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Abstract

We discuss regularity and separation from potential barriers of solutions of the Cahn-Hilliard equation with singular potentials. Then we show that the same results can be obtained also for the non-isothermal, conserved Caginalp system.

Key words: Cahn-Hilliard equation, Caginalp model, singular potentials, separation property, convergence to steady states
AMS-Classification: 35K55, 35B40, 80A22

1 Introduction

This note is devoted to the study of regularity and separation from singularities of solutions of the Cahn-Hilliard equation, the model which describes dynamics of two-phase systems, and its non-isothermal version due to Caginalp. The concentration difference of the two components (often called the order parameter of the system) is described by the nonlinear fourth order equation in a domain Ω:

$$u_t = \Delta(-\Delta u + f(u)) \text{ in } (0,T) \times \Omega, \tag{1.1}$$

where u denotes the order parameter, $\Omega \subset R^3$ is a bounded smooth domain, and $f(u)$ is the derivative of a double-well potential $F(u)$, whose wells correspond to the phases of the material. The equation is supplemented with Neumann boundary conditions and an initial datum

$$\partial_n u|_{\partial\Omega} = \partial_n \Delta u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0. \tag{1.2}$$

The thermodynamically relevant potential F suggested by Cahn and Hilliard [5] is of the form

$$F(u) = \frac{\theta}{2}\Big((1+u)\ln(1+u) + (1-u)\ln(1-u)\Big) - \frac{\theta_c}{2}u^2, \tag{1.3}$$

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where $u(t,x) = \pm 1$ denote the pure phases. The logarithmic terms are related to the entropy of the system, $\theta_c > 0$ is the critical temperature. If $\theta \geq \theta_c$ then f is convex and the mixed phase is stable. On the other hand, if $0 < \theta < \theta_c$, then F has indeed the double well form and phase separation occurs.

This equation was studied by many authors during the last years either with Neumann or dynamic boundary conditions, see, e.g., [12] for more references. In the case of a regular (nonsingular, e.g., cubic) nonlinear term the problem is well understood, however, the order parameter need not remain in the physically relevant region [-1,1], see [12, Remark 2.1]. This problem does not occur when singular nonlinearities f are taken into account.

The first existence and uniqueness result for the problem (1.1–1.3) was obtained by Elliott and Luckhaus [8], who constructed u as a limit of solutions of a regularized problem with $F(u)$ replaced by bounded functions. The result was then improved by Debusche and Dettori [7] who approximated $F(u)$ by polynomials. A different way was used in Miranville and Zelik [10], who approximate the equation by a generalized Cahn-Hilliard equation based on a microforce balance, and a direct proof using the general theory of monotone operators was given by Abels and Wilke [1]. The choice of the nonlinearities was extended to the class

$$f \in C^1(-1,1), \ \lim_{s\to -1} f(s) = -\infty, \ \lim_{s\to +1} f(s) = +\infty, \quad f'(s) \geq -d \text{ for some } d \geq 0. \tag{1.4}$$

It was proved in these papers that solutions stay in the region $-1 \leq u(t,x) \leq 1$ for all $t > 0, \ x \in \Omega$, and that the strict inequality

$$-1 < u(t,x) < 1 \text{ holds for almost every } t > 0, \ x \in \Omega, \tag{1.5}$$

and that for every $t > 0$, the set $\{x \in \Omega; \ |u(t,x)| = 1\}$ has zero measure. This enabled to develop the theory of exponential attractors. The separation from the singularities has been also shown in many papers dealing with proper dynamic boundary conditions, see [10] and references therein. However, in the case of Neumann boundary conditions, the problem of separation of the solution from the singularities of the nonlinearity f for all (positive) times, which is crucial for reducing the problem to the one with regular potentials with the well developed theory, is open for singularities, which are not strong enough in the sense that $f(s)$ tends only slowly to the limits in (1.4), as, e.g., the logarithmic nonlinearity. It was proved in [10], that the separation holds true for nonlinearities satisfying

$$|f'(z)| \leq c(f(z)^2 + 1), \tag{1.6}$$

which is satisfied for nonlinearities of the type

$$f(u) = \phi(u)(1-u^2)^{-\alpha}, \ \alpha > 1, \ \phi(\pm 1) \neq 0. \tag{1.7}$$

As claimed in [10, Remark 7.1], this can be relaxed to $\alpha > \frac{3}{7}$. Our assumption (1.20) satisfies (1.7) with $\alpha \geq \frac{1}{3}$. Abels and Wilke [1] proved that the $\omega-$limit set of any solution is separated from singularities, and that $\text{dist}(u(t), \omega(u_0)) \to 0$ as $t \to \infty$ in $L^\infty(\Omega)$, so that the solution separates from singularities for sufficiently large time. It allows to show that the solution converges to a single stationary state provided that the nonlinearity is analytic.

