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Abstract

We consider a coupled PDE-ODE system describing the motion of the rigid body in a container filled with the incompressible, viscous fluid. The fluid and the rigid body are coupled via Navier's slip boundary condition. We prove that the local in time strong solution is unique in the larger class of weak solutions on the interval of its existence. This is the first weak-strong uniqueness result in the area of fluid-structure interaction.

1 Introduction

A solid body motion in a fluid is a widespread phenomenon in nature, being one of the most classical problem of fluid mechanics. The understanding of the correct mathematical description of fluid-structure interaction has several important applications in many branches, such as in civil engineering, aerospace engineering, nuclear engineering, ocean engineering, biomechanics and etc..

As well-known [11] the fluid motion fulfills the Navier-Stokes equations and the solid motion is described by a system of ordinary differential equations of momentum conservation laws. The fluid and the solids can be coupled through a standard non-slip boundary condition: the continuity of velocities of the fluid and the solids at the fluid-body interfaces. Such approach has been investigated by many authors [3]-[7], [16, 24]. However, it has been shown in [14], [15], [23] that the non-slip condition exhibits an unrealistic phenomenon: two smooth solids can not touch each other. The non-slip condition prescribes the adherence of fluid particles to the solid boundaries and, as a consequence of a regularity of the fluid velocity, permits the creation of fine boundary layer that does not allow the contact of the solids.

Another method for coupling of the fluid and of the bodies admits the slippage of fluid particles at the boundaries, which is described by Navier's boundary condition. The first step in this direction of the study of Navier's condition was done by Neustupa, Penel [21], [22], who demonstrate that the collision with a wall can occur for a prescribed movement of a solid ball, when the slippage was allowed on both boundaries. We refer for a discussion of Navier's boundary condition to Introduction of [19]. In this last work a local in time existence result was demonstrated for the motion of the fluid and an elastic structure with prescribed Navier's condition on the boundaries. Also we mention the article [12] where a local existence up collisions of a weak solution for a fluid-solid structure was proved. The existence of a strong solution in $2D$ case was proven by Wang [27].

The global in time existence of the weak solution was proven in [2] for a mixed case, when Navier's condition was given on the solid boundary and the non-slip condition on the domain boundary. This result admits the collisions of the solid with the domain boundary. Recently the local in time existence of the strong solution for the mixed case was demonstrated in [1]. The main purpose of this article is to show the weak-strong uniqueness result.

2 Preliminaries

We shall investigate the motion of a rigid body inside of a viscous incompressible fluid. The fluid and the body occupy a bounded open domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3). Let the body be a connected open set $S_0 \subset \Omega$ at the initial time $t = 0$. The fluid fills the domain $F_0 = \Omega \backslash \overline{S_0}$ at $t = 0$.

The Cartesian coordinates $\mathbf{y}$ of points of the body at $t = 0$ are called the Lagrangian coordinates. The motion of any material point $\mathbf{y} = (y_1, .., y_N)^T \in S_0$ is described by two functions

$$t \to \mathbf{q}(t) \in \mathbb{R}^N \quad \text{and} \quad t \mapsto \mathbb{Q}(t) \in SO(N) \qquad \text{for} \quad t \in [0, T],$$

where $\mathbf{q} = \mathbf{q}(t)$ is the position of the body mass center at a time t and $SO(N)$ is the rotation group in $\mathbb{R}^N$, i.e. the $\mathbb{Q} = \mathbb{Q}(t)$ is a matrix, satisfying $\mathbb{Q}(t)\mathbb{Q}(t)^T = \mathbb{I}$, $\mathbb{Q}(0) = \mathbb{I}$ with $\mathbb{I}$ being the identity matrix. Therefore, the trajectories of all points of the body are described by a preserving orientation isometry

$$\mathbf{B}(t, \mathbf{y}) = \mathbf{q}(t) + \mathbb{Q}(t)(\mathbf{y} - \mathbf{q}(0)) \qquad \text{for any} \quad \mathbf{y} \in S_0 \tag{2.1}$$

and the body occupies the set

$$S(t) = \{\mathbf{x} \in \mathbb{R}^N : \ \mathbf{x} = \mathbf{B}(t, \mathbf{y}), \quad \mathbf{y} \in S_0\} = \mathbf{B}(t, S_0) \tag{2.2}$$

at any time t. The velocity of the body, called *rigid velocity*, is defined as

$$\frac{d}{dt}\mathbf{X}(t,\mathbf{y}) = \mathbf{u}_s(t,\mathbf{x}) = \mathbf{a}(t) + \mathbb{P}(t)(\mathbf{x} - \mathbf{q}(t)) \qquad \text{for all} \quad \mathbf{x} \in S(t), \tag{2.3}$$

where $\mathbf{a} = \mathbf{a}(t) \in \mathbb{R}^N$ is the translation velocity and $\mathbb{P} = \mathbb{P}(t)$ is the angular velocity. The velocity $\mathbf{u}_s$ has to be compatible with $\mathbf{B}$ in the sense

$$\frac{d\mathbf{q}}{dt} = \mathbf{a} \quad \text{and} \quad \frac{d\mathbb{Q}}{dt}\mathbb{Q}^T = \mathbb{P} \quad \text{in } [0,T]. \tag{2.4}$$

The angular velocity $\mathbb{P}$ is a skew–symmetric matrix, i.e. there exists a vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t) \in \mathbb{R}^N$, such that

$$\mathbb{P}(t)\mathbf{x} = \boldsymbol{\omega}(t) \times \mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{R}^N. \tag{2.5}$$

We define the fluid domain as $\Omega_F(t) = \Omega \setminus \overline{S(t)}$. For simplicity we admit that the densities of the fluid and the rigid body are equal to 1. We consider the following problem modeling the motion of the rigid body in viscous incompressible fluids.
Find $(\mathbf{u}, p, \mathbf{q}, \mathbb{Q})$ such that

$$\left.\begin{array}{l} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \operatorname{div}\left(\mathbb{T}(\mathbf{u},p)\right), \\ \operatorname{div}\, \mathbf{u} = 0 \end{array}\right\} \text{ in } \Omega_F(t) \times (0,T), \tag{2.6}$$

$$\left.\begin{array}{l} \frac{d^2}{dt^2}\mathbf{q} = -\int_{\partial S(t)} \mathbb{T}(\mathbf{u},p)\mathbf{n}\, d\gamma(\mathbf{x}), \\ \frac{d}{dt}(J\boldsymbol{\omega}) = -\int_{\partial S(t)} (\mathbf{x} - \mathbf{q}(t)) \times \mathbb{T}(\mathbf{u},p)\mathbf{n}\, d\gamma(\mathbf{x}) \end{array}\right\} \text{ in } (0,T), \tag{2.7}$$

$$(\mathbf{u} - \mathbf{u}_s)\cdot\mathbf{n} = 0, \qquad \beta(\mathbf{u}_s - \mathbf{u})\cdot\boldsymbol{\tau} = \mathbb{T}(\mathbf{u},p)\mathbf{n}\cdot\boldsymbol{\tau} \qquad \text{on } \partial S(t), \tag{2.8}$$

$$\mathbf{u}(0,.) = \mathbf{u}_0 \quad \text{in } \Omega; \qquad \mathbf{q}(0) = \mathbf{q}_0, \qquad \mathbf{q}'(0) = \mathbf{a}_0, \qquad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0, \tag{2.9}$$

where $\mathbf{n}(x)$ is the unit *interior* normal at $\mathbf{x} \in \partial S(t)$, i.e. the vector $\mathbf{n}$ is directed inside of $S(t)$. The surface measure over a moving surface $\partial S(t)$ is indicated by $d\gamma$. $\mathbb{J}$ is the matrix of the inertia moments of the body $S(t)$ related to its mass center, calculated as

$$\mathbb{J} = \int_{S(t)} \left(|\mathbf{x} - \mathbf{q}(t)|^2 \mathbb{I} - (\mathbf{x} - \mathbf{q}(t)) \otimes (\mathbf{x} - \mathbf{q}(t))\right) d\mathbf{x}.$$

In (2.6) $\mathbf{u}$ is the fluid velocity; $\mathbb{T}$ is the stress tensor and $\mathbb{D}$ is the deformation-rate tensor, which are defined as

$$\mathbb{T} = -pI + 2\mu\,\mathbb{D}\mathbf{u} \qquad \text{and} \qquad \mathbb{D}\mathbf{u} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right),$$

with p being the fluid pressure and $\mu > 0$ being the constant viscosity of the fluid.

