ in (477) sufficiently small (in comparison with a), we arrive at inequality

$$\mathcal{E}(\varrho, \vartheta, \mathbf{u}\,|\,r, \Theta, \mathbf{U})(\tau) \le c\Big(\mathcal{E}(\varrho_0, \vartheta_0, \mathbf{u}_0\,|\,r_0, \Theta_0, \mathbf{U}_0) + \|\mathbf{f}-\mathbf{g}\|^2_{L^\infty(Q_T)} + \int_0^\tau \mathcal{E}(\varrho, \vartheta, \mathbf{u}\,|\,r, \Theta, \mathbf{U})\ \mathrm{d}t\Big).$$

This yields the statement of Theorem 44 by virtue of the Gronwall inequality reported in Theorem 21.

12 Longtime Behavior of Weak Solutions

In this section, longtime behavior of (finite energy) weak solutions to the Navier-Stokes-Fourier system (323)–(332) will be examined under the thermodynamic stability conditions (378). Most of the material of this section is taken from the monograph [44]. There are two characteristic features that are used in future analysis:

- the system is energetically insulated meaning the total energy and the total mass of the fluid are constants of motion determined by the choice of initial data at least in the case of conservative (gradient) external forces;
- the total entropy of the system is nondecreasing in time.

These two properties give rise to a family of a priori estimates and substantially influence the behavior of the system for large times.

12.1 Equilibrium Solutions

Equilibrium solution is called any finite energy time-independent weak solution $(\varrho, \vartheta, \mathbf{u})$ of the Navier-Stokes-Fourier system (323)–(332) with the time-independent potential force

$\mathbf{f} = \nabla_x F$, where $F = F(x)$ is a time-independent scalar function (a potential). We start with several observations:

1. *Any equilibrium solution minimizes the entropy production rate*
   Indeed, one deduces from the total entropy balance (345) that

$$\int_0^\tau \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \mathrm{d}x\mathrm{d}t = 0. \tag{478}$$

   Therefore relation (478) together with the form of the stress tensor (328) and the heat flux (329) yields

$$\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} = 0, \ \ \nabla_x \vartheta = 0, \tag{479}$$

   for any equilibrium $(\varrho, \vartheta, \mathbf{u})$. As the velocity field vanishes on the boundary of Ω, the equalities in (479) together with the Korn inequality reported in Theorem 9 imply

$$\mathbf{u} \equiv 0, \ \vartheta = \tilde{\vartheta} = \text{const.} > 0 \text{ for any equilibrium state.}$$

2. It follows from the above discussion that any equilibrium solution corresponds to the zero velocity, time-independent, and a spatially homogeneous constant temperature. With this information at hand, we deduce from the momentum equation (340) that any equilibrium solution $\tilde{\varrho} = \tilde{\varrho}(x), \ \tilde{\vartheta}$ satisfies

$$\nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = \tilde{\varrho} \nabla_x F, \ \ \tilde{\varrho} \geq 0, \ \tilde{\vartheta} = const > 0, \tag{480}$$

   with the constraints

$$M_0 = \int_\Omega \tilde{\varrho} \, \mathrm{d}x, \ \ E_{0,F} = \int_\Omega \left(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \right) \mathrm{d}x, \ M_0 > 0, \ E_{0,F} \in \mathbb{R}, \tag{481}$$

   where the first constraint follows from the continuity equation (339) and the second one from the total energy conservation (352).

Next, the unique solvability of problem (480)–(481) will be discussed. The crucial role in the analysis of this problem plays the strict positivity of the equilibrium density. This property can be achieved by the thermodynamic stability condition $(378)_1$ extended to $\varrho = 0$ as follows:

$$\lim_{\varrho \to 0} \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0 \ \text{ for any fixed } \ \vartheta > 0. \tag{482}$$

Let us fix constant $\tilde{\vartheta} > 0$, $F \in W^{1,\infty}(\Omega)$ and suppose that $\tilde{\varrho} \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ verifies (480) a.e. in Ω. Then necessarily $\tilde{\varrho}(x) > 0$ for all $x \in \Omega$.

Indeed, by virtue of $(379)_1$, $\tilde{\varrho}$ satisfies equation

$$\mathcal{P}_{\tilde{\vartheta}}(\tilde{\varrho}) = F + c_{\tilde{\vartheta}}, \tag{483}$$

on any positivity component $\{x \in \Omega | \tilde{\varrho}(x) > 0\} \subset \Omega$, where $c_{\tilde{\vartheta}}$ is a constant that may depend on the specific positivity component and where

$$[0, \infty) \ni z \mapsto P_{\tilde{\vartheta}}(z) \equiv \partial_\varrho H_{\tilde{\vartheta}}(z, \tilde{\vartheta}) \in P_{\tilde{\vartheta}}([0, \infty)) \subset \mathbb{R}$$

is an invertible (increasing) application such that $\lim_{z \to 0} P_{\tilde{\vartheta}}(z) = -\infty$ by virtue of the thermodynamic stability condition (378)$_1$ extended by (482). Therefore, the right-hand side of (483) is bounded in contrast with the left-hand side, which tends to minus infinity for $\tilde{\varrho}$ approaching zero. Consequently, $\tilde{\varrho}$ must remain bounded away from zero on Ω.

Finally, *equilibrium solutions* $(\tilde{\varrho}, \tilde{\vartheta})$ *maximize the total entropy functional*

$$(\varrho, \vartheta) \to \int_\Omega \varrho s(\varrho, \vartheta)\, \mathrm{d}x$$

in the class of all measurable functions $\varrho \geq 0$, $\vartheta > 0$ verifying constraints

$$\int_\Omega \varrho\, \mathrm{d}x = \int_\Omega \tilde{\varrho}\, \mathrm{d}x, \ \int_\Omega \Big(\varrho e(\varrho, \vartheta) - \varrho F\Big)\, \mathrm{d}x = \int_\Omega \Big(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F\Big)\, \mathrm{d}x. \tag{484}$$

In order to see this property, use the definition of Helmholtz function and (484), where F is replaced by using formula (483) to get

$$\tilde{\vartheta} \int_\Omega \Big(\tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) - \varrho s(\varrho, \vartheta)\Big)\, \mathrm{d}x = \int_\Omega \Big(H_{\tilde{\vartheta}}(\varrho, \vartheta) - H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})\Big)\, \mathrm{d}x + \int_\Omega (\tilde{\varrho} - \varrho) F\, \mathrm{d}x$$

$$= \int_\Omega \Big(H_{\tilde{\vartheta}}(\varrho, \vartheta) - (\varrho - \tilde{\varrho}) \frac{\partial H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})}{\partial \varrho} - H_{\tilde{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta})\Big)\, \mathrm{d}x. \tag{485}$$

The most right integral is however nonnegative and equal to zero if and only if $(\varrho, \vartheta) = (\tilde{\varrho}, \tilde{\vartheta})$ by virtue of (355) or alternatively by virtue of Lemma 14.