It is the aim of this note to enlarge the set of nonlinearities which yields the immediate separation property, and to improve the regularity of solutions. Unfortunately, the separation property for the logarithmic potential is still out of reach, since obviously this potential satisfies neither (1.6) above nor (1.20) below.

The regularity of solutions satisfying the immediate separation property is examined at the end of Section 3.

In Section 4, we show that the same result holds also for the nonisothermal problem. The model proposed by Caginalp [3], [4], consists of the following system of equations for the phase-field variable u and the temperature ϑ:

$$u_t = \Delta(-\Delta u + f(u) + 2\vartheta),\ u(0) = u_0, \tag{1.8}$$

$$\vartheta_t + \frac{1}{2}u_t = \Delta\vartheta,\ \vartheta(0) = \vartheta_0. \tag{1.9}$$

The problem is complemented with the Neumann boundary conditions (1.2) and also

$$\partial_n\vartheta|_{\partial\Omega} = 0. \tag{1.10}$$

For the sake of simplicity, we normalized the physical constants to 1, and assume that $|\Omega| = 1$.

This system was studied many times with regular potentials, but with f satisfying (1.4) we have to establish the existence result first. We also examine the long-time behavior of solutions in the case that the nonlinearity is analytic in the interval (-1,1). We show that any solution of the Caginalp system converges to a single stationary state, a solution of the problem

$$\Delta\Big(\Delta u_\infty - f(u_\infty)\Big) = 0 \text{ on } \Omega, \tag{1.11}$$

$$\partial_n u_\infty = \partial_n\Delta u_\infty = 0 \text{ on } \partial\Omega, \tag{1.12}$$

$$\vartheta_\infty = const. \tag{1.13}$$

In the case of regular potentials, the convergence to equilibria was proved in [6] for the problem with dynamic boundary conditions, and in [9] with Neumann boundary conditions. Maximal attractors and inertial sets were examined in [2].

Our main result is the following

Theorem 1.1 *Let f satisfy (1.4). Then for every u_0 satisfying*

$$u_0 \in L^\infty(\Omega) \textit{ with } \|u_0\|_{L^\infty(\Omega)} \le 1 \textit{ and } \frac{1}{|\Omega|}\int_\Omega u_0 \,\mathrm{d}x \in (-1,1), \tag{1.14}$$

there is a unique solution $u \in L^\infty((0,\infty);L^\infty(\Omega)) \cap L^2((0,\infty);H^2(\Omega))$ of (1.1-1.2) such that for any $\varepsilon > 0$,

$$u \in L^\infty((\varepsilon,\infty);W^{2,6}(\Omega)), \tag{1.15}$$

$$f(u) \in L^\infty((\varepsilon,\infty);L^6(\Omega)), \tag{1.16}$$

$$u_t \in L^\infty((\varepsilon,\infty);H^{-1}(\Omega)) \cap L^2((\varepsilon,\infty);H^1(\Omega)), \tag{1.17}$$

$$\nabla u(t) \in C^{\frac{1}{2}}(\overline{\Omega}) \textit{ for all } t \ge \varepsilon, \tag{1.18}$$

$$u \in C^\mu((\varepsilon,T);C^\nu(\Omega)) \textit{ with } \nu \in (0,\frac{1}{2}],\ \mu = \frac{5-2\nu}{16}. \tag{1.19}$$

Moreover, if there exists $\delta \in (0,1)$ such that f satisfies, in addition to (1.4),

$$f(s) \leq -C(1+s)^{-\frac{1}{3}} \text{ for } -1 < s < -1+\delta, \quad f(s) \geq C(1-s)^{-\frac{1}{3}} \tag{1.20}$$

for $1-\delta < s < 1$, then

$$\|u(t)\|_{C(\overline{\Omega})} < 1 \text{ for all } t \geq \varepsilon. \tag{1.21}$$

Remark. In comparison with [1] we have immediate separation property, not only when the solution reaches a small neighborhood of a set of stationary solutions, but we need a stronger nonlinearity. On the other hand, our nonlinearity is allowed to be weaker than that in [10] (see (1.6)) to get separation for all positive times. This is achieved by improving regularity of solutions, and examining the behavior of solutions around possible points with $|u(t,x)| = 1$.

For the Caginalp system (1.8), (1.9) we have

Proposition 1.1 *Let the assumptions of Theorem 1.1 hold. Moreover, let $\vartheta_0 \in H^1(\Omega)$. Then there exists a unique solution (u,ϑ) of the problem (1.8)-(1.10), (1.2) such that u satisfies the assertions of Theorem 1.1, and*

$$\vartheta \in L^2((0,\infty); H^2(\Omega)) \cap L^\infty((0,\infty); H^1(\Omega)), \tag{1.22}$$

$$\|\vartheta\|_{L^r((t,t+1);W^{2,6}(\Omega))} \leq C_\varepsilon, \tag{1.23}$$

$$\|\vartheta_t\|_{L^r((t,t+1);L^6(\Omega))} \leq C_\varepsilon, \tag{1.24}$$

independently of $t \geq \varepsilon$, and for any $1 \leq r < \infty$. The constant C_ε depends on ε but is independent of r.