Let us introduce the definition of weak solutions for system (2.6)-(2.9). To begin with we define the space [9, 20]:

$$V^{0,2}(\Omega) = \{\mathbf{v} \in L^2(\Omega) : \ \operatorname{div} \mathbf{v} = 0 \quad \text{in } \mathcal{D}'(\Omega), \qquad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{in } H^{-1/2}(\partial\Omega)\},$$

where $\mathbf{n}$ is the unit normal to the boundary of Ω. Let $\mathcal{M}(\Omega)$ be the space of bounded Radon measures. Let

$$BD_0(\Omega) = \left\{\mathbf{v} \in L^1(\Omega): \ \mathbb{D}\mathbf{v} \in \mathcal{M}(\Omega), \qquad \mathbf{v} = 0 \quad \text{on} \ \ \partial\Omega\right\}$$

be the space of functions of bounded deformation. Let S be an open connected subset of Ω with the boundary $\partial S \in C^2$. We introduce the following space of vector functions

$$\begin{aligned} KB(S) \ \ = \ \ \{&\mathbf{v} \in BD_0(\Omega): \ \mathbb{D}\mathbf{v} \in L^2(\Omega\backslash\overline{S}), \qquad \mathbb{D}\mathbf{v} = 0 \ \text{ a.e. on } S, \\ &\operatorname{div}\mathbf{v} = 0 \quad \text{in} \ \ \mathcal{D}'(\Omega)\} \, . \end{aligned}$$

Now, we can give a definition of the weak solutions of (2.6)-(2.9).

Definition 2.1 *The triple* $\{\mathbf{B}, \mathbf{u}\}$ *is a weak solution of system* (2.6)-(2.9)*, if the following conditions are satisfied:*

1) The function $\mathbf{B}(t,\cdot): \mathbb{R}^N \to \mathbb{R}^N$ *is a preserving orientation isometry* (2.1)*, which defines a time dependent set* $S(t)$ *by* (2.2)*. The isometry* $\mathbf{B}$ *is compatible with* $\mathbf{u} = \mathbf{u}_s$ *on* $S(t)$*: the functions* $\mathbf{q}$, $\mathbb{Q}$ *are absolutely continuous on* $[0, T]$ *and satisfy equalities* (2.3)-(2.5)*;*

2) The function $\mathbf{u} \in L^2(0, T; KB(S(t))) \cap L^\infty(0, T; V^{0,2}(\Omega))$ *satisfies the integral equality*

$$\int_0^T dt \int_{\Omega\backslash\partial S(t)} \{\mathbf{u}\boldsymbol{\psi}_t + (\mathbf{u}\otimes\mathbf{u}) : \mathbb{D}\boldsymbol{\psi} - 2\mu_f \, \mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi} \, \} d\mathbf{x}$$

$$= -\int_\Omega \mathbf{u}_0 \boldsymbol{\psi}(0,\cdot) \, d\mathbf{x} + \int_0^T dt \int_{\partial S(t)} \beta(\mathbf{u}_s - \mathbf{u}_f)(\boldsymbol{\psi}_s - \boldsymbol{\psi}_f) \, d\gamma, \quad (2.10)$$

which holds for any test function $\boldsymbol{\psi}$*, such that*

$$\begin{aligned} \boldsymbol{\psi} \ &\in \ L^{2(N-1)}(0, T; KB(S(t))), \\ \boldsymbol{\psi}_t \ &\in \ L^2(0, T; L^2(\Omega\backslash\partial S(t))), \qquad \boldsymbol{\psi}(T,\cdot) = 0. \end{aligned} \quad (2.11)$$

By $\mathbf{u}_s(t,\cdot)$, $\boldsymbol{\psi}_s(t,\cdot)$ *and* $\mathbf{u}_f(t,\cdot)$, $\boldsymbol{\psi}_f(t,\cdot)$ *we denote the trace values of* $\mathbf{u}$, $\boldsymbol{\psi}$ *on* $\partial S(t)$ *from the "rigid" side* $S(t)$ *and the "fluid" side* $F(t)$*, respectively.*

Let us recall the global solvability result proved in [2].

Theorem 2.1 *Let the boundaries be* $\partial\Omega \in C^{0,1}$, $\partial S_0 \in C^2$*. Let us assume that* $\overline{S_0} \subset \Omega$ *and*

$$\mathbf{u}_0 \in V^{0,2}(\Omega), \qquad \mathbb{D}\mathbf{u}_0 = 0 \qquad in \ \ \mathcal{D}'(S_0).$$

Then problem (2.6)-(2.9) *possesses a weak solution* $\{\mathbf{B}, \mathbf{u}\}$*, such that the isometry* $\mathbf{B}(t,\cdot)$ *is Lipschitz continuous with respect to* $t \in [0, T]$,

$$\mathbf{u} \in C_{\text{weak}}(0, T; V^{0,2}(\Omega)) \cap L^2(0, T; KB(S(t)))$$

and for a.a. $t \in (0, T)$ *the following energy inequality holds*

$$\frac{1}{2}\int_\Omega |\mathbf{u}|^2(t) \ d\mathbf{x} + \int_0^t dr \left\{ \int_{\Omega_F(t)} 2\mu \, |\mathbb{D}\,\mathbf{u}|^2 \ d\mathbf{x} + \int_{\partial S(r)} \beta|\mathbf{u} - \mathbf{u}_s|^2 \ d\gamma \right\} \le \frac{1}{2}\int_\Omega |\mathbf{u}_0|^2 \ d\mathbf{x}.$$

Let us introduce on the fluid domain $\Omega_F(t)$ for $t \in (0,T)$ the following function spaces:

$$L^2(\Omega_F^1(t)), \qquad L^\infty(\Omega_F^1(t)), \qquad W^{1,\infty}(\Omega_F^1(t)) \qquad \text{for a.e. } t \in (0,T)$$

and

$$L^2(0,T;L^2(\Omega_F(t))), \qquad L^2(0,T;H^k(\Omega_F(t))), \qquad H^k(0,T;L^2(\Omega_F(t))) \qquad \text{with } \; k = 1 \text{ or } 2.$$

Also we recall the local existence result for the strong solution obtained in [1].

Theorem 2.2 *Let the boundaries be $\partial\Omega$, $\partial S_0 \in C^2$. Suppose that $\overline{S_0} \subset \Omega$*

$$\mathbf{u}_0 \in KB(S_0).$$

Then problem (2.6)-(2.9) *possesses a local-in-time strong solution* $(\mathbf{B}, \mathbf{u})$ *such that the isometry* $\mathbf{B}(t,\cdot)$ *is Lipschitz continuous with respect to $t \in [0,T]$, and there exists a maximal $T_0 > 0$ such that* (2.6)-(2.9) *has a unique strong solution* $(\mathbf{u}, p, \mathbf{a}(t), \boldsymbol{\omega}(t))$ *which for all $T < T_0$ satisfies the following energy inequality:*

$$\|\mathbf{u}\|_{L^2(0,T;H^2(\Omega_F(t))} + \|p\|_{L^2(0,T;H^1(\Omega_F(t))} + \|\mathbf{a}\|_{H^1(0,T)} + \|\boldsymbol{\omega}\|_{H^1(0,T)} \le C.$$

Here and below we denote by C generic constants depending only on the data of our problem (2.6)-(2.9).