The above discussion leads to the following theorem:

Theorem 45 (see [44, Theorem 4.1]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Assume that the thermodynamic functions p, e, and s are continuously differentiable in $(0, \infty)^2$ and that they satisfy relations (326), (378) together with condition (482). Let $F \in W^{1,\infty}(\Omega)$. Then for given constants $M_0 > 0$, $E_{0,F}$, there exists at most one solution $\tilde{\varrho}, \tilde{\vartheta}$ of static problem (480) in the class of locally Lipschitz functions subject to the constraints (481). In addition, $\tilde{\varrho}$ is strictly positive in Ω, and, moreover,*

$$\int_\Omega \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta})\, \mathrm{d}x \geq \int_\Omega \varrho s(\varrho, \vartheta)\, \mathrm{d}x$$

for any couple $\varrho \geq 0$, $\vartheta > 0$ of measurable functions satisfying (484).

Remark 23. 1. If the solution of problem (480)–(481) with $F \in W^{1,\infty}(\Omega)$ exists, then $\tilde{\varrho} \in W^{1,\infty}(\Omega)$, and it is given by the formula

$$\tilde{\varrho}(x) = \mathcal{P}_{\tilde{\vartheta}}^{-1}\Big(F(x) + c\Big)$$

where $c \in \mathbb{R}$ and $\tilde{\vartheta} > 0$ are determined through implicit relations

$$\int_\Omega \mathcal{P}_{\tilde{\vartheta}}^{-1}\Big(F(x) + c\Big)\,\mathrm{d}x = M_0,$$

$$\int_\Omega \Big[\mathcal{P}_{\tilde{\vartheta}}^{-1}\Big(F(x) + c\Big) e\Big(\mathcal{P}_{\tilde{\vartheta}}^{-1}\Big(F(x) + c\Big), \tilde{\vartheta}\Big) - \mathcal{P}_{\tilde{\vartheta}}^{-1}\Big(F(x) + c\Big) F(x)\Big]\,\mathrm{d}x = E_{0,F}$$

with $\mathcal{P}_{\tilde{\vartheta}}(\cdot) = \partial_\varrho H_{\tilde{\vartheta}}(\cdot, \tilde{\vartheta})$.

2. The result is based on strict positivity of the equilibrium density, which follows from the assumptions $(378)_1$ and (482). A simple example $p(\varrho, \vartheta) = a\varrho^\gamma$ with $a > 0, \gamma > 1$ shows that the solution of (480) may not be strictly positive in Ω at least for small values of the total mass M_0. Indeed, the function

$$\tilde{\varrho}(x) = \left(\left[\frac{\gamma - 1}{a\gamma}(F(x) + c)\right]^+\right)^{\frac{1}{\gamma - 1}}, \; c \in \mathbb{R}$$

represents a classical solution of (480). In addition, it can be shown that, in general, the solutions of (480) are not uniquely determined by the total mass M_0 (see [24] and Remark after Theorem 30).

3. Existence theory of finite energy solutions with specific constitutive laws for p, e satisfying assumptions (369)–(375) was built in Theorems 40and 41. It is to be noticed that these assumptions obey Gibbs relations (326), thermodynamic stability conditions (341), as well as the additional condition (482). Theorem 45 therefore applies to this situation.

The following lemma concludes this section by the observation that boundedness of the entropy and the total mass of a static state imply bounds of its norm, at least when the pressure and internal energy satisfy assumptions (369)–(375) (needed for the existence theory in Theorems 40, 41). This result will be useful in the sequel.

Lemma 18 (see [44, Lemma 5.5]). *Let the thermodynamic functions p, e, and s be given through* (369)–(375). *Let* $\tilde{\varrho}$, $\tilde{\vartheta}$ *be a solution of the problem* (480) *such that*

$$\int_\Omega \tilde{\varrho}\,\mathrm{d}x = M_0, \quad \int_\Omega \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta})\,\mathrm{d}x \equiv \tilde{S}_0$$

for certain constants $M_0 > 0$, $\tilde{S}_0 \in \mathbb{R}$. *Then there exist constants* $\underline{\varrho}$, $\underline{\vartheta}$, $\overline{\varrho}$, $\overline{\vartheta}$ *depending only on* M_0, $\tilde{S}_0$, *and* $\|F\|_{L^\infty(\Omega)}$ *such that*

$$0 < \underline{\vartheta} < \tilde{\vartheta} < \overline{\vartheta}, \;\; 0 < \underline{\varrho} < \tilde{\varrho}(x) < \overline{\varrho} \text{ for all } x \in \Omega.$$

12.2 Longtime Behavior of Conservative System

Until the end of Sect. 12, hypotheses (369)–(377) with $\gamma = 5/3$, $1/2 \le \beta \le 1$ are assumed. (These values were considered in [44].)

In this situation, existence of finite energy weak solutions on time interval $(0, T)$ is guaranteed by Theorem 40. Moreover, pressure and internal energy obey all assumptions of Theorem 45 dealing with the static states.

First observation is that the weak solutions constructed in Theorem 40 on time interval $(0, T)$ can be extended to the time interval $(0, \infty)$. This is subject of the following theorem:

Theorem 46 (see [44, Theorems 4.4 and 4.5]). *Let the hypotheses of Theorem 40 be satisfied and, in addition,*

$$\mathbf{f} \in L^{\infty}((0, \infty) \times \Omega); \mathbb{R}^3), \quad \beta \in [\frac{1}{2}, 1].$$

Then there holds: If $0 < T_1 < T_2$ *and if* $(\varrho_1, \vartheta_1, \mathbf{u}_1)$ *is a finite energy weak solution constructed in Theorem 40 on time interval* $[0, T_1)$*, then there exists a weak solution* $(\varrho_2, \vartheta_2, \mathbf{u}_2)$ *with the same properties as stated in Theorem 40 on the time interval* $[0, T_2)$ *such that*

$$(\varrho_2, \vartheta_2, \mathbf{u}_2)|_{[0,T_1)} = (\varrho_1, \vartheta_1, \mathbf{u}_1).$$

Theorem 47 (see [44, Theorem 4.5]). *Let the hypotheses of Theorem 46 be satisfied. Let* $\{\varrho, \mathbf{u}, \vartheta\}$ *be a weak solution of the system* (323)–(332) *on time interval* $[0, \infty)$ *constructed in Theorem 46, where*

$$\mathbf{f} = \nabla_x F, \quad F = F(x), \ F \in W^{1,\infty}(\Omega).$$

Then there exist $\tilde{\varrho} = \varrho(x)$, $\tilde{\vartheta} = \text{const} > 0$ *solving the static problem* (480)–(481) *such that*

$$\varrho(t, .) \to \tilde{\varrho} \textit{ in } L^{\frac{5}{3}}(\Omega), \tag{486}$$

$$(\varrho\mathbf{u})(t, .) \to 0 \textit{ in } L^{\frac{5}{4}}(\Omega; \mathbb{R}^3), \tag{487}$$

$$\vartheta(t, .) \to \tilde{\vartheta} \textit{ in } L^4(\Omega) \tag{488}$$

as $t \to \infty$.