Remark. For (1.22)-(1.24) it is sufficient that f satisfies (1.4). With these estimates and (1.15) at hand, we arrive at

Proposition 1.2 *Let f, u_0 satisfy (1.4),(1.14), and $\vartheta_0 \in H^1(\Omega)$. Then for any solution u, ϑ of the problem (1.8)-(1.10), (1.2), the trajectories*

$$\cup_{t\geq T} u(t), \quad \cup_{t\geq T}\vartheta(t), \ T > \varepsilon, \tag{1.25}$$

are precompact in the space $W^{s,6}(\Omega)$ for any $s \in (0,2)$.

The convergence result is formulated in the following

Proposition 1.3 *Let the assumptions of Proposition 1.2 be satisfied, and, in addition, let f be real analytic in (-1,1). Let (u,ϑ) be a solution of the problem (1.8)-(1.10), (1.2). Then there exists a function u_∞-a classical solution of the problem (1.11), (1.12), and $\vartheta_\infty = \int_\Omega \vartheta_0 \mathrm{d}x$ such that*

$$u(t) \to u_\infty, \quad \vartheta(t) \to \vartheta_\infty \ \text{ in } C^\nu(\Omega), \ \nu < \frac{3}{2} \ \text{ as } t \to \infty.$$

Remark. Under the assumptions of Proposition 1.3 we have also separation of u from the singularities of f, but only for large times, i.e., when the solution arrives into a small neighborhood of the set of stationary solutions. The immediate separation property holds again under the additional assumption (1.20). The solutions converge to equilibria in a stronger topology than it was proved in [1] for the isothermal case and in [9] for a regular nonlinearity.

2 Preliminaries

Denote $\langle v \rangle = \int_\Omega v(x)\ \mathrm{d}x$. It is easy to see that, due to the boundary conditions, the integral mean of the solution is a conserved quantity,

$$\langle u(t) \rangle = \int_\Omega u(t)\ \mathrm{d}x = \int_\Omega u_0\ \mathrm{d}x = \langle u_0 \rangle \text{ for all } t \geq 0, \tag{2.1}$$

and that the time derivative u_t has zero mean:

$$\langle u_t(t) \rangle = \int_\Omega u_t(t)\ \mathrm{d}x = 0. \tag{2.2}$$

Then we can rewrite equation (1.1) in the following equivalent form

$$(-\Delta_N)^{-1} u_t = \Delta_N u - f(u) + \langle f(u) \rangle, \tag{2.3}$$

where $(-\Delta_N)^{-1}$ is the inverse to the negative Laplace operator with the Neumann boundary conditions, which is well defined on the space $\{v \in L^2(\Omega); \langle v \rangle = 0\}$.

Multiplying (2.3) by u_t, and integrating by parts, we get

$$\int_\Omega |(-\Delta_N)^{-\frac{1}{2}} u_t|^2\ \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t} E(u(t)), \tag{2.4}$$

where the free energy functional

$$E(v) = \int_\Omega \Big(\frac{1}{2} |\nabla v|^2 + F(v) \Big)\ \mathrm{d}x \tag{2.5}$$

represents the starting point in the derivation of the Cahn-Hilliard equation.

The following existence Theorem is proved in [1].

Theorem 2.1 [see [1, Theorem 1.2]] *Let f satisfy (1.4). Let the initial function u_0 satisfy*

$$u_0 \in H^1(\Omega) \ \textit{and} \ \ E(u_0) < \infty. \tag{2.6}$$

Then there is a unique solution $u \in L^\infty((0,\infty); H^1(\Omega))$ of (1.1-1.2) with $u_t \in L^2((0,\infty); H^{-1}(\Omega))$ such that

$$\kappa u \in L^\infty((0,\infty); H^2(\Omega)) \tag{2.7}$$

$$\kappa f(u) \in L^\infty((0,\infty); L^2(\Omega)) \tag{2.8}$$

$$\kappa u_t \in L^\infty((0,\infty); H^{-1}(\Omega)) \cap L^2((0,\infty); H^1(\Omega)), \tag{2.9}$$

where $\kappa(t) = \left(\frac{t}{1+t}\right)^{\frac{1}{2}}$.