The aim of our paper is to prove the weak-strong uniqueness result for system (2.6)-(2.9). More precisely, we prove that on the interval $(0,T_0)$ where the strong solution exists, the strong solution is unique in the class of weak solutions given by Definition 2.1. To the best of our knowledge this is the first weak-strong uniqueness in the area of fluid-structure interaction. Uniqueness of weak solutions in $2D$ case for the fluid-rigid body system with no-slip coupling condition was proved in [18].

Let us end this section by a well-known *Reynolds transport theorem* from the fluid mechanics theory, which will be often used in our calculations on moving domains.

Lemma 2.1 *Let $V(t)$ be a time dependent volume moved by a smooth velocity $\mathbf{v} = \mathbf{v}(t,\mathbf{x})$. Then*

$$\frac{d}{dt}\int_{V(t)} f(t,\mathbf{x})\,dx = \int_{V(t)} \frac{df}{dt}\,d\mathbf{x} \tag{2.12}$$

for any smooth function $f = f(t,\mathbf{x})$. Here $\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla)f$ is the total time derivative.

3 Weak-strong uniqueness

Let $(\mathbf{u}_1, \mathbf{a}_1, \boldsymbol{\omega}_1)$ be the triplet consisting of the fluid velocity, the translation rigid body velocity and the angular rigid body velocity connected to the weak solution $(\mathbf{u}_1, \mathbf{B}_1)$ (see Definition 2.1), i.e.

$$\mathbf{B}_1(t, \mathbf{y}) = \mathbf{q}_1(t) + \mathbb{Q}_1(t)(\mathbf{y} - \mathbf{q}_0),$$

where $\mathbf{a}_1 = \mathbf{q}_1'(t) \in \mathbb{R}^3$ and $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_1(t) \in \mathbb{R}^3$ is associated with the skew–symmetric matrix $\mathbb{P}_1 = \mathbb{Q}_1'\mathbb{Q}_1^T$, satisfying the property

$$\boldsymbol{\omega}_1(t) \times \mathbf{x} = \mathbb{P}_1(t)\mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{R}^3. \tag{3.1}$$

We denote

$$S_1(t) = \mathbf{B}_1(t, S_0)$$

the domain of the rigid body at the time t and

$$\Omega_F^1(t) = \Omega \setminus S_1(t)$$

the corresponding fluid domain.

Moreover, let $(\mathbf{u}_2, \mathbf{a}_2, \boldsymbol{\omega}_2)$ be the strong solution given by Theorem 2.2 with the corresponding rigid deformation $\mathbf{B}_2$:

$$\mathbf{B}_2(t, \mathbf{y}) = \mathbf{q}_2(t) + \mathbb{Q}_2(t)(\mathbf{y} - \mathbf{q}_0)$$

where $\mathbf{a}_2 = \mathbf{q}_2'$ and $\boldsymbol{\omega}_2$ is associated with the skew–symmetric matrix $\mathbb{P}_2 = \mathbb{Q}_2'\mathbb{Q}_2^T$, satisfying

$$\boldsymbol{\omega}_2(t) \times \mathbf{x} = \mathbb{P}_2(t)\mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{R}^3. \tag{3.2}$$

Also as before, we denote

$$S_2(t) = \mathbf{B}_2(t, S_0)$$

the domain of the rigid body at the time t and

$$\Omega_F^2(t) = \Omega \setminus S_2(t)$$

the corresponding fluid domain.

In this article our main objective is to demonstrate the following weak-strong uniqueness theorem.

Theorem 3.1 *We will prove the weak-strong uniqueness result, i.e.*

$$(\mathbf{u}_1, \mathbf{a}_1, \boldsymbol{\omega}_1) = (\mathbf{u}_2, \mathbf{a}_2, \boldsymbol{\omega}_2) \qquad \text{on the interval } (0, T_0)$$

where the strong solution $(\mathbf{u}_2, \mathbf{a}_2, \boldsymbol{\omega}_2)$ *exists.*

The demonstration of this theorem we divide on few steps, proving auxiliary Lemmas 3.1-3.4. The major difficulty in the study of this uniqueness result consists from the fact that the fluid domains of $\mathbf{u}_1$ and $\mathbf{u}_2$ are a priori different and therefore we need to transform the strong solution to the fluid domain of the weak solution in order to compare them.

Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two time dependent changes of variables defined in Appendix 4. Furthermore we define the inverse transform of $\mathbf{X}_2$, i.e.

$$\mathbf{Y}_2(t,\cdot) = \mathbf{X}_2(t,\cdot)^{-1}.$$

It is easy to see

$$\mathbf{Y}_2(t,\mathbf{x}_2) = \mathbf{q}_0 + \mathbb{Q}_2^T(t)(\mathbf{x}_2 - \mathbf{q}_2(t)), \qquad \mathbf{x}_2 \in \partial S_2(t).$$

Finally we define the transformation $\widetilde{\mathbf{X}}_1 : \Omega_F^2(t) \to \Omega_F^1(t)$ in the following way:

$$\widetilde{\mathbf{X}}_1(t,\mathbf{x}_2) = \mathbf{X}_1(t,\mathbf{Y}_2(t,\mathbf{x}_2))$$

and let $\widetilde{\mathbf{X}}_2(t,\mathbf{x}_1)$ be its inverse, i.e.

$$\widetilde{\mathbf{X}}_2(t,\cdot) = \widetilde{\mathbf{X}}_1(t,\cdot)^{-1}.$$

In the neighborhoods of $S_1(t)$ and $S_2(t)$ the transformations $\widetilde{\mathbf{X}}_2$ and $\widetilde{\mathbf{X}}_1$ are rigid. They are given with the following expressions:

$$\begin{cases} \widetilde{\mathbf{X}}_1(t,\mathbf{x}_2) = \mathbf{q}_1(t) + \mathbb{Q}_1(t)\mathbb{Q}_2^T(t)(\mathbf{x}_2 - \mathbf{q}_2(t)) & \text{in the neighborhood of } S_2(t), \\ \widetilde{\mathbf{X}}_2(t,\mathbf{x}_1) = \mathbf{q}_2(t) + \mathbb{Q}_2(t)\mathbb{Q}_1^T(t)(\mathbf{x}_1 - \mathbf{q}_1(t)) & \text{in the neighborhood of } S_1(t). \end{cases} \tag{3.3}$$

Furthermore we put $\mathbb{Q} = \mathbb{Q}_2\mathbb{Q}_1^T$. Now we define a transformed solution of the strong solution $(\mathbf{u}_2, p_2, \mathbf{a}_2, \boldsymbol{\omega}_2)$, where p_2 is a respective pressure (see Appendix 4):

$$\begin{cases} \mathbf{U}_2(t,\mathbf{x}_1) = \mathcal{J}_{\widetilde{\mathbf{X}}_1(t,\mathbf{X}_2(t,\mathbf{x}_1))}\mathbf{u}_2(t,\mathbf{X}_2(t,\mathbf{x}_1)), & P_2(t,\mathbf{x}_1) = p_2(t,\widetilde{\mathbf{X}}_2(t,\mathbf{x}_1)), \\ \mathbf{A}_2(t) = \mathbb{Q}^T(t)\mathbf{a}_2(t), & \boldsymbol{\Omega}_2(t) = \mathbb{Q}^T(t)\boldsymbol{\omega}_2(t), \end{cases} \tag{3.4}$$

where $\mathcal{J}_{\widetilde{\mathbf{X}}_1(t,\mathbf{X}_2(t,\mathbf{x}_1))} = \frac{\partial \widetilde{\mathbf{X}}_{1_i}}{\partial \mathbf{x}_{1_j}}$. Using formulas (3.3) we see that the following equalities hold (see also [17]):

$$\mathbf{n}_1 = \mathbb{Q}^T\mathbf{n}_2, \qquad \mathbb{T}(\mathbf{u}_2,p_2)\mathbf{n}_2 = \mathbb{Q}\mathbb{T}(\mathbf{U}_2,P_2)\mathbf{n}_1. \tag{3.5}$$