Sketch of the proof. The main idea of the proof is to show that a norm implying convergence (486)–(488) is dominated by the "distance" of the trajectory $\{\varrho, \mathbf{u}, \vartheta\}$ from the equilibrium state $\{\tilde{\varrho}, 0, \tilde{\vartheta}\}$ by means of the relative energy functional. In view of inequality (471) (or alternatively in view of Lemma 14), the theorem will be proved once we achieve

$$\mathcal{E}(\varrho(t, \cdot), \vartheta(t, \cdot), \mathbf{u}(t, \cdot) \,|\, \tilde{\varrho}, \tilde{\vartheta}, 0) \to 0 \text{ as } t \to \infty.$$

12.2.1 Weak Compactness of the Set of Weak Solutions

The following stability result can be shown in the same way as the similar stability result proved in Theorem 42.

Lemma 19 (see [44, Theorem 4.2]). *Let the assumptions of Theorem 46 be satisfied. Let* $\mathbf{f}_n \in L^\infty((0,T)\times\Omega)$,

$$\|\mathbf{f}_n\|_{L^\infty((0,T)\times\Omega)} \le c. \tag{489}$$

Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ *be a sequence of finite energy weak solutions to the system* (323)–(332)*, with* $\mathbf{f} = \mathbf{f}_n$*, such that*

$$\operatorname{ess}\sup_{\tau\to 0} \int_\Omega \left(\frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + \varrho_n e(\varrho_n, \vartheta_n)\right)(\tau, .)\, \mathrm{d}x \le \overline{E}, \tag{490}$$

$$\operatorname{ess}\inf_{\tau\to 0} \int_\Omega \varrho_n s(\varrho_n, \vartheta_n)(\tau, .)\, \mathrm{d}x \ge \underline{S}, \tag{491}$$

and

$$\int_\Omega \varrho_n \, \mathrm{d}x \ge \underline{M}, \tag{492}$$

uniformly in n, where

$$\underline{S} > \underline{M} S_\infty,\ S_\infty = \lim_{Z\to\infty} S(Z) \ge -\infty, \tag{493}$$

and $\underline{M} > 0$, $\overline{E} > 0$*. Finally, suppose that one of the following alternatives holds: either*

$$\varrho_n(0) \equiv \varrho_{0,n} \to \varrho_0 \textit{ in } L^1(\Omega), \tag{494}$$

or

$$\mathrm{div}_x \mathbf{u}_n \to 0 \textit{ in } L^1((0,T)\times\Omega). \tag{495}$$

Then, passing to a subsequence if necessary, we have

$$\mathbf{f}_n \rightharpoonup^* \mathbf{f} \ \textit{ in } L^\infty((0,T)\times\Omega;\mathbb{R}^3),$$

$$\varrho_n \to \varrho \ \textit{ in } L^1((0,T)\times\Omega) \cap C_{\mathrm{week}}([0,T]; L^{\frac{5}{3}}(\Omega)),$$

$$\vartheta_n \rightharpoonup^* \vartheta \ \textit{ in } L^\infty(0,T; L^4(\Omega)) \textit{ and strongly in } L^1((0,T)\times\Omega),$$

$$\nabla_x \vartheta_n \rightharpoonup \nabla\vartheta \textit{ in } L^2(0,T; L^2(\Omega;\mathbb{R}^3)),$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \ \textit{ in } L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

where the trio $(\varrho, \vartheta, \mathbf{u})$ *is again a weak solution of the system driven by the force* $\mathbf{f}$.

Moreover,

$$\int_\Omega \left(\frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + \varrho_n e(\varrho_n, \vartheta_n)\right)\,\mathrm{d}x \to \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)\right)\,\mathrm{d}x \ \textit{in}\ L^\infty(0,T),$$

$$\int_\Omega \varrho_n s(\varrho_n, \vartheta_n)(\tau, .)\,\mathrm{d}x \to \int_\Omega \varrho s(\varrho, \vartheta)(\tau, .)\,\mathrm{d}x \ \textit{for a.a.}\ \tau \in (0,T).$$

Remark 24. 1. Hypotheses (494), (495) are of rather different character. Assumption (494) prevents possible spatial oscillations of the density field that may be imposed through the initial data. The conclusion of the theorem is then important for existence theory for the initial-boundary value problems. Hypothesis (495) is satisfied when $\{\varrho_n, \theta_n, \mathbf{u}_n\}$ represent suitable time shifts of a single trajectory, which provides useful information of the longtime behavior of the corresponding single solution.

2. The meaning of (493) is to avoid degenerate states with zero temperature. Since the entropy can be always normalized so that $S_\infty \in \{0, -\infty\}$, condition (493) reduces to strict positivity of $\underline{S}$ in the former case and to finiteness of $\underline{S}$ in the latter.

12.2.2 Time Shifts of the Weak Solution

Let $(\varrho, \vartheta, \mathbf{u})$ be a finite energy weak solution determined by Theorem 46. We introduce sequences

$$\varrho_n(t,x) = \varrho(t+n,x),\ \vartheta_n(t,x) = \vartheta(t+n,x),\ \mathbf{u}_n(t,x) = \mathbf{u}(t+n,x),\ t \in (0,T),\ x \in \Omega.$$

It is a routine matter to show that $(\varrho_n, \vartheta_n, \mathbf{u}_n)$ verifies hypotheses of Lemma 19.

In particular, it follows from the dissipation balance in the form (347) and the fact that the total entropy $\tau \mapsto \int_\Omega [\varrho s(\varrho, \vartheta)](\tau, x)\,\mathrm{d}x$ is non decreasing (see (349)) that

$$\int_0^T \|\mathbf{u}_n\|^2_{W^{1,q}(\Omega;\mathbb{R}^9)} \to 0,\ \int_0^T \|\nabla_x \vartheta^n\|^2_{L^2(\Omega;\mathbb{R}^3)} \to 0,\ q = \frac{8}{5-\beta}.$$