Remark The regularization effect of the equation allows us to consider also the initial values from the space $L^\infty(\Omega)$. We may use the procedure of approximation of u_0 by functions satisfying (2.6) as in the proof of Theorem 2.2 in [7], which asserts that for $u_0 \in L^2(\Omega)$ satisfying $\langle u_0 \rangle \in (-1,1)$, and $\|u_0\|_{L^\infty(\Omega)} \leq 1$ there is a unique solution in the space

$$u \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H^2(\Omega)),\ \ \kappa u_t \in L^2((0,\infty); H^1(\Omega)). \tag{2.10}$$

It is also shown in [7] that $\kappa^\alpha u_t \in L^\infty((0,\infty); H^{-1}(\Omega))$ with $\alpha > 2$. So, the choice of a weaker initial value u_0, satisfying the assumptions of Theorem 1.1 does not affect our considerations for $t \geq \varepsilon$.

3 Proof of Theorem 1.1

With the above Theorem 2.1 at hand, we can improve the regularity of solutions.

The statement (1.17) follows immediately from (2.9). Consequently, given ε, there exists a constant C_ε such that

$$\|u_t(t)\|_{H^{-1}(\Omega)} \le C_\varepsilon, \text{ i.e., } \|(-\Delta_N)^{-1}u_t(t)\|_{H^1(\Omega)} \le C_\varepsilon \text{ for all } t \ge \varepsilon, \tag{3.1}$$

$$\|u_t\|_{L^2((\varepsilon,T);H^1(\Omega))} \le C_\varepsilon. \tag{3.2}$$

Here and in what follows, C_ε denotes a constant that depends on ε, and c denotes a generic constant which may vary even within one line.

Similarly, directly from Theorem 2.1, we have

$$\|u(t)\|_{H^2(\Omega)} \le C_\varepsilon, \quad \||\nabla u(t)|\|_{L^6(\Omega)} \le c\||\nabla u(t)|\|_{H^1(\Omega)} \le C_\varepsilon, \quad \|f(u(t))\|_{L^2(\Omega)} \le C_\varepsilon \tag{3.3}$$

for all $t \ge \varepsilon$, where the second inequality follows from the imbedding theorem.

Multiplying (2.3) by $-f(u(t))^5$, and integrating over Ω, we get

$$\int_\Omega (-\Delta_N)^{-1}u_t(t)(-f(u(t))^5)\mathrm{d}x$$

$$= 5\int_\Omega f(u(t))^4 f'(u(t))|\nabla u(t)|^2\mathrm{d}x + \int_\Omega f(u(t))^6\mathrm{d}x - \langle f(u(t))\rangle \int_\Omega f(u(t))^5\mathrm{d}x$$

$$\ge \|f(u(t))\|^6_{L^6(\Omega)} - \|f(u(t))\|_{L^1}\|f(u(t))\|^5_{L^6(\Omega)} - 5d\||\nabla u(t)|^2\|_{L^3(\Omega)}\|f(u(t))^4\|_{L^{\frac{3}{2}}(\Omega)},$$

$$= \|f(u(t))\|^6_{L^6(\Omega)} - \|f(u(t))\|_{L^1}\|f(u(t))\|^5_{L^6(\Omega)} - 5d\||\nabla u(t)|\|^2_{L^6(\Omega)}\|f(u(t))\|^4_{L^6(\Omega)},$$

where we used (1.4).

On the other hand,

$$\int_\Omega (-\Delta_N)^{-1}u_t(t)(-f(u(t))^5)\ \mathrm{d}x \le \|(-\Delta_N)^{-1}u_t(t)\|_{L^6(\Omega)}\|f(u(t))^5\|_{L^{\frac{6}{5}}(\Omega)}$$

$$\le \|(-\Delta_N)^{-1}u_t(t)\|_{H^1(\Omega)}\|f(u(t))\|^5_{L^6(\Omega)} \le C_\varepsilon\|f(u(t))\|^5_{L^6(\Omega)},$$

where the last step follows by (3.1). It means that

$$\|f(u(t))\|^6_{L^6(\Omega)} \le \|f(u(t))\|_{L^1}\|f(u(t))\|^5_{L^6(\Omega)} + 5d\||\nabla u(t)|\|^2_{L^6(\Omega)}\|f(u(t))\|^4_{L^6(\Omega)}$$

$$+C_\varepsilon\|f(u(t))\|^5_{L^6(\Omega)}.$$

Hence, taking into account (3.3), we arrive at

$$\|f(u(t))\|_{L^6(\Omega)} \le C_\varepsilon \text{ for all } t \ge \varepsilon. \tag{3.4}$$

Consequently, from (2.3), (3.1), and the imbedding $H^1(\Omega) \subset L^6(\Omega)$ we deduce

$$\Delta u(t) \in L^6(\Omega),$$

which, together with (3.3), implies

$$\|u(t)\|_{W^{2,6}(\Omega)} \le C_\varepsilon, \ t \ge \varepsilon, \tag{3.5}$$

and (1.15) follows. Then also

$$\|\nabla u(t)\|_{W^{1,6}(\Omega)} \le C_\varepsilon, \tag{3.6}$$

and from the imbedding theorem

$$\nabla u(t) \in (C^{\frac{1}{2}}(\overline{\Omega}))^3, \ \ \||\nabla u(t)|\|_{C^{\frac{1}{2}}(\overline{\Omega})} \le C_\varepsilon, \ t \ge \varepsilon, \tag{3.7}$$

which yields (1.18).