Let us compute how the slip boundary condition is transformed, using the transformed solution (3.4). Let $\mathbf{u}_s^1$, $\mathbf{u}_s^2$ be the velocity of the bodies $S_1(t)$ and $S_2(t)$, respectively (see (2.3)). We define the transformed rigid velocity

$$\mathbf{U}_s^2(t,\mathbf{x}_1) = \mathbb{Q}^T\mathbf{u}_s^2(t,\mathbf{X}_2(t,\mathbf{x}_1)) = \mathbf{A}_2(t) + \boldsymbol{\Omega}_2(t) \times (\mathbf{x}_1 - \mathbf{q}_1(t)).$$

We use (3.5) to verify that $\mathbf{U}_2$ satisfies the slip boundary condition:

$$\begin{cases} \mathbf{U}_2(t,\mathbf{x}_1)\cdot\mathbf{n}_1 = \mathbb{Q}^T\mathbf{u}_2(t,\mathbf{x}_2)\cdot\mathbf{n}_1 = \mathbf{u}_2(t,\mathbf{x}_2)\cdot\mathbf{n}_2 \\ \qquad\qquad = \mathbf{u}_s^2(t,\mathbf{x}_2)\cdot\mathbf{n}_2 = \mathbf{U}_s^2(t,\mathbf{x}_1)\cdot\mathbf{n}_1 \qquad \text{on } \partial S_1(t), \\ \\ \mathbb{T}(\mathbf{U}_2,P_2)\mathbf{n}_1\cdot\boldsymbol{\tau}_1 = \mathbb{Q}^T\mathbb{T}(\mathbf{u}_2,p_2)\mathbf{n}_2\cdot\mathbb{Q}^T\boldsymbol{\tau}_2 \\ \qquad\qquad = \beta(\mathbf{u}_s^2-\mathbf{u}_2)\cdot\boldsymbol{\tau}_2 = \beta(\mathbf{U}_s^2-\mathbf{U}_2)\cdot\boldsymbol{\tau}_1 \qquad \text{on } \partial S_1(t). \end{cases}$$

Now we can prove the following lemma.

Lemma 3.1 *The transformed solution* $(\mathbf{U}_2, P_2, \mathbf{A}_2, \boldsymbol{\Omega}_2)$ *of the strong solution* $(\mathbf{u}_2, p_2, \mathbf{a}_2, \boldsymbol{\omega}_2)$, *defined by* (3.4), *satisfy the following system of equations on the fluid domain* $\Omega_F^1(t)$*:*

$$\left.\begin{aligned} &\partial_t\mathbf{U}_2 + (\mathbf{U}_2\cdot\nabla)\mathbf{U}_2 - \triangle\mathbf{U}_2 + \nabla P_2 = (\mathcal{L}-\triangle)\mathbf{U}_2 - \mathcal{M}\mathbf{U}_2 \\ &\qquad\qquad\qquad\qquad -\tilde{\mathcal{N}}\mathbf{U}_2 - (G+\nabla)P_2, \\ &\text{div } \mathbf{U}_2 = 0 \end{aligned}\right\} \text{ in } \Omega_F^1(t)\times(0,T), \tag{3.6}$$

$$\left.\begin{aligned} &(\mathbf{U}_2-\mathbf{U}_s^2)\cdot\mathbf{n}_1 = 0, \\ &\mathbb{T}(\mathbf{U}_2,P_2)\mathbf{n}_1\cdot\boldsymbol{\tau}_1 = \beta(\mathbf{U}_s^2-\mathbf{U}_2)\cdot\boldsymbol{\tau}_1 \end{aligned}\right\} \text{ on } \partial S_1(t)\times(0,T), \tag{3.7}$$

$$\mathbf{A}_2' = -\widetilde{\boldsymbol{\omega}}\times\mathbf{A}_2 - \int_{\partial S_1(t)} \mathbb{T}(\mathbf{U}_2,P_2)\mathbf{n}_1\,d\gamma(\mathbf{x}_1) \qquad \text{in } (0,T), \tag{3.8}$$

$$(J_1\boldsymbol{\Omega}_2)' = -\widetilde{\boldsymbol{\omega}}\times(J_1\boldsymbol{\Omega}_2) - \int_{\partial S_1(t)} \{(\mathbf{x}_1-\mathbf{q}_1(t))\times\mathbb{T}(\mathbf{U}_2,P_2)\mathbf{n}_1\}\;d\gamma(\mathbf{x}_1) \qquad \text{in } (0,T), \tag{3.9}$$

where the matrix J_1 *and the vector* $\widetilde{\boldsymbol{\omega}}$ *are defined by*

$$J_1 = \mathbb{Q}^T J_2\mathbb{Q} \qquad \textit{and} \qquad \widetilde{\boldsymbol{\omega}}\times\mathbf{x} = \mathbb{Q}^T\mathbb{Q}'\mathbf{x}. \tag{3.10}$$

Proof. Equations (3.6) and (3.7) follow from the standard calculations, we refer to Appendix 4 and the above considerations. The interested reader can find complete details of these calculations, for example, in the articles [8, 17]. Let us prove (3.8). We calculate

$$\begin{aligned} \mathbf{A}_2' &= (\mathbb{Q}^T\mathbf{a}_2)' = \mathbb{Q}^T\mathbf{a}_2' + (\mathbb{Q}^T)'\mathbf{a}_2 = -\int_{\partial S_1(t)} \mathbb{T}(\mathbf{U}_2,P_2)\mathbf{n}_1\,d\gamma - \mathbb{Q}^T\mathbb{Q}'\mathbb{Q}^T\mathbf{a}_2 \\ &= -\int_{\partial S_1(t)} \mathbb{T}(\mathbf{U}_2,P_2)\mathbf{n}_1\,d\gamma - \widetilde{\boldsymbol{\omega}}\times\mathbf{A}_2. \end{aligned}$$

Let us now deduce (3.9). We have that

$$\begin{aligned} \mathbb{Q}^T(J_2\boldsymbol{\omega}_2)' &= -\mathbb{Q}^T\int_{\partial S_2(t)} \{(\mathbf{x}_2-\mathbf{q}_2(t))\times\mathbb{T}(\mathbf{u}_2,p_2)\mathbf{n}_2\}\;d\gamma(\mathbf{x}_2) \\ &= -\int_{\partial S_1(t)} \{(\mathbf{x}_1-\mathbf{q}_1(t))\times\mathbb{T}(\mathbf{U}_2,P_2)\mathbf{n}_1\}\;d\gamma(\mathbf{x}_1) \end{aligned}$$

and on the other hand:

$$\begin{aligned}\mathbb{Q}^T(J_2\boldsymbol{\omega}_2)' &= \mathbb{Q}^T(\mathbb{Q}J_1\mathbb{Q}^T\boldsymbol{\omega}_2)' \\ &= \mathbb{Q}^T\mathbb{Q}'J_1\boldsymbol{\Omega}_2 + (J_1\boldsymbol{\Omega}_2)' = \widetilde{\boldsymbol{\omega}} \times (J_1\boldsymbol{\Omega}_2) + (J_1\boldsymbol{\Omega}_2)'.\end{aligned}$$

Therefore combining these last two relations we derive (3.9). Hence this lemma is proven. ■

As a consequence of the previous Lemma 3.1 we can give the weak formulation for $(\mathbf{U}_2, P_2, \mathbf{A}_2, \boldsymbol{\Omega}_2)$.