With this information at hand, application of Lemma 19 yields existence of functions $\tilde{\varrho} = \tilde{\varrho}(x)$ and $\tilde{\vartheta} = \tilde{\vartheta}(t)$ such that

$$\varrho_n \to \tilde{\varrho} \text{ in } L^1((0,T)\times\Omega) \cap C_{\mathrm{weak}}(0,T;L^{5/3}(\Omega)),$$
$$\vartheta_n \to \tilde{\vartheta} \text{ in } L^2(0,T;W^{1,2}(\Omega)),$$

where $(\tilde{\varrho}(t,x), \tilde{\vartheta}(t))$ is an equilibrium state (480), (481). In accordance with Theorem 45, the equilibrium solution is uniquely determined by the constants of motion (481), whence $\tilde{\varrho}(t,x) = \tilde{\varrho}(x)$, $\tilde{\vartheta}(t) = \tilde{\vartheta} = const$. Moreover, according to Lemma 18, there are numbers $0 < \underline{\vartheta} < \overline{\vartheta} < \infty$, $0 < \underline{\varrho} < \overline{\varrho} < \infty$ (determined by M_0, E_0, $\tilde{S}_0$ and $\|F\|_{W^{1,\infty}(\Omega)}$) such that

$$\underline{\varrho} \le \tilde{\varrho} \le \overline{\varrho}, \quad \underline{\vartheta} \le \tilde{\vartheta} \le \overline{\vartheta}.$$

12.2.3 Relative Energy Function

The above convergence relations yield

$$\int_\Omega \Big(\frac{1}{2}\varrho_n \mathbf{u}_n^2 + H_{\tilde\vartheta}(\varrho_n, \vartheta_n) - \varrho_n F \Big) \, \mathrm{d}x \to \int_\Omega \Big(H_{\tilde\vartheta}(\tilde\varrho, \tilde\vartheta) - \tilde\varrho F \Big) \, \mathrm{d}x;$$

whence, recalling (483),

$$[\mathcal{E}(\varrho_n, \mathbf{u}_n, \vartheta_n | \tilde\varrho, \tilde\vartheta, 0)] \to 0.$$

Moreover, according to Lemma 14,

$$\mathcal{E}(\varrho_n, \vartheta_n, \mathbf{u}_n \mid \tilde\varrho, \tilde\vartheta, 0)(\tau) \geq c \int_\Omega \Big(\int_\Omega \varrho_n \mathbf{u}_n^2 \, \mathrm{d}x + [1]_{\mathrm{res}} + [\varrho_n]_{\mathrm{res}}^{5/3} + [\vartheta_n]_{\mathrm{res}}^4 + [\varrho_n - \tilde\varrho]_{\mathrm{ess}}^2 + [\vartheta_n - \tilde\vartheta]_{\mathrm{ess}}^2 \Big) \, \mathrm{d}x$$

in terms of notation (471)–(472), where $c = c(\underline{\varrho}, \overline{\varrho}, \underline{\vartheta}, \overline{\vartheta}) > 0$. This finishes the proof of Theorem 47.

The following theorem asserts that the set of equilibria is a kind of attractor for all trajectories emanating from a set of bounded total mass and energy. It means that all trajectories approach the set of equilibria uniformly with growing time. As the total mass and energy are constants of motion, we cannot expect the attractor to be bounded or even compact in the associated energy norm. It is basically the only situation when the energetically insulated Navier-Sokes-Fourier system possesses an attractor.

Theorem 48 (see [44, Theorem 5.1]). *Let the assumptions of Theorem 47 be satisfied. Let* $\underline{M} > 0$, $\overline{E}_F$, $\underline{S}$ *be given, with* $\underline{S} > \underline{M} S_\infty$, $S_\infty = \lim_{Z\to\infty} S(Z) \geq -\infty$. *Then for any* $\varepsilon > 0$, *there exists a time* $T = T(\varepsilon)$ *such that*

$$\|\varrho(t,.) - \tilde\varrho\|_{L^{\frac{5}{3}}(\Omega)} \leq \varepsilon,$$

$$\|(\varrho\mathbf{u})(t,.)\|_{L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)} \leq \varepsilon,$$

$$\|\vartheta(t,.) - \tilde\vartheta\|_{L^4(\Omega)} \leq \varepsilon$$

for a.a. $t \geq T(\varepsilon)$, *for any weak solution* $\{\varrho, \mathbf{u}, \vartheta\}$ *of the Navier-Stokes-Fourier system defined on* $(0,\infty)$ *constructed in Theorem 46 and satisfying*

$$\int_\Omega \varrho(t,.) \, \mathrm{d}x > \underline{M}, \; t \in (0,\infty), \tag{496}$$

$$\int_\Omega (\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \varrho F)(t,.) \, \mathrm{d}x < \overline{E}_F, \; t \in (0,\infty), \tag{497}$$

$$\operatorname*{ess\,inf}_{t>0} \int_\Omega \varrho s(\varrho,\vartheta)(t,.) \, \mathrm{d}x > \underline{S}, \tag{498}$$

where $(\tilde{\varrho}, \tilde{\vartheta})$ *is a solution of the static problem* (480) *determined uniquely by the condition*

$$\int_\Omega \tilde{\varrho} \, \mathrm{d}x = \int_\Omega \varrho \, \mathrm{d}x,$$

$$\int_\Omega \Big(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \Big) \, \mathrm{d}x = \int_\Omega \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \Big) \, \mathrm{d}x.$$

Remark 25. 1. The total mass and the total energy are constant in time, so the specific choice of the initial time does not play any role; the interval $(0, \infty)$ may be replaced by (T, ∞). In general, the case $S_\infty = -\infty$, is possible, so the meaning of the condition (498) is to avoid degenerate states with vanishing absolute temperature.
2. The rate of decay to the set of static solutions characterized by the mapping $\varepsilon \to T(\varepsilon)$ depends on $\underline{M}$, $\overline{E}_F$, and the structural properties of the constitutive functions.
3. Condition (498) is automatically satisfied if the fluid obeys the third thermodynamical law $\lim_{Z \to \infty} S(Z) = 0$.

12.3 Longtime Behavior for Time-Dependent Forcing: Blow Up of Energy

The choice of time-independent nonconservative driving force

$$\mathbf{f} = \mathbf{f}(x), \ \mathbf{f} \in L^\infty(\Omega; \mathbb{R}^3) \text{ such that } \mathbf{f} \not\equiv \nabla_x F$$

reflects a constant supply of the mechanical energy into the system that is, in accordance with second law of thermodynamics, irreversibly converted to heat. As the boundary of Ω is thermally insulated, the system accumulates the energy, therefore, inevitably

$$E(t) = \int_\Omega \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \Big)(t, .) \, \mathrm{d}x \to \infty \text{ as } t \to \infty.$$

To avoid blow up of $E(t)$ in the general situation of time-dependent forcing term, the function $\mathbf{f}$ must behave like gradient of a scalar potential when time tends to infinity, or $\mathbf{f}$ must rapidly oscillate as time tends to infinity. The former situation is described in Theorem 49 and the latter in Theorem 50 in the next section.