The regularity class (1.19) is a consequence of the following estimates, where we used the fact that $u_t \in L^4(0,T;L^2(\Omega)$ by (2.9) and the interpolation between $H^{-1}(\Omega)$ and $H^1(\Omega)$:

Let $\nu \in (0, \frac{1}{2}]$, $p = \frac{3}{1-\nu}$ (thus $p \le 6$), and $\theta = \frac{3}{p} - \frac{1}{2}$, and $t_1, t_2 \ge \varepsilon > 0$. Then

$$\|u(t_1) - u(t_2)\|_{C^\nu(\Omega)} \le \|u(t_1) - u(t_2)\|_{W^{1,p}(\Omega)}$$

$$\le \|u(t_1) - u(t_2)\|^{\frac{1}{2}}_{L^p(\Omega)} \cdot \|u(t_1) - u(t_2)\|^{\frac{1}{2}}_{W^{2,p}(\Omega)}$$

$$\le c\|u(t_1) - u(t_2)\|^{\frac{1}{2}}_{L^p(\Omega)} \le c\Big(\int_{t_1}^{t_2} \|u_t\|_{L^p(\Omega)}\Big)^{\frac{1}{2}}$$

$$\le c\Big(\int_{t_1}^{t_2} \|u_t\|^\theta_{L^2(\Omega)} \cdot \|u_t\|^{1-\theta}_{L^6(\Omega)}\Big)^{\frac{1}{2}}$$

$$\le c\Big(\Big\|\|u_t\|^\theta_{L^2(\Omega)}\Big\|_{L^{\frac{4}{\theta}}(t_1,t_2)} \cdot \Big\|\|u_t\|^{1-\theta}_{L^6(\Omega)}\Big\|_{L^{\frac{2}{1-\theta}}(t_1,t_2)} \cdot \|1\|_{L^{\frac{4}{2+\theta}}(t_1-t_2)}\Big)^{\frac{1}{2}}$$

$$\le c|t_1 - t_2|^{\frac{2+\theta}{8}} = c|t_1 - t_2|^{\frac{5-2\nu}{16}}$$

Hence $u \in C^\mu((\varepsilon, T); C^\nu(\Omega))$ with $\nu \in (0, \frac{1}{2}]$, $\mu = \frac{5-2\nu}{16}$, and $\varepsilon > 0$.

The last goal is to show the separation property (1.21). To this end, we examine points (t_0, x_0) where, possibly, $|u(t_0, x_0)| = 1$. Assume $x_0 \in \Omega$. Let $u(t_0, x_0) = \min\{u(t_0, x); x \in \Omega\} = -1$. It means that $\nabla u(t_0, x_0) = 0$, and by (1.18) there exists $\mathcal{U} \subset \Omega$, a neighborhood of x_0, such that

$$|\nabla u(t_0, x)| \le c|x - x_0|^{\frac{1}{2}}. \tag{3.8}$$

Then

$$u(t_0, x) - (-1) = u(t_0, x) - u(t_0, x_0) = \int_0^1 \frac{\partial}{\partial s} u(t_0, sx + (1-s)x_0) \mathrm{d}s$$

$$\le |x - x_0| \int_0^1 |\nabla u(t_0, sx + (1-s)x_0)| \mathrm{d}s$$

$$= |x - x_0| \int_0^1 \nabla u(t_0, sx + (1-s)x_0) - \nabla u(t_0, x_0)| \mathrm{d}s$$

$$\leq c|x - x_0| \int_0^1 s^{\frac{1}{2}} |x - x_0|^{\frac{1}{2}} \mathrm{d}s \leq c|x - x_0|^{\frac{3}{2}},$$

by (3.7). This yields

$$u(t_0, x) \leq -1 + c|x - x_0|^{\frac{3}{2}}, \ x \in \mathcal{U}. \tag{3.9}$$

If $x_0 \in \partial\Omega$, then again $\nabla u(t_0, x_0) = 0$ since

$$\nabla u(t_0, x_0) = \nabla_{\partial\Omega} u(t_0, x_0) + \partial_\nu u(t_0, x_0)\nu(x_0),$$

where $\nabla_{\partial\Omega}$ denotes the surface gradient on $\partial\Omega$ and $\nu(x_0)$ is the outer unit normal in x_0. If the boundary $\partial\Omega$ is sufficiently smooth, then there exists an extension $\tilde{u}$ of $u(t_0, .)$ defined on a neighborhood $\mathcal{V}$ of x_0 with an equivalent $W^{2,6}$-norm, such that $\tilde{u}(x_0) = -1, \ \nabla\tilde{u}(x_0) = 0$. Then the same argument as above applies, and (3.9) holds in $\mathcal{V} \cap \Omega$. Alternatively, the construction given in the proof of [1, Proposition 6.1] to deal with the boundary points in the case of C^3 boundary can be used to get (3.9).