Corollary 3.1 *Let us denote by*

$$\mathbf{F} = (\mathcal{L} - \triangle)\mathbf{U}_2 - \mathcal{M}\mathbf{U}_2 - \tilde{\mathcal{N}}\mathbf{U}_2 - (G + \nabla)P_2.$$

Then the transformed solution $(\mathbf{U}_2, P_2, \mathbf{A}_2, \boldsymbol{\Omega}_2)$ *satisfies the following equality:*

$$\begin{aligned}&\int_0^T dt \int_{\Omega\backslash\partial S_1(t)} \Big(\mathbf{U}_2 \cdot \boldsymbol{\psi}_t + (\mathbf{u}_1 \otimes \mathbf{U}_2) : \mathbb{D}\boldsymbol{\psi}\Big)\, d\mathbf{x}_1 - \int_0^T dt \int_{\Omega_F^1(t)} (\mathbf{U}_2 - \mathbf{u}_1) \cdot \nabla\mathbf{U}_2 \cdot \boldsymbol{\psi}\ \ d\mathbf{x}_1 \\ &-2\mu_f \int_0^T dt \int_{\Omega\backslash\partial S_1(t)} \big(\mathbb{D}\mathbf{U}_2 : \mathbb{D}\boldsymbol{\psi} + \mathbf{F}\cdot\boldsymbol{\psi}\big)\, d\mathbf{x}_1 = \int_0^T dt \left\{\int_{\partial S_1(t)} \beta(\mathbf{U}_s^2 - \mathbf{U}_2)\cdot(\boldsymbol{\psi}_s - \boldsymbol{\psi}_f)\, d\gamma(\mathbf{x}_1)\right\} \\ &- \int_\Omega \mathbf{u}_0\boldsymbol{\psi}(0,\cdot)\, d\mathbf{x}_1 + \int_0^T (\widetilde{\boldsymbol{\omega}} \times (J_1\boldsymbol{\Omega}_2)\cdot\boldsymbol{\psi}_\omega + \widetilde{\boldsymbol{\omega}} \times \mathbf{A}_2 \cdot \boldsymbol{\psi}_h)\ dt, \qquad (3.11)\end{aligned}$$

which holds for any test function $\boldsymbol{\psi}$ *satisfying* (2.11). *Let us note that this function* $\boldsymbol{\psi}$ *is a rigid one on* $S_1(t)$, *that is*

$$\boldsymbol{\psi}(t,\mathbf{x}) = \boldsymbol{\psi}_h(t) + \boldsymbol{\psi}_\omega \times (\mathbf{x} - \mathbf{q}_1(t)) \qquad \textit{for } \mathbf{x} \in S_1(t).$$

Proof. Using the Reynolds transport theorem (2.12) we can write the inertial term as:

$$\begin{aligned}&\int_0^T dt \int_{\Omega_F^1(t)} (\partial_t\mathbf{U}_2\cdot\boldsymbol{\psi} + \mathbf{U}_2\cdot\nabla\mathbf{U}_2\cdot\boldsymbol{\psi})\ d\mathbf{x}_1 \\ =\ & -\int_0^T dt \int_{\Omega_F^1(t)} \big(\mathbf{U}_2\cdot\partial_t\boldsymbol{\psi} + \mathbf{u}_1\cdot\nabla(\mathbf{U}_2\cdot\boldsymbol{\psi}) - \mathbf{U}_2\cdot\nabla\mathbf{U}_2\cdot\boldsymbol{\psi}\big)\, d\mathbf{x}_1 - \int_\Omega \mathbf{u}_0\boldsymbol{\psi}(0,\cdot)\, d\mathbf{x}_1 \\ =\ & -\int_0^T dt \int_{\Omega_F^1(t)} \big(\mathbf{U}_2\cdot\partial_t\boldsymbol{\psi} + \mathbf{u}_1\otimes\mathbf{U}_2 : \mathbb{D}\boldsymbol{\psi} + (\mathbf{u}_1 - \mathbf{U}_2)\cdot\nabla\mathbf{U}_2\cdot\boldsymbol{\psi}\big)\, d\mathbf{x}_1 - \int_\Omega \mathbf{u}_0\boldsymbol{\psi}(0,\cdot)\, d\mathbf{x}_1.\end{aligned}$$

The rest of the proof follows directly from Lemma 3.1 in a classical way. The details can be found in Appendix A.1. of the article [2]. ■

Before proceeding with the proof we need to prove the following two Lemmas that give us estimates for the additional terms arriving from transformation in (3.6)-(3.9).

Lemma 3.2 *For the vector $\widetilde{\boldsymbol{\omega}}$ defined by* (3.10) *the following estimate holds:*

$$|\widetilde{\boldsymbol{\omega}}(t)| \le C|\boldsymbol{\omega}_1(t) - \boldsymbol{\omega}_2(t)|, \qquad t \in [0, T_0],$$

where C depends only the initial energy, precisely $C = C(\boldsymbol{\omega}_2(0), \boldsymbol{\omega}_1(0))$.

Proof. First we estimate the term $|\mathbb{Q}(t) - \mathbb{I}|$. We have

$$|\mathbb{Q}(t) - \mathbb{I}| = |\mathbb{Q}_2(t)\mathbb{Q}_1(t)^T - \mathbb{Q}_1(t)\mathbb{Q}_1(t)^T| \le C|\mathbb{Q}_2(t) - \mathbb{Q}_1(t)| \le C|\boldsymbol{\omega}_1(t) - \boldsymbol{\omega}_2(t)|.$$

Here we used the fact that $\mathbb{Q}_1^T$ is bounded in $C([0,T]; SO(3))$.

Furthermore, we estimate the time derivative $\mathbb{Q}'$. We can write

$$\begin{aligned} \mathbb{Q}' &= \mathbb{Q}_2'\mathbb{Q}_1^T + \mathbb{Q}_2(\mathbb{Q}_1^T)' = \mathbb{Q}\mathbb{Q}^T\mathbb{Q}_2'\mathbb{Q}_1^T - \mathbb{Q}_2\mathbb{Q}_1^T\mathbb{Q}_1'\mathbb{Q}_1^T \\ &= \mathbb{Q}(\mathbb{Q}^T\mathbb{Q}_2' - \mathbb{Q}_1')\mathbb{Q}_1^T. \end{aligned}$$

Therefore we have to estimate the term $\mathbb{Q}^T\mathbb{Q}_2' - \mathbb{Q}_1'$:

$$|\mathbb{Q}^T\mathbb{Q}_2' - \mathbb{Q}_1'| \le |\mathbb{Q}^T - \mathbb{I}||\mathbb{Q}_2'| + |\mathbb{Q}_1' - \mathbb{Q}_2'|.$$

Since $\mathbb{Q}_i' = \mathbb{P}_i\mathbb{Q}_i$, $i = 1, 2$ by (3.1)-(3.2) we conclude:

$$|\mathbb{Q}'(t)| \le C|\boldsymbol{\omega}_1(t) - \boldsymbol{\omega}_2(t)|,$$

where we also used that $\mathbb{Q}_i'$ are bounded by the Lipschitz continuity of $\mathbf{B}_i(t, \cdot)$ for $t \in [0, T_0]$. Finally, since a skew–symmetric matrix $\mathbb{P}_{\widetilde{\boldsymbol{\omega}}} = \mathbb{Q}^T\mathbb{Q}'$ is associated with $\widetilde{\boldsymbol{\omega}}$ (see (3.10)) we obtain the result of this lemma. ∎

Lemma 3.3 *The following estimate holds:*

$$\begin{aligned} \|(\mathcal{L} - \triangle)\mathbf{U}_2 &- \mathcal{M}\mathbf{U}_2 - \tilde{\mathcal{N}}\mathbf{U}_2 - (G + \nabla)P_2\|_{L^2(0,T_0;L^2(\Omega_F^1(t)))} \\ &\le C\left(||\mathbf{a}_1 - \mathbf{a}_2||_{L^2(0,T_0)} + ||\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2||_{L^2(0,T_0)}\right). \end{aligned}$$

where C depends only on $\|\mathbf{U}_2\|_{L^2(0,T_0;H^2(\Omega_F^1(t)))}$, $\|P_2\|_{L^2(0,T_0;H^1(\Omega_F^1(t)))}$ and $\|\mathbf{U}_2\|_{L^\infty(0,T_0;H^1(\Omega_F(t)))}$.