The main theorem of this section reads:

Theorem 49 (see [44, Theorem 5.2]). *Let the assumptions of Theorem 46 be satisfied.*

Then for any finite energy weak solution of the Navier-Stokes-Fourier system defined on the interval $(0, \infty)$ *constructed in Theorem 46, one of the following alternatives holds:*

- *Either*

$$E(t) = \int_\Omega \Big(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \Big)(t, .) \, \mathrm{d}x \to \infty \textit{ for } t \to \infty, \tag{499}$$

- *or there is a constant* E_∞ *such that*

$$E(t) \leq E_\infty \text{ for a.a. } t > 0. \tag{500}$$

Moreover, in the latter case, each sequence of times $\tau_n \to \infty$ *contains a subsequence such that*

$$\mathbf{f}_n(t,x) = \mathbf{f}(t+\tau_n, x)$$

satisfies

$$\mathbf{f}_n \to \nabla_x F \text{ weakly-(*) in } L^\infty((0,T\times\Omega);\mathbb{R}^3) \text{ for any fixed } T > 0, \tag{501}$$

where the limit

$$F = F(x),\ \ F \in W^{1,\infty}(\Omega)$$

may depend on the choice of $\{\tau_n\}_{n=1}^\infty$.

Sketch of the proof of Theorem 49. The proof of this Theorem follows from the compactness Lemma 19 and Lemma 18. Assume that there is a solution $\{\varrho, \mathbf{u}, \vartheta\}$ such that

$$E(\tau_n) \leq \overline{E} < \infty \text{ for } \tau_n \to \infty.$$

Then, due to the structural properties of e, s, the total entropy is bounded

$$\int_\Omega \varrho s(\varrho,\vartheta)(\tau_n, .)\,\mathrm{d}x \leq \overline{S},$$

and, as the total entropy is nondecreasing in time, assume

$$\lim_{t\to\infty} \int_\Omega \varrho s(\varrho,\vartheta)(t,.)\,\mathrm{d}x = \overline{S}. \tag{502}$$

For time shifts

$$\varrho_n(t,x) = \varrho(\tau_n + t, x),\ \ \mathbf{u}_n(t,x) = \mathbf{u}(\tau_n+t,x),\ \ \vartheta_n(t,x) = \vartheta(\tau_n+t,x)$$

it implies, together with the entropy balance (341), that $\mathbf{u}_n,\ \nabla_x\mathbf{u}_n,\ \nabla\vartheta_n \to 0$ in, say $L^1((0,T)\times\Omega;\mathbb{R}^3)$. Then application of the compactness Lemma 19 gives (weak) convergence of $\{\varrho_n, \vartheta_n, \mathbf{f}_n\}$ to a limit $\{\tilde\varrho, \tilde\vartheta, \mathbf{f}\}$ satisfying, in the sense of distributions,

$$\nabla_x p(\tilde\varrho,\tilde\vartheta) = \tilde\varrho\mathbf{f},\ \ \int_\Omega \tilde\varrho\,\mathrm{d}x = M_0,\ \ \int_\Omega \tilde\varrho s(\tilde\varrho,\tilde\vartheta)(t,.)\,\mathrm{d}x = \overline{S}.$$

The entropy is a strictly increasing function of temperature, so the last equality implies that $\tilde{\vartheta}$ is independent of t. Then, in accordance with Theorem 45, $\mathbf{f}$ is a gradient of a scalar function $F(x)$.

The last point is to show that the energy cannot oscillate, i.e.,

$$\limsup_{t\to\infty} E(t) = \infty, \ \liminf_{t\to\infty} E(t) < \infty$$

is excluded. If this is valid, then the continuity of the energy implies that for any $K > 0$, there exists a sequence of times $\tau_n \to \infty$ such that $E(\tau_n) = K$. Now, define again time shifts of solutions and deduce, as above, that they converge to a static solution satisfying

$$\nabla_x p(\tilde{\varrho}, \tilde{\vartheta}) = \tilde{\varrho} \nabla_x F, \ \tilde{\vartheta} = const > 0,$$

$$\int_\Omega \tilde{\varrho} \, \mathrm{d}x = M_0, \ \int_\Omega \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta})(t, .) \, \mathrm{d}x = \overline{S}. \tag{503}$$

and

$$\int_\Omega \left(\tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\varrho} F \right) \mathrm{d}x = K. \tag{504}$$

However, by virtue of Lemma 18, relations (503) and (504) are not compatible for arbitrary (large) K, which concludes the proof of Theorem 49.

Examples of external forces which drive the energy to infinity are given in the following corollary.

These examples are direct consequences of Theorem 46. The fact that the blowup $E(t) \to \infty$ implies the blowup of the thermal energy $E_{\mathrm{th}}(\tau) = \int_\Omega \varrho(\tau, x) e(\varrho(\tau, x), \vartheta(\tau, x)) \, \mathrm{d}x$ is formulated in Corollary 4.

Corollary 3. *Let the assumptions of Theorem 46 be satisfied. Let $\mathbf{f} \in L^\infty((0,T) \times \Omega; \mathbb{R}^3)$ satisfies one of the following conditions:*

(i) $\mathbf{f} = \mathbf{f}(x)$, $\mathbf{f} \not\equiv \nabla_x F$;
(ii) $\mathbf{f}$ *is time periodic, nonconstant in time,* $\mathbf{f}(t + T, x) = \mathbf{f}(t, x)$ *for all* t, x;
(iii) $\mathbf{f}$ *is almost periodic, nonconstant in time*;
(iv) $\mathbf{f}$ *is asymptotic periodic (almost periodic) nonconstant in time, meaning*

$$\sup_{x\in\Omega} |\mathbf{f}(t,x) - \mathbf{g}(t,x)| \to 0 \ \textit{as } t \to \infty, \ \textit{where } \mathbf{g} \textit{ is periodic (almost periodic) nonconstant in time.}$$

Then

$$E(t) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right)(t, .) \, \mathrm{d}x \to \infty \ \textit{as } t \to \infty$$

for any finite energy weak solution $\{\varrho, \mathbf{u}, \vartheta\}$ *of the Navier-Stokes-Fourier system defined on* $(0, \infty) \times \Omega$ *constructed in Theorem 46.*

Remark 26. 1. The first condition together with Theorem 45 gives a complete description of the longtime behavior of the energetically insulated Navier-Stokes-Fourier system driven by a time-independent external force.

2. In contrast with the static case, the function $\mathbf{f}(t,x) = \nabla_x F(t,x)$ with F periodic and nonconstant in time satisfies condition (ii), which leads to the explosion of the total energy. With the help of Corollary 3, it is possible to construct forces that tend to zero when time goes to infinity and vanish on a large set, but still drive the energy of the system to infinity. See [44, Example 5.1].

The following result shows that boundedness of the internal energy implies boundedness of the total energy.