In the same way, we deduce that if there is a point (t_0, x_0) such that $u(t_0, x_0) = 1 = \max\{u(t_0, x); x \in \Omega\}$, then

$$u(t_0, x) \geq 1 - c|x - x_0|^{\frac{3}{2}}, \ x \in \mathcal{U}. \tag{3.10}$$

Assume now that f satisfies (1.20), and $u(t_0, x_0) = -1$. We already know from [1], or [10] that $u(t_0, x) \in (-1, 1)$ for a.e. $x \in \Omega$. By (1.20) and (3.9) we deduce that

$$f(u(t_0, x)) \leq -c(1 + u(t_0, x))^{-\frac{1}{3}} \leq c|x - x_0|^{-\frac{1}{2}}, \text{ for a.e. } x \in \mathcal{U}.$$

Since $[x \mapsto |x - x_0|^{-\frac{1}{2}}] \notin L^6(\mathcal{U})$, this yields a contradiction to (3.4).

In the same way we get a contradiction when $u(t_0, x_0) = 1$ at some point (t_0, x_0), $t_0 \geq \varepsilon, \ x_0 \in \Omega$. Theorem 1.1 has been proved.

Remarks.

1. To prove the separation property, it is sufficient to show

$$\|(-\Delta_N)^{-1} u_t\|_{L^\infty(\Omega)} \leq c \text{ for all } t \geq \varepsilon. \tag{3.11}$$

Then it follows from the equation (2.3) that $\Delta u = (-\Delta)^{-1} u_t - \langle f(u) \rangle + f(u) \leq 0$ almost everywhere in some neighborhood of (t_0, x_0) with $u(t_0, x_0) = -1$ because of the singularity of $f(u)$ at $u = -1$. This is in contradiction with $u(t_0, x_0) = -1 = \min\{u(t_0, x); x \in \Omega\}$. In the same way we would get $\Delta u \geq 0$ a.e. in some neighborhood of a maximum point (t_0, x_0). Hence the immediate separation holds in this case without any additional assumptions on f.

2. The condition (1.6) implies that $u_t \in L^\infty((\varepsilon, \infty); L^2(\Omega))$, so that (3.11) holds. Indeed, differentiate (2.3) with respect to t, multiply by $\Delta_N u_t$, and integrate over Ω. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \|u_t(t)\|^2_{L^2(\Omega)} \leq -\|\Delta_N u_t(t)\|^2_{L^2(\Omega)} + \frac{1}{2} \|f(u(t))_t\|^2_{L^2(\Omega)} + \frac{1}{2} \|\Delta_N u_t(t)\|^2_{L^2(\Omega)}$$

$$\le c\frac{1}{2}\|f'(u(t))u_t(t)\|^2_{L^2(\Omega)} \le c\|f'(u(t))^2\|_{L^{\frac{3}{2}}(\Omega)}\|u_t^2(t)\|_{L^3(\Omega)}$$

$$\le \|f(u(t))\|^4_{L^6(\Omega)}\|u_t(t)\|^2_{H^1(\Omega)},$$

and (3.11) follows from (1.16) and (1.17). The same procedure with the bounds $\|f(u)\|_{L^8((t,t+1);L^\infty(\Omega))} \le c$, $\|u_t\|_{L^4((t,t+1);L^2(\Omega))} \le c$ was applied in [10]. The condition (1.6) can be relaxed showing that $\|u_t(t)\|_{H_2^{-1+s}} \le c$ for $s > \frac{1}{2}$, $t \ge \varepsilon$, and employing the imbedding theorem. This can be achieved by multiplying the time derivative of (2.3) by $(-\Delta_N)^s u_t$ instead of $\Delta_N u_t$, and realizing that

$$\|f'(u(t))u_t(t)\|_{H_2^{-1+s}} \le c\|f'(u(t))\|_{H_3^{-1+s}(\Omega)}\|u_t(t)\|_{H_2^1(\Omega)}$$

$$\le c\|f'(u(t))\|_{L^p(\Omega)}\|u_t(t)\|_{H^1(\Omega)} \text{ with } p > 2.$$

See [13, sect. 4.4, Theorem 2, and sect. 2.2.3, Remark 2]. Hence we have (1.21) if $f'(u)(t) \in L^p(\Omega)$ for some $p > 2$, $t \ge \varepsilon$, in particular, if $|f'(u)| \le c(|f(u)|^q + 1)$ for some $q < 3$. This is still a stronger assumption than (1.20).