Proof. First we estimate transformations $\widetilde{\mathbf{X}}_2$ and $\widetilde{\mathbf{X}}_1$. Since these transformations are rigid in the neighborhood of the rigid body we have:

$$\begin{aligned} \widetilde{\mathbf{X}}_2(t, \mathbf{x}_1) &= \mathbf{q}_2 + \mathbb{Q}(\mathbf{x}_1 - \mathbf{q}_1) \\ &= \mathbf{q}_2 - \mathbf{q}_1 + \mathbb{Q}\big(\mathbf{x}_1 - (I - \mathbb{Q}^T)\mathbf{q}_1\big) \qquad \text{on } \partial S_1(t). \end{aligned}$$

The functions $\mathbf{q}$ and $\mathbb{Q}$ satisfy differential equations (2.4), then the following estimates hold for $t \in [0, T_0]$:

$$|(\mathbf{q}_1 - \mathbf{q}_2)(t)| = |\int_0^t (\mathbf{a}_1 - \mathbf{a}_2)(r)dr| \le C\|\mathbf{a}_1 - \mathbf{a}_2\|_{L^2(0,T_0)},$$

$$|(\mathbb{Q}_1 - \mathbb{Q}_2)(t)| = |\int_0^t (\mathbb{Q}_1'(r) - \mathbb{Q}_2'(r))dr| \leq C\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_{L^2(0,T_0)}.$$

Using the proof of Lemma 3.2 we can get the following estimates on $\partial S_1(t)$ for $t \in [0, T_0]$:

$$\begin{aligned} |\widetilde{\mathbf{X}}_2(t) - \mathrm{id}| &\leq |\mathbf{q}_1(t) - \mathbf{q}_2(t)| + C|\mathbb{Q}_1(t) - \mathbb{Q}_2(t)| \\ &\leq C(\|\mathbf{a}_1 - \mathbf{a}_2\|_{L^2(0,T_0)} + \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_{L^2(0,T_0)}), \end{aligned}$$

$$|\partial_t \widetilde{\mathbf{X}}_2(t)| \leq C(|\mathbf{a}_1(t) - \mathbf{a}_2(t)| + |\boldsymbol{\omega}_1(t) - \boldsymbol{\omega}_2(t)|).$$

Using the previous estimates and a standard construction of change of variables connected to the rigid motion $\widetilde{\mathbf{X}}_2$ (see the articles [17, 25] or for slightly different point of view we refer to Proposition 1 and Corollary 1 of [18]), one gets the following estimates:

$$\left.\begin{aligned} &\|\widetilde{\mathbf{X}}_2(t,.) - \mathrm{id}\|_{W^{2,\infty}(\Omega_F^1(t))} \leq C(\|\mathbf{a}_1 - \mathbf{a}_2\|_{L^2(0,T_0)} + \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_{L^2(0,T_0)}), \\ &\|\partial_t \widetilde{\mathbf{X}}_2(t,.)\|_{W^{1,\infty}(\Omega_F^1(t))} \leq C(|\mathbf{a}_1(t) - \mathbf{a}_2(t)| + |\boldsymbol{\omega}_1(t) - \boldsymbol{\omega}_2(t)|), \end{aligned}\right\} \quad t \in [0, T_0]. \tag{3.12}$$

Analogous estimates can be derived for $\widetilde{\mathbf{X}}_1$.

To finish the proof we use the formulas for the transformed differential operators (4.1)-(4.4). Estimates (3.12) imply:

$$\begin{aligned} \|g_{ij}(t) - \delta_{ij}\|_{W^{1,\infty}(\Omega_F(t))} &+ \|g^{ij}(t) - \delta_{ij}\|_{W^{1,\infty}(\Omega_F(t))} + \|\Gamma_{ij}^k(t)\|_{L^\infty(\Omega_F(t))} \\ &\leq C(|\mathbf{a}_1 - \mathbf{a}_2|_{L^2(0,T_0)} + |\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2|_{L^2(0,T_0)}), \qquad t \in [0, T_0]. \end{aligned}$$

The proof of this lemma follows from the fact that

$$\mathbf{U}_2 \in L^2(0, T_0; H^2(\Omega_F^1(t))) \cap L^\infty(0, T_0; H^1(\Omega_F(t))), \qquad P_2 \in L^2(0, T_0; H^1(\Omega_F^1(t))).$$

■

Let us give a principal lemma of our article from which Theorem 3.1 follows.

Lemma 3.4 *We have that*

$$(\mathbf{u}_1, p_1, \mathbf{a}_1, \boldsymbol{\omega}_1) = (\mathbf{U}_2, P_2, \mathbf{A}_2, \boldsymbol{\Omega}_2) \qquad \textit{and} \qquad (\mathbf{u}_1, \mathbf{B}_1) = (\mathbf{u}_2, \mathbf{B}_2).$$

Proof. First we subtract equality (3.11) for $(\mathbf{U}_2, P_2, \mathbf{A}_2, \boldsymbol{\Omega}_2)$ from equality (2.10) for $(\mathbf{u}_1, p_1, \mathbf{a}_1, \boldsymbol{\omega}_2)$. In the obtained identity for the difference

$$(\mathbf{u}, p, \mathbf{a}, \boldsymbol{\omega}) = (\mathbf{u}_1 - \mathbf{U}_2, p_1 - P_2, \mathbf{a}_1 - \mathbf{A}_2, \boldsymbol{\omega}_1 - \boldsymbol{\Omega}_2)$$

we can take the test function $\boldsymbol{\psi}(r) = \mathbf{u}(1 - sgn^{\varepsilon}_{+}(r-t))$ for any fixed $t \in (0,T)$ and pass on $\varepsilon \to 0$, that gives the identity:

$$\begin{aligned} &\int_0^t dr \int_{\Omega\setminus\partial S_1(r)} \Big(-\frac{1}{2}\partial_t|\mathbf{u}|^2 - \mathbf{u}_1 \otimes \mathbf{u} : \mathbb{D}\mathbf{u}\Big)\, d\mathbf{x} + \int_0^t dr \int_{\Omega^1_F(r)} \mathbf{u}\cdot\nabla\mathbf{U}_2\cdot\mathbf{u}\, d\mathbf{x} \\ &+ \int_{\Omega\setminus\partial S_1(t)} |\mathbf{u}(t)|^2\, d\mathbf{x} - \int_{\Omega_F(0)} |\mathbf{u}_0|^2\, d\mathbf{x} - 2\mu_f \int_0^t dr \int_{\Omega\setminus\partial S_1(r)} \big(|\mathbb{D}\mathbf{u}|^2 + \mathbf{F}\cdot\mathbf{u}\big)\, d\mathbf{x} \\ = \; &\int_0^t dr \int_{\partial S(r)} \beta|\mathbf{u}_s - \mathbf{u}|^2 d\gamma + \int_0^t (\widetilde{\boldsymbol{\omega}} \times (J_1\boldsymbol{\Omega}_2)\cdot\boldsymbol{\omega} + \widetilde{\boldsymbol{\omega}} \times \mathbf{A}_2\cdot\mathbf{a})\; dr. \end{aligned} \tag{3.13}$$

The main difficulty in the study of this relation is to estimate the difference of the convective terms.