Corollary 4. *Let the assumption of Theorem 46 be satisfied. Let $\{\varrho, \mathbf{u}, \vartheta\}$ be a global finite energy weak solution on $[0,\infty)$ to the Navier-Stokes-Fourier system constructed in Theorem 46 such that*

$$E(t) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,.)\,\mathrm{d}x \to \infty \text{ as } t \to \infty. \tag{505}$$

Then

$$\operatorname*{ess\,lim\,sup}_{t\to\infty} \int_\Omega \varrho e(\varrho,\vartheta)(t,.)\,\mathrm{d}x = \infty.$$

In fact, if $\operatorname{ess\,lim\,sup}_{t\to\infty} \int_\Omega \varrho e(\varrho,\vartheta)\,\mathrm{d}x < \infty$, then also

$$\sup \|\varrho(t,.)\|_{L^{\frac{5}{3}}(\Omega)} \le c,$$

and the total entropy is bounded, which in turn yields a sequence of times $\tau_n \to \infty$ such that

$$\int_{\tau_n}^{\tau_n+1} \|\mathbf{u}(t,.)\|_{L^6(\Omega;\mathbb{R}^3)} \to 0.$$

These two relations imply $\operatorname{ess\,lim\,inf}_{t\to\infty} E(t) < \infty$, in contrast with (505).

12.4 Longtime Behavior: Stabilization to Equilibria for Rapidly Oscillating Driving Forces

An example of nontrivial driving forces that, in contrast with the examples in the last section, stabilize the system, is given in this section. The previous discussion may suggest that almost all time-dependent driving forces imposed on the energetically insulated Navier-Stokes-Fourier system result in the blowup of the energy for time tending to infinity. Instead of forces that converge to a conservative form or simply vanish, also rapidly oscillating forces may stabilize the system. This means that the condition (501) allows for some interesting exceptions and that thanks to rapid oscillations the solutions may converge to the homogeneous static state as time goes to infinity. The specific choice of the driving force was studied in [44], where the following result was proved:

Theorem 50 (see [44, Theorem 5.3]). *Let the assumptions of Theorem 46 be satisfied. Let the driving force take the form*

$$\mathbf{f}(t,x) = \omega(t^{\beta})\mathbf{w}(x), \quad t > 0,\ x \in \Omega,$$

where $\mathbf{w} \in W^{1,\infty}(\Omega)$, $\mathbf{w} \neq 0$, *and*

$$\omega \in L^{\infty}(R),\ \omega \neq 0,\ \sup_{\tau>0}\left|\int_0^{\tau} \omega(t)\mathrm{d}t\right| < \infty, \tag{506}$$

are given functions, with $\beta > 2$.

Then any global-in-time finite energy weak solution of the Navier-Stokes-Fourier system constructed in Theorem 46 satisfies

$$\varrho\mathbf{u}(t,\cdot) \to 0 \textit{ in } L^{5/4}(\Omega;\mathbb{R}^3) \textit{ as } t \to \infty, \tag{507}$$

$$\vartheta(t,\cdot) \to \tilde{\vartheta} \textit{ in } L^{4}(\Omega) \textit{ as } t \to \infty, \tag{508}$$

$$\varrho(t,\cdot) \to \tilde{\varrho} \textit{ in } L^{5/3}(\Omega) \textit{ as } t \to \infty, \tag{509}$$

where $\tilde{\varrho}, \tilde{\vartheta}$ *are positive constants,*

$$\tilde{\varrho} = \frac{1}{|\Omega|}\int_{\Omega} \varrho \mathrm{d}x.$$

Proof of Theorem 50. The proof is based on the energy estimates obtained by means of the total dissipation balance and on the analysis of possible oscillations of the driving force $\mathbf{f}$.

The idea is to apply Lemma 19 on the sequence of time shifts

$$\varrho_n(t,x) = \varrho(t+n,x),\ \vartheta_n(t,x) = \vartheta(t+n,x),$$

$$\mathbf{u}_n(t,x) = \mathbf{u}(t+n,x),\ \mathbf{f}_n(t,x) = \omega((t+n)^{\beta})\mathrm{w}(x),$$

$t \in (0,T),\ x \in \Omega.$

To this end, it should be shown that

$$\mathbf{f}_n \rightharpoonup^{*} 0 \text{ in } L^{\infty}((0,T)\times\Omega;\mathbb{R}^3), \tag{510}$$

$$E(\tau) = \int_{\Omega}\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(\tau,\cdot)\mathrm{d}x \to E_{\infty} \text{ for } \tau \to \infty, \tag{511}$$

and, exactly as (502), observed that

$$\int_{\Omega} \varrho s(\varrho,\vartheta)(\tau,\cdot)\mathrm{d}x \to S_{\infty} \text{ as } \tau \to \infty. \tag{512}$$

With (510)–(512) at hand, application of Lemma 19 yields

$$\varrho(\tau_n + \cdot, \cdot) \to \tilde{\varrho} \text{ in } C_{\text{week}}([0, 1]; L^{5/3}(\Omega)),$$

$$\mathbf{u}(\tau_n + \cdot, \cdot) \to 0 \text{ in, say, } L^1((0, 1) \times \Omega),$$

$$\vartheta(\tau_n + \cdot, \cdot) \to \tilde{\vartheta} \text{ in, say, } L^1((0, 1) \times \Omega)$$

for any $\tau_n \to \infty$, where $\tilde{\varrho}, \tilde{\vartheta}$ is the (constant) solution to the stationary problem (480), uniquely determined by

$$\tilde{\varrho}|\Omega| = M_0, \quad \tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta})|\Omega| = E_\infty.$$

To finish the proof, it remains to show convergence (507)–(509). This follows from (511), (512), and the coercivity of Helmholtz function established in Lemma 14 (see (354) and (359)). Hence, it is sufficient to show (510) and (511).

Proof of (510). It is enough to see that

$$\int_0^1 \omega((t + n)^\beta)\psi(t) \, \mathrm{d}t \to 0 \text{ as } n \to \infty \text{ for any } \psi \in C_c^\infty(0, 1).$$

This is a consequence of hypothesis (506), and $\beta > 2$:

$$\int_0^1 \omega((t+n)^\beta)\psi(t) \, \mathrm{d}t = -\frac{1}{\beta}\int_0^1 \mathcal{O}((t+n)^\beta)\Big[\psi'(t+n)^{1-\beta} + (\beta - 1)(t+n)^{-\beta}\psi(t)\Big] \, \mathrm{d}t \to 0, \tag{513}$$

where

$$\mathcal{O}(\tau) = \int_0^\tau \omega(t) \, \mathrm{d}t$$

is bounded according to (506).

The convergence (511) follows from the total energy balance (343) and the following relation

$$\int_0^\tau \omega(t^\beta) \int_\Omega \mathbf{w}(x)(\varrho\mathbf{u})(t, x) \, \mathrm{d}x\mathrm{d}t \to I_\infty \in \mathbb{R} \text{ for } \tau \to \infty. \tag{514}$$

Proof of (514). First, deduce energy estimates and then uniform bounds via an iteration process.