3. In one space dimension, the imbedding $H^1(\Omega) \subset C(\Omega)$ and (1.17) ensures that

$$|(-\Delta)^{-1}u_t(t,x)| \le M \text{ for all } t \ge \varepsilon,\ x \in \Omega.$$

4. In the two-dimensional case, $H^1(\Omega) \subset L^p(\Omega)$ for any $p > 0$, and we can proceed as in the proof of Theorem 1.1 to show that $f(u) \in L^\infty((\varepsilon,\infty), L^p(\Omega))$, and, accordingly, we get separation property for any negative power $f(u) \sim (1-u^2)^{-\alpha}$, $\alpha > 0$. However, this still does not solve the problem for the logarithmic potential (1.3). It was solved in [10, Theorem 7.2] for f satisfying the additional assumption $|f'(u)| \le \exp(C_1|f(u)| + C_2)$, which is satisfied by the logarithmic nonlinearity. Using the Orlicz imbedding theorem, the authors obtained estimates allowing to prove that $(-\Delta)^{-1}u_t$ is bounded in the same way as in the case $f' \le c(f^2+1)$.

3.1 Regularity

Assume $f \in C^{2+\mu}(-1,1)$, $\mu > 0$, and satisfies (1.20) and (1.4). Let $\varepsilon > 0$ be given. Since (1.15) holds for any $\varepsilon > 0$, there is $\gamma \in (0,1)$ such that

$$\|u(t)\|_{C(\overline{\Omega})} \le 1-\gamma \text{ for all } t \ge \varepsilon/2, \text{ and}$$

$$f(u) \in L^\infty((\varepsilon/2,\infty); W^{2,6}(\Omega)),\quad \Delta f(u) \in L^\infty((\varepsilon/2,\infty); L^6(\Omega)).$$

First, we apply the maximal regularity of parabolic equations to the equation

$$u_t + \Delta^2 u = \Delta f(u) \text{ with boundary condtitions (1.2)},$$

in L^p spaces. We take some $t_0 \in (\varepsilon/2, \varepsilon)$ such that $u(t_0) \in W^{2,6}(\Omega)$. Then the solution u with the initial value $u(t_0)$ belongs to the space $L^2((t_0,T), W^{4,6}(\Omega))$. Then we can find $t_1 \le \varepsilon$ with $u(t_1) \in W^{4,6}(\Omega)$ as an initial value of u, which yields that

$$u \in L^r((\varepsilon,T); W^{4,6}(\Omega)),\quad u_t \in L^r((\varepsilon,T); L^6(\Omega)),\quad 1 \le r < \infty,\quad T \in (\varepsilon,\infty).$$

Imbedding theorems then yield

$$u \in C((\varepsilon, T); W^{3,6}(\Omega)), \text{ and, conseqently, } u \in C((\varepsilon, T); C^{2,\frac{1}{2}}(\Omega)).$$

Then, assuming $f \in C^{2+\mu}(-1,1)$, $\mu > 0$, also $f(u) \in C((\varepsilon, T); C^{2,\mu}(\Omega))$, $\Delta f(u) \in C((\varepsilon, T); C^{\mu}(\Omega))$, and subsequent application of the maximal regularity theorem, this time in the space of Hölder continuous functions, yields a classical solution u of (1.1).

4 Nonisothermal case

In this section, we prove Propositions 1.1 and 1.3.

To obtain an existence result corresponding to Theorem 2.1, we reformulate (1.8), (1.9) in terms of enthalpy $e = u + \frac{1}{2}\vartheta$ and u as follows:

$$u_t = \Delta(-\Delta u + f(u) + u - 2e), \tag{4.1}$$

$$e_t - \Delta e = -\frac{1}{2}\Delta u. \tag{4.2}$$

This allows us to repeat the existence proof in [1], based on the Lipschitz perturbation of monotone operators. To see it, denote by Bu, $u \in L^2((0,T); H^1(\Omega))$, the unique solution of (4.2) in the space $L^2((0,T); H^1(\Omega))$ with the Neumann boundary conditions and the integral mean given by the (conserved) initial function e_0. The operator B is a bounded linear operator from $L^2((0,T); H^1(\Omega))$ into itself, it has an explicit representation

$$B = (-\Delta_N)^{\frac{1}{2}}(\partial_t - \Delta_N)^{-1}(-\Delta_N)^{\frac{1}{2}},$$

and we can rewrite (4.1) in the form

$$u_t + \Delta(\Delta u - f(u) - du) = \Delta(-du + u + 2Bu). \tag{4.3}$$

Assuming (1.4), $f(u) + du$ on the left hand side is monotone, and the right hand side is Lipschitz continuous from $L^2((0,T); H^1(\Omega))$ into $L^2((0,T); H^{-1}(\Omega))$. With the initial condition (2.6), we can proceed in the same way as in [1] (see [1, Theorem 3.1]) to get the existence of a unique solution to (4.3) satisfying (2.7)-(2.9). Regularity of the enthalpy e, and, consequently, of the temperature ϑ follows from the maximal regularity of parabolic equation (4.2) with the right hand side $-\frac{1}{2}\Delta u$ where u satisfies (2.7), (2.10). For $e_0 \in H^1(\Omega)$ we have