Let us combine the convective terms with the fluid acceleration term, then we have:

$$\begin{aligned} &\int_0^t dr \int_{\Omega^1_F(t)} \Big(-\frac{1}{2}\partial_t|\mathbf{u}|^2 - \mathbf{u}_1\otimes\mathbf{u} : \nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{U}_2\cdot\mathbf{u}\Big)\, d\mathbf{x} + \int_{\Omega^1_F(t)} |\mathbf{u}(t)|^2\; d\mathbf{x} - \int_{\Omega_F(0)} |\mathbf{u}_0|^2\; d\mathbf{x} \\ &= \frac{1}{2}\Big(\int_{\Omega^1_F(t)} |\mathbf{u}(t)|^2\; d\mathbf{x} - \int_{\Omega_F(0)} |\mathbf{u}_0|^2\; d\mathbf{x}\Big) + \int_0^t dr \int_{\Omega^1_F(t)} \mathbf{u}\cdot\nabla\mathbf{U}_2\cdot\mathbf{u}\;\; d\mathbf{x}. \end{aligned}$$

By integration by parts the last term in the right hand side of this identity is written as:

$$\int_0^t dr \int_{\Omega^1_F(t)} \mathbf{u}\cdot\nabla\mathbf{U}_2\cdot\mathbf{u}\; d\mathbf{x} = -\int_0^t dr \int_{\Omega^1_F(t)} \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{U}_2\; d\mathbf{x} + \int_0^t dr \int_{\partial\Omega^1_F(t)} (\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{U}_2)\; d\gamma. \tag{3.14}$$

The first term in the right-hand side of (3.14) can be estimated in the standard way (see e.g. Temam [26]) by using the interpolation:

$$\begin{aligned} \Big|\int_0^t dr \int_{\Omega^1_F(t)} \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{U}_2\; d\mathbf{x}\Big| \;&\le\; \int_0^t \|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{U}_2\|_{L^4}\; dr \\ &\le\; C\int_0^t (\|\mathbf{u}\|^{1/4}_{L^2}\|\nabla\mathbf{u}\|^{3/4}_{L^2} + \|\mathbf{u}\|_{L^2})\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{U}_2\|_{L^4}\; dr \\ &\le\; C\varepsilon\int_0^t \|\nabla\mathbf{u}\|^2_{L^2}\; dr + \frac{C}{\varepsilon}\int_0^t \|\mathbf{u}\|^2_{L^2}\big(\|\mathbf{U}_2\|^2_{L^4} + \|\mathbf{U}_2\|^8_{L^4}\big)\; dr. \end{aligned}$$

The second term in the right-hand side of (3.14) is estimated as follows:

$$
\begin{aligned}
& |\int_0^t dr \int_{\partial\Omega_F^1(r)} (\mathbf{u}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{U}_2)\ d\gamma| \\
\leq\ & C\int_0^t (|\mathbf{a}(r)| + |\boldsymbol{\omega}(r)|)\|\mathbf{u}(r)\|_{L^2(\partial S_1(r))}\|\mathbf{U}_2(r)\|_{L^2(\partial S_1(r))}\ dr \\
\leq\ & C\int_0^t (|\mathbf{a}(r)| + |\boldsymbol{\omega}(r)|)\|\mathbb{D}\mathbf{u}(r)\|_{L^2(\Omega_F^1(r))}\|\mathbb{D}\mathbf{U}_2(r)\|_{L^2(\Omega_F^1(r))}\ dr \\
\leq\ & \frac{C}{\varepsilon}\int_0^t \|\mathbf{u}(r)\|^2_{L^2(\Omega)}\|\mathbb{D}\mathbf{U}_2(r)\|^2_{L^2(\Omega_F^1(r))}\ dr + C\varepsilon\int_0^t \|\mathbb{D}\mathbf{u}(r)\|^2_{L^2(\Omega_F^1(r))}\ dr \\
\leq\ & \frac{C}{\varepsilon}\int_0^t \|\mathbf{u}(r)\|^2_{L^2(\Omega)}\ dr + C\varepsilon\int_0^t \|\mathbb{D}\mathbf{u}(r)\|^2_{L^2(\Omega_F^1(r))}\ dr
\end{aligned}
$$

Moreover, applying (2.12) of the Reynolds transport theorem we have

$$
\frac{1}{2}\frac{d}{dt}\int_{S_1(t)} |\mathbf{u}|^{\mathbf{2}}\, d\mathbf{x} = \frac{1}{2}\int_{S_1(t)} \partial_t|\mathbf{u}|^2 d\mathbf{x} + \int_{S_1(t)} \mathbf{u}_1\otimes\mathbf{u} : \nabla\mathbf{u}\, d\mathbf{x}.
$$

Finally, the remainder terms in (3.13) can be estimated by using Lemmas 3.3 and 3.2.

By putting this estimates together we conclude:

$$
\begin{aligned}
\|\mathbf{u}(t)\|^2_{L^2(\Omega)} \ &+\ 2\int_0^t \|\mathbb{D}\mathbf{u}\|^2_{L^2(\Omega)}\ dr \leq \|\mathbf{u}_0\|^2_{L^2(\Omega)} \\
&\quad +C\int_0^t \varepsilon\|\mathbb{D}\mathbf{u}\|^2_{L^2(\Omega)}\ dr + C\int_0^t (|\mathbf{a}(r)|^2 + |\boldsymbol{\omega}(r)|^2)\ dr \\
&+\ \frac{C}{\varepsilon}\int_0^t \|\mathbf{u}\|^2_{L^2(\Omega_F^1(r))}\big(\|\mathbf{U}_2(r)\|^2_{L^4(\Omega_F^1(r))} + \|\mathbf{U}_2(r)\|^8_{L^4(\Omega_F^1(r))} + 1\big)\ dr \\
&+\ C\int_0^t (|\mathbf{a}(r)| + |\boldsymbol{\omega}(r)|)\|\mathbf{u}(r)\|_{L^2(\Omega_F(r))}\ dr.
\end{aligned}
$$

Using Young's inequality and taking ε such that $\|\mathbb{D}\mathbf{u}\|$ term can be absorbed in the left-hand side we get:

$$
\|\mathbf{u}(t)\|^2_{L^2(\Omega)} \leq C\int_0^t \|\mathbf{u}(r)\|^2_{L^2(\Omega)}\big(1 + \|\mathbf{U}_2(r)\|^2_{L^4(\Omega_F^1(r))} + \|\mathbf{U}_2(r)\|^8_{L^4(\Omega_F^1(r))}\big)\ dr.
$$

Hence we finish the proof by applying the integral Gronwall's inequality and conclude that $\mathbf{u} = 0$. ■

4 Appendix - Local transformation

Since the fluid domain depends on the motion of the rigid body, we transform the problem to a fixed domain. We define the local transformation as in Takahaski [25]. Let us point that such type of transformation firstly was suggested by Inoue and Wakimoto [8] and then extensively used in the context of strong solution to fluid-rigid body systems (see e.g. [17, 25]). Here we just briefly repeat the main facts about this transformation for the convenience of the reader. Let us just emphasize that our case is slightly different since we are not transforming to the fixed cylindrical domain, but form one moving domain to the other. However, the essential fact for this transformation is that the change of variable is volume preserving diffeomorphism - which is true also on our case.