Denoting

$$U(t) = \int_\Omega \mathbf{w} \cdot (\varrho\mathbf{u})(t, .) \, \mathrm{d}x,$$

proceed as in (513) to get

$$\int_{T-1}^{T}\omega(t^\beta)\int_\Omega \mathbf{w}(x)(\varrho\mathbf{u})(t,x)\,\mathrm{d}x\mathrm{d}t = \int_{T-1}^{T}\omega(t^\beta)U(t)\,\mathrm{d}t \leq$$
$$\frac{1}{\beta}\left|\mathcal{O}(t^\beta)t^{1-\beta}U(t)\right|_{T-1}^{T} + \frac{1}{\beta}\left|\int_{T-1}^{T}\mathcal{O}(t^\beta)\left[(1-\beta)t^{-\beta}U(t)+t^{1-\beta}\frac{\mathrm{d}}{\mathrm{d}t}U(t)\right]\mathrm{d}t\right|.$$

Hence, (514) follows provided that U, $\frac{\mathrm{d}}{\mathrm{d}t}U$ are proved to be bounded functions.

$$|U(t)| \leq \sqrt{M_0}||\mathbf{w}||_{L^\infty(\Omega;\mathbb{R}^3)}||\sqrt{\varrho}\mathbf{u}(t,\cdot)||_{L^2(\Omega;\mathbb{R}^3)}.$$

Take a test function $\varphi = \psi(t)\mathbf{w}(x)$ in the momentum equation (324) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t)=\int_\Omega(\varrho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\mathbf{w}+p(\varrho,\vartheta)\mathrm{div}_x\mathbf{w}-\mathbb{S}:\nabla_x\mathbf{w})\,(t,\cdot)\,\mathrm{d}x+\int_\Omega\left(\varrho|\mathbf{w}|^2\right)(t,\cdot)\omega(t^\beta)\,\mathrm{d}x \tag{515}$$

for a.a. $t\in(0,\infty)$. To get uniform bounds for $\frac{\mathrm{d}}{\mathrm{d}t}U$, the total dissipation balance (346) is used. Now, fix $\tilde\vartheta > 0$, $\tilde\varrho = \frac{M_0}{|\Omega|}$, $M_0 = \int_\Omega \varrho\,\mathrm{d}x$ and rewrite equation (346) in terms of relative energy $\mathcal{E}(\varrho,\vartheta,\mathbf{u}\,|\,\tilde\varrho,\tilde\vartheta,0)$. Denoting $D(t) = \mathcal{E}(\varrho,\vartheta,\mathbf{u}\,|\,\tilde\varrho,\tilde\vartheta,0)$ and $Q(t) = \int_\Omega\frac{\tilde\vartheta}{\vartheta}\left(\mathbb{S}:\nabla\mathbf{u}-\frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta}\right)\,\mathrm{d}x$, the equation (335) can be rewritten as follows:

$$D(\tau)+\int_0^\tau Q(t)\,\mathrm{d}t \leq C(\tilde\vartheta,M_0,E_0,\tilde S_0)+\int_0^\tau\omega(t^\beta)U(t)\,\mathrm{d}t. \tag{516}$$

The next goal is to establish uniform bounds for D, which then imply bounds for Q and the time derivative of U.

The coercivity of Helmholtz function $H_{\tilde\vartheta}$ gives

$$\int_\Omega|(\varrho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\mathbf{w}+p(\varrho,\vartheta)\mathrm{div}_x\mathbf{w})\,(t,\cdot)|\,\mathrm{d}x \leq c_1E(t)\leq c_2(1+D(t))$$

for a.a. $t\in(0,\infty)$. Also, writing

$$\mathbb{S}:\nabla_x\mathbf{w} = \sqrt{\frac{\mu(\vartheta)}{\vartheta}}[\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}]:[\nabla_x\mathbf{w}+\nabla_x^t\mathbf{w}-\frac{2}{3}\mathrm{div}_x\mathbf{w}\mathbb{I}]\sqrt{\mu(\vartheta)\vartheta}$$
$$+\sqrt{\eta(\vartheta)/\vartheta}\mathrm{div}_x\mathbf{u}\,\mathrm{div}_x\mathbf{w}\sqrt{\eta(\vartheta)\vartheta},$$

yields

$$\left(\int_\Omega\mathbb{S}:\nabla_x\mathbf{w}\,\mathrm{d}x\right)^2 \leq$$
$$c\|\mathbf{w}\|^2_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\int_\Omega\frac{\mu(\vartheta)}{\vartheta}|\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}|^2\,\mathrm{d}x\int_\Omega\mu(\vartheta)\vartheta\,\mathrm{d}x$$
$$+c\|\mathbf{w}\|^2_{W^{1,\infty}(\Omega;\mathbb{R}^3)}\int_\Omega\frac{\eta(\vartheta)}{\vartheta}|\mathrm{div}_x\mathbf{u}|^2\,\mathrm{d}x\int_\Omega\eta(\vartheta)\vartheta\,\mathrm{d}x.$$

Assumptions (376) on μ and η give

$$\int_\Omega (\mu(\vartheta) + \eta(\vartheta))\vartheta(t,\cdot)\,\mathrm{d}x \le c_1 \int_\Omega (1 + \vartheta^2)(t,\cdot)\,\mathrm{d}x \le c_2(1 + \sqrt{D(t)}).$$

Altogether, the previous estimates imply

$$D(\tau) + \int_0^\tau Q(t)\,\mathrm{d}t \le c_1 + \int_0^\tau \omega(t^\beta)U(t)\,\mathrm{d}t, \tag{517}$$

$$D(\tau) \ge 0,\ \ Q(\tau) \ge 0,\ \ |U(t)| \le c_2\sqrt{D(t)}, \tag{518}$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)\right| \le c_3 D(\tau) + c_4\sqrt{Q(\tau)}\sqrt{1 + \sqrt{D(\tau)}} + c_5 \tag{519}$$

for a.a. $\tau \in (0,\infty)$, where constants c_i, $i = 1,\dots,5$ depend only on M_0, E_0, $\tilde{S}_0$ and on the norms $\|\mathbf{w}\|_{W^{1,\infty}(\Omega;\mathbb{R}^3)}$ and $\|\omega\|_{L^\infty(R)}$. Moreover, the entropy balance equation (345) gives

$$\int_0^\tau Q(t)\,\mathrm{d}t \le |\tilde{S}_0| + \int_\Omega \varrho|s(\varrho,\vartheta)|(\tau)\,\mathrm{d}x \le |\tilde{S}_0| + c(\overline{\varrho},\overline{\vartheta}) + D(\tau). \tag{520}$$

Uniform Bounds. Next, estimates (517)–(520) are used to obtain a uniform bound on D. The first step in the proof is to obtain a bound $D(\tau) \le c\tau^2$. Then an iteration procedure, where by repeating the same argument many times gives better and better bounds in each step, and after finitely many steps the uniform bound is obtained.