$$e \in L^2((0,\infty); H^2(\Omega)) \cap L^\infty((0,\infty); H^1(\Omega)), \tag{4.4}$$

$$\kappa e_t \in L^\infty((0,\infty); H^{-1}(\Omega)) \cap L^2((0,\infty); H^1(\Omega)), \tag{4.5}$$

so

$$\|\vartheta(t)\|_{H^1(\Omega)} \le 2\|e(t)\|_{H^1(\Omega)} + 2\|u(t)\|_{H^1(\Omega)} \le C_\varepsilon \text{ for all } t \ge \varepsilon. \tag{4.6}$$

A result analogous to Theorem 1.1 can be deduced from (1.8) rewritten in the form

$$(-\Delta_N)^{-1}u_t - 2\vartheta = \Delta_N u - f(u) + \langle f(u)\rangle - 2\langle\vartheta\rangle, \tag{4.7}$$

with the left hand side in $L^\infty((\varepsilon,\infty),L^6(\Omega))$. Then, multiplying (4.7) by $f(u)^5$, we get, in the same way as in the previous section,

$$\|f(u(t))\|_{L^6(\Omega)} \leq C_\varepsilon, \quad \|u(t)\|_{W^{2,6}(\Omega)} \leq C_\varepsilon \quad \text{for } t \geq \varepsilon. \tag{4.8}$$

Consequently, we get also the separation property (1.21) provided that f satisfies (1.20). Using again the maximal regularity of parabolic equation for the enthalpy e with the right hand side in $L^\infty((\varepsilon,\infty);L^6(\Omega))$ yields finally that the temperature satisfies (1.23), (1.24), and Proposition 1.1 follows.

4.1 Long-time behavior-convergence to equilibrium

If f satisfies (1.4), and in addition f is real analytic in (-1,1), we can also prove convergence of any solution (with u_0 as in (1.14) and $\vartheta_0 \in H^1(\Omega)$) to a single equilibrium $(u_\infty,\vartheta_\infty)$, a solution of the stationary problem (1.11)-(1.13).

In this case, multiplying the equation (1.8) by $(-\Delta_N)^{-1}u_t$, and (1.9) by $4(-\Delta_N)^{-1})(\vartheta + \frac{1}{2}u_t)$, and adding, we deduce the energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\nabla u|^2 + F(u) + 2\vartheta^2 \,\mathrm{d}x =$$

$$-\int_\Omega |(-\Delta_N)^{-\frac{1}{2}}u_t|^2 + 4|(-\Delta_N^{-\frac{1}{2}}(\vartheta_t + \frac{1}{2}u_t)|^2 \,\mathrm{d}x.$$

This implies in particular that $J(u,\vartheta) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u) + 2\vartheta^2 \,\mathrm{d}x$ is a strict Lyapunov functional. Since $\langle\vartheta(t)\rangle = \langle\vartheta_0\rangle$ for all $t \geq 0$ because of (1.10), and $v(t) = \vartheta(t) - \langle\vartheta_0\rangle$ is a solution of the parabolic equation (1.9) with the right hand side u_t satisfying (2.2), (2.9), we get that $v(t) \to 0$ for $t \to \infty$, say, in $L^2(\Omega)$. It follows from Proposition 1.2 that the trajectory

$$\cup_{t\geq T}\vartheta(t),$$

is precompact in the space $W^{s,6}(\Omega)$ for any $s < 2$, and, by imbedding, $W^{s,6}(\Omega) \subset C^{s-\frac{1}{2}}(\Omega)$. Then we have also

$$\vartheta(t) \to \langle\vartheta_0\rangle \text{ as } t \to \infty \text{ in } C^\nu(\Omega),\ \nu < \frac{3}{2}.$$

To show convergence of $u(t)$ to a single stationary point u_∞, we proceed in the same way as in [9] for the Caginalp model with a regular potential. The proof is based on the fact that solutions of the stationary problem are separated from the singular points of the nonlinearity (see [1, Proposition 6.1], or [9, Lemma 3.1]), and hence f is real analytic on some neighborhood of the set of equilibria. An application of the Łojasiewicz-Simon inequality on the energy functional E given by (2.5), together with the convergence of temperature ϑ yields integrability of the time derivative of u. We get $u_t \in L^1((T,\infty);H^{-1}(\Omega))$ for some $T > \varepsilon$, and, consequently, the convergence of $u(t)$ in $H^{-1}(\Omega)$ for $t \to \infty$. With the compactness of the trajectory of u at hand (see (1.25)), we arrived at Proposition 1.3.

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