Let us $\delta(t) = dist(S(t), \partial\Omega)$. We fix δ_0 such that $\delta(t) > \delta_0$ and define the solenoidal velocity field $\Lambda(t, \mathbf{x})$ such that $\Lambda = 0$ in the $\delta_0/4$ neighborhood of $\partial\Omega$, $\Lambda = \mathbf{a}(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{q}(t))$ in the $\delta_0/4$ neighborhood of $S(t)$. Let us define the flow $\mathbf{X}(t) : \Omega \to \Omega$ as the unique solution of the system

$$\frac{d}{dt}\mathbf{X}(t, \mathbf{y}) = \Lambda(t, \mathbf{X}(t, \mathbf{x})), \qquad \mathbf{X}(0, \mathbf{y}) = \mathbf{y}, \qquad \forall\, \mathbf{z} \in \Omega.$$

We denote $\mathbf{Y}$ the inverse of $\mathbf{X}$, i.e.

$$\mathbf{Y}(t, \cdot) = \mathbf{X}(t, \cdot)^{-1}.$$

Let us write the unknown functions $(\mathbf{u}, p, \boldsymbol{\omega}, \mathbf{a}, \mathbb{T})$ by the change of variables $\mathbf{x} \to \mathbf{y}$. Then in the system of coordinates $\mathbf{y} \in \Omega$ we obtain the new unknown functions

$$\left.\begin{array}{ll} \mathbf{U}(t, \mathbf{y}) = \mathcal{J}_Y(t, \mathbf{X}(t, \mathbf{y}))\mathbf{u}(t, \mathbf{X}(t, \mathbf{y})), & P(t, \mathbf{y}) = p(t, \mathbf{X}(t, \mathbf{y})), \\ \Xi(t) = \mathbb{Q}^t(t)\boldsymbol{\omega}(t), & \xi(t) = \mathbb{Q}^t(t)\mathbf{a}(t), \\ \mathcal{T}(\mathbf{U}(t, \mathbf{y}), P(t, \mathbf{y})) = \mathbb{Q}^T(t)\mathbb{T}(\mathbb{Q}(t)\mathbf{U}(t, \mathbf{y}), P(t, \mathbf{y}))\mathbb{Q}(t) & \end{array}\right\} \text{ for } t \in [0, T], \;\; \mathbf{y} \in \Omega_0,$$

The Jacobian of this change of variables $\mathbf{x} \to \mathbf{y}$ is denoted by

$$\mathcal{J}_Y(t, \mathbf{X}(t, \mathbf{y})) = \left(\frac{\partial Y_i}{\partial y_j}\right),$$

In the sequel we derive a system which satisfies the new unknown functions $(\mathbf{U}, P, \Xi, \xi, \mathcal{T})$. Our system (2.6)-(2.9) is written in terms of the variables $(t, \mathbf{x})$, therefore we have to rewrite these equations in terms of the new variables $\mathbf{Y} = \mathbf{Y}(t, \mathbf{x})$.

Let us first note that the determinant of the Jacobian $\mathcal{J}_Y$ equals to 1, since is a divergence free vector field. Hence using these change of variables we have

$$\begin{aligned} \int_{\partial S(t)} \mathbb{T}(\mathbf{u}, p)\mathbf{n}(t)\; d\gamma(\mathbf{x}) \;&=\; \mathbb{Q}\int_{\partial S(0)} \mathcal{T}(\mathbf{U}, P)\mathbf{N}\; d\sigma(\mathbf{y}), \\ \int_{\partial S(t)} (\mathbf{x} - \mathbf{q}(t)) \times \mathbb{T}(\mathbf{u}, p)\mathbf{n}(t)\; d\gamma(\mathbf{x}) \;&=\; \mathbb{Q}\int_{\partial S(0)} \mathbf{y} \times \mathcal{T}(\mathbf{U}, P)\mathbf{N}\; d\sigma(\mathbf{y}), \end{aligned}$$

where $d\sigma$ indicates the surface measure over non-moving surface $\partial S(0)$.

In the sequel we derive the equations which satisfy these new unknown functions $(\mathbf{U}, P, \Xi, \xi, \mathcal{T})$. Let us introduce the metric covariant tensor

$$g_{ij} = X_{k,i}X_{k,j}, \qquad X_{k,i} = \frac{\partial X_k}{\partial y_i}$$

the metric covariant tensor

$$g^{ij} = Y_{i,k} Y_{j,k} \qquad Y_{i,k} = \frac{\partial Y_i}{\partial x_k}$$

and the Christoffel symbol (of the second kind)

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}). \qquad g_{il,j} = \frac{\partial g_{il}}{\partial y_j}$$

It is easy to observe that in particular it holds

$$\Gamma^k_{ij} = Y_{k,l} X_{l,ij}. \qquad X_{l,ij} = \frac{\partial X_l}{\partial y_i \partial y_j}$$

Hence under the change of variables $\mathbf{x} \to \mathbf{y}$ the operator $\mathcal{L}$ is the transformed Laplace operator and it is given by

$$(\mathcal{L}\mathbf{u})_{ij} = \sum_{j,k=1}^{n} \partial_j (g^{jk} \partial_k \mathbf{u}_i) + 2 \sum_{j,k,l=1}^{n} g^{kl} \Gamma^i_{jk} \partial_l \mathbf{u}_j + \sum_{j,k,l=1}^{n} \Big(\partial_k (g^{kl} \Gamma^i_{kl}) + \sum_{m=1}^{n} g^{kl} \Gamma^m_{jl} \Gamma^i_{km} \Big) \mathbf{u}_j. \tag{4.1}$$

The convection term is transformed into

$$(\mathcal{N}\mathbf{u})_i = \sum_{j=1}^{n} \mathbf{u}_j \partial_j \mathbf{u}_i + \sum_{j,k+1}^{n} \Gamma^i_{jk} \mathbf{u}_j \mathbf{u}_k. \tag{4.2}$$

The transformation of time derivative and gradient are given by

$$(\mathcal{M}\mathbf{u})_i = \sum_{j=1}^{n} \dot{\mathbf{Y}}_j \partial_j \mathbf{u}_i + \sum_{j,k=1}^{n} \Big(\Gamma^i_{jk} \dot{\mathbf{Y}}_k + (\partial_k \mathbf{Y}_i)(\partial_j \dot{\mathbf{X}}_k) \Big) \mathbf{u}_j. \tag{4.3}$$

The gradient of pressure is transform as follows

$$(\mathcal{G}p)_i = \sum_{j=1}^{n} g^{ij} \partial_j p. \tag{4.4}$$

Therefore combining all formulas of transformed differential operators (4.1)-(4.4) we see that after the change of variables the system (2.6)-(2.9) is transformed into the following system

$$\left.\begin{array}{rcl} \mathbf{U}_t + (\mathcal{M} - \mathcal{L})\mathbf{U} & = & -\mathcal{N}(\mathbf{U}) - \mathcal{G}p, \\ \text{div } \mathbf{U} & = & 0 \end{array}\right\} \text{ in } \Omega_F(0) \times (0,T),$$

$$\left.\begin{array}{rcl} m \frac{d}{dt} \boldsymbol{\xi} & = & -m(\boldsymbol{\Xi} \times \boldsymbol{\xi}) - \int_{\partial S(0)} \mathcal{T}(\mathbf{U}, P)\mathbf{N} d\sigma, \\ I \frac{d}{dt} \boldsymbol{\Xi} & = & \boldsymbol{\Xi} \times (I\boldsymbol{\Xi}) - \int_{S(0)} \mathbf{y} \times \mathcal{T}(\mathbf{U}, P)\mathbf{N} d\sigma \end{array}\right\} \text{ in } (0,T),$$

$$\begin{aligned}(\mathbf{U}-\mathbf{U}_s)\cdot\mathbf{N} &= 0,\\ \beta(\mathbf{U}-\mathbf{U}_s)\cdot\tau &= -2(\mathbb{D}(\mathbf{U})\mathbf{N}\cdot\tau \qquad \text{on } \partial S(0),\end{aligned}$$

$$\begin{aligned}\mathbf{U} &= 0 \quad \text{on } \partial\Omega,\\ \boldsymbol{\xi}(0) &= \mathbf{a}(0) \quad \text{and} \quad \Xi(0)=\boldsymbol{\omega}(0),\end{aligned}$$

where $\mathbf{U}_s = (\boldsymbol{\Xi}(t)\times\mathbf{y}+\xi(t))$ is the transformed rigid velocity $\mathbf{u}_s$; $\mathbf{N}=\mathbf{N}(y)$ is the unit normal at $\mathbf{y}\in\partial S(0)$, directed inside of $S(0)$; $I=\mathbb{Q}^t\mathbb{J}\mathbb{Q}$ is the transformed inertia tensor which no longer depends on time (see details in the article [17]).

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