The initial bound on D, follows from (517), (518):

$$D(\tau) \le c\left(1 + \int_0^\tau (1 + \sqrt{D(t)})\,\mathrm{d}t\right);$$

whence

$$D(\tau) \le c\tau^2 \text{ for a.a. } \tau \in (1,\infty).$$

This estimate is a starting point for the iteration procedure described in what follows. Assume that the following estimate has been already proved:

$$D(\tau) \le c\tau^m \text{ for a.a. } \tau \in (1,\infty) \tag{521}$$

for a certain $m \in [0,2]$. Using (521) in (519) gives

$$\left|\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)\right|^2 \le c\left(\tau^{2m} + Q(\tau)\tau^{m/2}\right) \text{ for a.a. } \tau \in (1,\infty);$$

whence, thanks to (520) and (521)

$$\int_{T-1}^{T} \left| \frac{\mathrm{d}}{\mathrm{d}\tau} U(\tau) \right|^2 \mathrm{d}t \le c_1 \left(T^{2m} + T^{m/2} \int_0^T Q(t)\,\mathrm{d}t \right) \le c_2 (T^{2m} + T^{3m/2}) \le c_3 T^{2m} \tag{522}$$

provided $T > 2$.

On the other hand, with the bounded (see (506)) primitive function $\mathcal{O}$

$$\mathcal{O}(\tau) = \int_0^\tau \omega(t)\,\mathrm{d}t,$$

the following estimate holds

$$\beta \int_{T-1}^{T} \omega(t^\beta) U(t)\,\mathrm{d}t \le$$
$$\left| \mathcal{O}(t^\beta) t^{1-\beta} U(t) \right|_{T-1}^{T} + \left| \int_{T-1}^{T} \mathcal{O}(t^\beta) \left[(1-\beta) t^{-\beta} U(t) + t^{1-\beta} U'(t) \right] \mathrm{d}t \right|.$$

Therefore,

$$\left| \int_{T-1}^{T} \omega(t^\beta) U(t)\,\mathrm{d}t \right| \le c_1 \left(T^{1-\beta+m/2} + T^{-\beta+m/2} + T^{m+1-\beta} \right) \le c_2 T^{1-\beta+m} \tag{523}$$

using $|U(t)| \le c t^{m/2}$, and (522).

Finally it follows that

$$D(\tau) \le c \tau^{2-\beta+m}; \tag{524}$$

in particular, (521) implies (524). Hence, using the assumption $\beta > 2$, it holds, after finitely many steps

$$\mathrm{esssup}_{\tau \in (0,\infty)} D(\tau) < \infty.$$

Now, it follows from (523)

$$I(\tau) = \int_0^\tau \omega(\tau^\beta) U(t)\,\mathrm{d}t \to I_\infty \in \mathbb{R} \text{ for } \tau \to \infty,$$

and, using the total energy balance (343) also

$$E(\tau) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) (\tau, \cdot)\,\mathrm{d}x \to E_\infty \text{ for } \tau \to \infty.$$

□

Remark 27. 1. Even if the restriction $\beta > 2$ is probably not optimal, some uniform growth of frequency is necessary. Indeed, consider

$$\mathbf{f}(t,x) = \omega(nt)\mathbf{w} \text{ for } t \in (T_{n-1}, T_n),\ T_0 = 0,$$

where ω is a time-periodic function with zero mean, and the sequence of times T_n is chosen in such a way that

$$E(t) \geq n \text{ for } a.a.\ t \geq T_n.$$

Such sequence of times is possible to find applying repeatedly the existence Theorem and Corollary 3 to the problem on the intervals (T_n, ∞), with initial data

$$\varrho(T_n,.) \in L^{\frac{5}{3}}(\Omega),\ \varrho\mathbf{u}(T_n,.) \in L^{\frac{5}{4}}(\Omega;\mathbb{R}^3),\ \vartheta(T_n,.) \in L^4(\Omega),$$

where ϑ is uniquely determined by the equation

$$\varrho(T_n,.)s\big(\varrho(T_n,.),\vartheta(T_n,.)\big) = \varrho s(T_n,.).$$

2. Similar stability result was proved in [4] for unbounded driving forces, when the oscillations are so rapid that they in some sense prevail the growth in time, or the decay in time allows for slower oscillations, specifically,

$$\mathbf{f} = t^\delta \omega(t^\beta)\mathbf{w},$$

where ω and $\mathbf{w}$ satisfy the assumptions of Theorem 50, and

$$\delta > 0,\ \ \beta - 2\delta > 2 \ \text{ or } \ \delta \leq 0,\ \beta - \delta > 2.$$

The assertions (507)–(510) hold true for this kind of forcing terms. The proof of this result follows the same lines as that of Theorem 50; it is based on precise energy estimates together with careful analysis of possible oscillations of the driving force.

13 Conclusion

In spite of the fact that the theory of weak solutions to the compressible Navier-Stokes equations is a young topic, it already benefits of quite large comprehensive literature including monographs. The first results appeared in the pioneering seminal work of P.L. Lions [77] dealing with the equations in barotropic regime. The Lions' breakthrough was made possible due to the discovery of the so-called effective viscous flux identity and the renormalized transport theory developed previously by DiPerna and Lions in [18]. Another milestone in the understanding of weak solutions for these equations is Feiresl's monograph [30] containing a comprehensive treatment of the heat-conducting compressible fluids with weak formulation of the energy conservation in terms of the thermal energy balance. In the light of this work (that employs in addition to the techniques introduced by P.L. Lions new ideas related to the notion of oscillations defect measure), the Lions theory is a particular case of Feireisl's results. Monograph [88] contains an extensive material on weak solutions to the compressible Navier-Stokes equations in barotropic regime ranging from stationary to evolution problems and from bounded to unbounded domains with different boundary conditions containing comprehensive detailed proofs. The theory of weak solutions has been revisited in [93] in view of applications in the control theory. Monograph [33] introduces in Chapter 3 a theory of weak solutions to the complete Navier-Stoke-Fourier system with the energy conservation in terms of the entropy balance and the entropy production rate as a Radon measure (entropy weak solutions). Among others, this work reveals importance of the Helmholtz function (called sometimes ballistic free energy). This quantity plays an essential role in the book [44] devoted to the investigation of the longtime behavior of weak solutions. Thermodynamic stability conditions for entropy weak solutions to the Navier-Stokes-Fourier system as well as for the barotropic equations can be reformulated as a variational inequality called relative energy inequality (see [34,49,50]) that becomes a basic tool to prove the weak-strong uniqueness principle for these equations (see again [34, 49, 50]) and has many other applications, e.g., the investigation of various singular limits or deriving error estimates for various numerical schemes.

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Cross-References

► Blow-Up Criteria of Strong Solutions and Conditional Regularity of Weak Solutions
► Concepts of Solutions in the Thermodynamics of Compressible Fluids
► Symmetric Solutions to the Viscous Gas Equations
► Weak and Strong Solutions Of Equations Of Compressible Magnetohydrodynamics
► Weak Solutions to 2D and 3D Compressible Navier-Stokes Equations in Critical Cases
► Weak Solutions in the Intermediate Regularity Class
► Weak Solutions with Density Dependent Viscosities

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